Evaluation of Motion-Transmission Characteristics of Planar Six-Link Mechanisms with a Prismatic Pair*

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An index of motion-transmission characteristics $r$ of the planar six-link mechanism with a prismatic pair is presented through investigations of the sensitivities of the angular and linear displacements of the links to the deviations of the kinematic constants and the forces acting on the pairs due to the external forces on the links. Generally speaking, the average of the sensitivities $S$ and the average of the forces acting on the pairs $F$ decrease rapidly in the interval $[0, 0.3]$ and slowly in the $[0.3, 1]$ as $r$ increases. The $r-S$ and $r-F$ relations are approximated to the regions bounded by two rectangular hyperbolas. Consequently, it is possible to estimate the values of $S$ and $F$ for any mechanism at the given crank angle by means of $r$.

Key Words: Kinematics, Mechanism, Prismatic Pair, Planar Six-Link Mechanism, Transmission Index, Sensitivity, Force Acting on Pair

1. Introduction

For the analytical synthesis of mechanisms, a mathematical model is established so as to simulate the functional relation between the input and output quantities, and the values of its parameters are determined so that the functional relation may approximate the desired one with an acceptable accuracy. In addition, a constraint with respect to the motion-transmission characteristics must be satisfied, which evaluates whether the mechanism may operate smoothly or not, taking into consideration manufacturing and assembly errors as well as frictional forces acting on the pairs.

The motion-transmission characteristics of six-link mechanisms, which are composed of the planar four-link mechanism and the two-link chain driven by its coupler point, as shown in Fig. 1, and the ratios of whose link lengths to the longest one are about unity, may be evaluated by two transmission angles $\theta_2$.

However, in the case of the six-link mechanism, wherein both links of the component two-link chain are connected by means of the prismatic pair, as shown in Fig. 2, the second transmission angle $\theta_2$ is a constant value of 90° regardless of the relative displacement between them, thus it does not play the role of the index of the motion-transmission characteristics. Therefore, it is necessary to introduce a new index.

For example, the six-link dwell-motion mechanism synthesized by K. Hain** has poor motion-trans-

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mission characteristics in a position such that the distance from the stationary pairing point of the driving link to the slider is small. At this position, it is impossible to rotate the driving link of the experimental apparatus manufactured by the ordinary process in one direction. This mechanism and the relation between the input and output angles are shown in Figs. 3 and 4, respectively. The values $\theta_{\text{in}}, \varepsilon_i (i=1, 2)$ in Fig. 4 are the range of the input angle and the structural error of the dwell motion at two positions, respectively.

In this paper, an index of the motion-transmission characteristics of the six-link mechanism having one prismatic pair is investigated, as in preceding papers\(^{[10]}\), on the basis of the ratio of the deviations of the angular and linear displacements of the moving links to the small changes in the kinematic constants, namely the sensitivities, and the forces acting on the pairs due to the external forces on the links.

2. Sensitivity Analysis

Let O-xyz denote the stationary rectangular coordinate system. Let $\theta_0$ and $\theta_k$ denote respectively, the angular displacements of the driving link and the driven link of the six-link mechanism of the Stephenson type, whose stationary link DA is located on the x-axis as shown in Fig. 2. Furthermore, let $\theta_0$ and $\theta_k$ denote respectively, the angular displacements of links BC and CD, and let $s$ denote the linear displacement of sliders EF.

The link lengths $Z_0, Z_1, Z_2, Z_3, Z_4, Z_5$, the polar coordinates $Z_i, \alpha_i$ of the coupler point on the coupler link, and the rectangular coordinates $X_0, Y_0$ of the stationary pairing point G are adopted as nine kinematic constants.

2.1 Displacement analysis

Letting $(x_0, y_0)$ and $(x_e, y_e)$ denote respectively the rectangular coordinates of pairing points B and E, we have the following equations.

$$\begin{align}
x_0 &= Z_0 + Z_1 \cos \theta_0 = Z_2 \cos \theta_0 + Z_3 \cos \theta_1 \\
y_0 &= Z_1 \sin \theta_0 = Z_2 \sin \theta_0 + Z_3 \sin \theta_1 \\
x_e &= Z_1 \cos (\theta_0 + \alpha_0) + Z_2 \cos \theta_1 \\
y_e &= Z_1 \sin (\theta_0 + \alpha_0) + Z_2 \sin \theta_1 \\
&= y_0 + Z_3 \cos \theta_k + s \sin \theta_k
\end{align}$$

Eliminating $\theta_0$ from Eqs. (1) and (2), the formulas for determining the value of $\theta_0$ are obtained as follows.

$$\begin{align}
r_1 &= (x_0^2 + y_0^2 - Z_0^2 + Z_1^2) / 2Z_1 \\
r_2 &= (x_e^2 + y_e^2 - r_0^2) \\
\theta_0 &= 2 \tan^{-1} \left( \frac{y_0 + \sqrt{r_0^2 - x_0^2}}{x_0 + r_1} \right)
\end{align}$$

Then, the value of $\theta_0$ is determined using the following equations.

$$\begin{align}
x_c &= Z_0 \cos \theta_c, \quad y_c = Z_0 \sin \theta_c \\
\theta_c &= \tan^{-1} \left( \frac{y_c - y_0}{x_c - x_0} \right)
\end{align}$$

Eliminating $s$ for Eqs. (3) and (4), the formulas for determining the value of $\theta_k$ are obtained as follows.

$$\begin{align}
r_3 &= (x_e - x_0)^2 + (y_e - y_0)^2 - 2Z_3^2 \\
\theta_k &= -2 \tan^{-1} \left( \frac{x_e - x_0 \pm \sqrt{r_3}}{y_e - y_0 + Z_3} \right)
\end{align}$$
Then, the value of \( s \) is determined using the following equation.

\[
s = \frac{(x - x_0 + Z_s \sin \theta_h)}{\cos \theta_h}
\]  
(8)

2.2 Average of sensitivities

There exist nine kinematic constants. For example, with respect to the link length \( Z_s \), we derive the sensitivities of the angular displacements \( \theta_1 \), \( \theta_2 \), and \( \theta_8 \), and the linear displacement \( s \).

By partially differentiating both sides of Eqs. (1) through (4) with respect to \( Z_s \), we obtain the following equations:

\[
\begin{bmatrix}
Z_s S_1 & Z_s C_1 & 0 & 0 \\
-ZaC_1 & Z_s C_3 & 0 & 0 \\
Z_s S_4 & Z_s S_3 & (Z_s C_5 + sS_3) & C_6 \\
-Z_s C_4 & -Z_s C_3 & (Z_s S_4 - sC_3) & S_8
\end{bmatrix}
\begin{bmatrix}
\partial \theta_1 / \partial Z_s \\
\partial \theta_2 / \partial Z_s \\
\partial \theta_3 / \partial Z_s \\
\partial \theta_8 / \partial Z_s
\end{bmatrix}
\]

\[
= \begin{bmatrix}
C_1 \\
S_1 \\
C_3 \\
S_3
\end{bmatrix}
\]

where

\[ C_i = \cos \theta_i, \quad S_i = \sin \theta_i \quad (i = 2, 3, 6) \]
\[ C_1 = \cos (\theta_1 + a_1), \quad S_1 = \sin (\theta_1 + a_1) \]

By solving Eq. (9) for \( \partial \theta_1 / \partial Z_s, \partial \theta_2 / \partial Z_s, \partial \theta_3 / \partial Z_s \) and \( \partial \theta_8 / \partial Z_s \), the sensitivities with respect to \( Z_s \) are obtained. We adopt the average of the absolute values of twenty-four sensitivities of Eq. (10) exclusive of the other sensitivities to be always zero as a quantity for evaluating the motion-transmission characteristics.

\[
\begin{bmatrix}
\partial \theta_i / \partial Z_s \\
\partial \theta_i / \partial Z_s \\
\partial \theta_i / \partial Z_s \\
\partial \theta_i / \partial Z_s
\end{bmatrix}
\]

Here, in order to match the dimensions of sensitivities, those of the angular displacement with respect to the kinematic constants of the length are multiplied by the average of link lengths:

\[
Z_{ak} = \frac{1}{3} (Z + Z_0 + \sqrt{(x - x_0)/2})^2 + y^2 \\
+ \sqrt{(Z_0/2)^2 + \frac{1}{2} Z_0Z_0 \cos a_1)/7}
\]

(11)

Also, the sensitivities of the linear displacement with respect to the kinematic constant of the angle are divided by \( Z_{ak} \).

3. Forces Acting on Pairs

In the operation of the mechanism, the loads, the forces of inertia, and the gravitational forces act on the links as external forces, thus the torque required for balancing these forces is applied to the driving link. At the same time, the forces act on the revolute pairs and the force and the moment act on the prismatic pair.

3.1 Formulas

Let \( F_x, F_y, F_z \) and \( F_r \) respectively denote magnitudes of external forces on the links BC, CD, EF and FG, which act along lines passing through the points \( H_3, H_4, H_5 \) and \( H_6 \), to make the angles of \( \beta_2, \beta_3, \beta_4 \), and \( \theta_8 \) with respect to lines BC, CD, EF and FG. Moreover, let the input torque, the components along the \( x \)- and \( y \)-axes of the forces acting on three stationary pairs A, D and G, and the components along the binary links and their perpendiculars of the forces acting on the moving pairs of the ternary links B, C and E be denoted by the symbols shown in Fig. 5. Furthermore, let \( F_{in} \) and \( M_r \) denote the force and the moment acting on the prismatic pair \( F \), respectively.

Then, we have the formulas for estimating the orthogonal components of the forces acting on the pairs.

\[
\begin{align*}
F_{x} &= -F_y \sin \theta_8 / Z_3 \\
F_{y} &= -F_z \sin \theta_8 / Z_3 \\
F_{z} &= (F_x Z_3 - \sin \theta_8 / Z_3) - F_z \cos \theta_8 / Z_3 \\
F_{r} &= F_x \sin \theta_8 / Z_3 \\
F_{r} &= F_y \sin \theta_8 / Z_3 \\
F_{r} &= F_z \sin \theta_8 / Z_3 \\
F_{r} &= F_x \sin \theta_8 / Z_3 \\
F_{r} &= F_y \sin \theta_8 / Z_3 \\
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F_{r} &= F_z \sin \theta_8 / Z_3 \\
F_{r} &= F_x \sin \theta_8 / Z_3 \\
F_{r} &= F_y \sin \theta_8 / Z_3 \\
F_{r} &= F_z \sin \theta_8 / Z_3 \\
F_{r} &= F_x \sin \theta_8 / Z_3 \\
F_{r} &= F_y \sin \theta_8 / Z_3 \\
F_{r} &= F_z \sin \theta_8 / Z_3 \\
\end{align*}
\]

where

\( \theta_8 = \theta_8 + 90^\circ \)
Therefore, the forces acting at the revolute pairs A, B, C, D, E and G, and the force along the line EF at the prismatic pair F are given by the following equations.

\[
\begin{align*}
F_x &= \sqrt{F_{ax}^2 + F_{ay}^2}, \quad F_y = \sqrt{F_{bx}^2 + F_{by}^2} \\
F_c &= \sqrt{F_{cx}^2 + F_{cy}^2}, \quad F_d = \sqrt{F_{dx}^2 + F_{dy}^2} \\
F_e &= \sqrt{F_{ex}^2 + F_{ey}^2}, \quad F_f = \sqrt{F_{ex}^2 + F_{ey}^2} \\
F_c &= \sqrt{F_{cx}^2 + F_{cy}^2}
\end{align*}
\] (13)

### 3.2 Average of forces acting on pairs

Even though the load on the driven link is known, the forces acting on the pairs change in a complex manner according to the form of the links, the operating speed, the state of the installation of the mechanism, and so on. Therefore, it is impossible to generalize the properties of the forces acting on the pairs considering these influences.

For these reasons, we undertake the investigation using the average of seven forces acting on the pairs in the case where the magnitude, the direction, and the distance to the point of action, \((F_i, \beta, \text{and } h_i, \text{respectively, where } i=2,3,5 \text{ and } 6)\) of the external forces are given randomly in the following intervals:

\[
\begin{align*}
&[0, 1], \quad [0, 2\pi] \\
&[-2Z, Z], \quad [Z, 2Z]
\end{align*}
\] (14)

where \(Z = s\).

### 4. Index of Motion-Transmission Characteristics

In cases of ordinary nonuniform-motion mechanisms, the domains of motion whose input and output links are large, the absolute values of the sensitivities and the ratios of the forces acting on the pairs to the external forces generally need to be restricted to small values.

#### 4.1 Definition

The denominator of the formulas for estimating the sensitivities coincides with the determinant composed of the coefficients of Eq. (9).

\[
\Delta = \begin{vmatrix}
Z_aS_a & Z_aS_b & -(Z_aC_a + sS_a) & C_a \\
-Z_aC_b & -Z_aC_b & -(Z_aS_a - sC_a) & S_a \\
Z_bS_a & Z_bS_b & -(Z_bC_b + sS_b) & C_b \\
-Z_bC_a & -Z_bC_b & -(Z_bS_a - sC_a) & S_b
\end{vmatrix} \\
= Z_aZ_b \sin (\theta_a - \theta_b)
\] (15)

The absolute value of \(\Delta\) should be increased so that the absolute values of the sensitivities are decreased. This requirement is realized in the case where the ratios of two link lengths \(Z_a\) and \(Z_b\) to the average of the link lengths approach unity, and the linear displacement of the slider \(s\) and the absolute value of \(\sin (\theta_a - \theta_b)\) are not extremely small.

The difference between two angles \(\theta_a\) and \(\theta_b\) coincides with the transmission angle \(\mu_t\) of the component four-link mechanism of the planar six-link mechanism with a prismatic pair. Moreover, the factor \(s \sin (\mu_t)\) is also contained in the denominator of the formulas for the estimation of forces acting on the pairs. Therefore, this factor is considered useful as the main component of the index of motion-transmission characteristics.

Due to these reasons, we present a new index for estimating the motion-transmission characteristics as follows.

\[
\tau = \lambda_1 \lambda_2 \lambda_3 \sin (\mu_t)
\] (16)

\[
\lambda_1 = \begin{cases} 
1 & \text{if } Z_{a4} \geq Z_{a3} \\
(Z_{a4}/Z_{a3}) & \text{if } Z_{a4} < Z_{a3}
\end{cases}
\]

\[
\lambda_2 = \begin{cases} 
1 & \text{if } Z_{a3} \geq Z_{a2} \\
(Z_{a3}/Z_{a2}) & \text{if } Z_{a3} < Z_{a2}
\end{cases}
\]

\[
\lambda_3 = \begin{cases} 
1 & \text{if } s \geq Z_{a3} \\
(s/Z_{a3}) & \text{if } s < Z_{a3}
\end{cases}
\]

where \(Z_{a4}\) is the average of the link lengths of a component four-link mechanism.

\[
Z_{a4} = (Z_a + Z_b + Z_t + Z_s)/4
\] (17)

Furthermore, \(Z_{a3}\) is the smallest value of \(Z_a\) and \(Z_b\), and \(Z_{a2}\) is given by the following equation:

\[
Z_{a2} = \sqrt{(Z_a - Z_b)/2}^2 + y_b^2 + \sqrt{(Z_b/2)^2 + Z_l^2 - Z_4Z_a \cos \alpha_t}/3
\] (18)

#### 4.2 Simulation

Let \(U_i (i=1 \sim 9)\) denote the set of nine uniform random numbers in the interval \([0, 1]\). Then, we determine the values for four link lengths \(Z_{a4} (i=1 \sim 4)\) from \(U_i (i=1 \sim 4)\) using Garrett's formula (9) as follows:

\[
Z_{a4} = \begin{cases} 
0.6U_i + 0.2 & \text{if } 0 \leq U_i < 0.5 \\
3U_i - 1 & \text{if } 0.5 \leq U_i < 1
\end{cases}
\] (i=1 \sim 4)

Let \(C-x' y')\) denote the rectangular coordinate system wherein the origin, the \(x'\)-axis and \(y'\)-axis coincide with the center of the revolute pair \(C\), the central line of the coupler link \(BC\), and the line perpendicular to \(BC\), respectively.

We determine the coordinates \((x', y')\) of the coupler point in the square wherein the center is \((Z_4/2, 0)\) and the lengths of four sides are \(4Z_{a4}\), as follows:

\[
x' = 4Z_{a4}(U_4 - 0.5) + 0.5Z_4 \\
y' = 4Z_{a4}U_5 - 0.5
\] (20)

Then, the kinematic constants \(Z_4\) and \(\alpha_t\) are given by the following equations.

\[
Z_4 = \sqrt{x'^2 + y'^2} \\
\alpha_t = \tan^{-1}(y'/x')
\] (21)

We determine the link length \(Z_4\) from \(U_4\) by Eq. (19), and the coordinates \((x_c, y_c)\) of the stationary revolute pair \(G\) from \(U_5\) and \(U_6\), using the following equations.

\[
x_c = 4Z_{a4}(U_4 - 0.5) + 0.5Z_4 \\
y_c = 4Z_{a4}(U_5 - 0.5)
\] (22)
For ten thousand planar six-link mechanisms with a prismatic pair obtained in the manner mentioned above, we calculated the average of the absolute values of sensitivities at various positions the input angle $\theta_i$ takes from 0° to 360° in increments of 10°, and investigated the relation between them and the index of motion-transmission characteristics $r$. The relation between $r$ and $S$ is shown in Fig. 6.

With respect to the planar six-link mechanism obtained in the same manner as in the investigation of sensitivities, we calculated the average of the forces acting on the pairs $F$; these forces were simulated using a hundred sets of four external forces on moving links to be randomized according to the intervals of the magnitude, the direction, and the distance to the point of action, (Eq.(14)). The relation between $r$ and $F$ is shown in Fig. 7.

Equations (23) represents rectangular hyperbolas which are determined for the approximation of the upper and lower bounds of the $r$-$S$ relation by means of the maximum and minimum values of $S$ in the eighteen subintervals obtained by dividing the interval $[0.1, 1]$ equally.

\[
S = 0.56/r + 0.24 \\
S = 0.10/r + 0.28
\]  

(23)

The rectangular hyperbolas approximating the upper and lower bounds of the relation between $r$ and $F$ are obtained as follows.

\[
F = 0.57/r + 0.62 \\
F = 0.12/r + 0.50
\]  

(24)

The curves of Eqs.(23) and (24) are shown in Figs. 6 and 7, respectively, and the widths of the regions bounded by two curves are about 0.8 when $r$ is 0.5. Therefore, these equations are useful for the approximate of the values of $S$ and $F$ from the value of $r$.

4.3 Position and form of prismatic pair

When we connect two links to rotate about the pairing points E and G on two other links, by means of the prismatic pair, a substantial part of the pair may be formed at the position where the distances from the pairing points E and G to the central line of slider are $Z_5$ and $Z_7$, respectively, and the distance measured along the central line from E to the center of the slider F is $Z_a$, as shown in Fig. 8.

From a practical view, even if the elements of the prismatic pair are precisely manufactured and slide relatively smoothly, it is desirable that the prismatic pair is formed at a position as close as possible to the segment EG. Thus, we determine the values of kinematic constants so that the center of the slider may coincide with $F^*$ namely, $Z_6=Z_6^*$, $Z_a=0$ and $Z_7=0$; this is to ensure that selflocking of slider of an actual size does not occur during the operation of the mechanism.

The planar six-link mechanism with a prismatic pair shown in Fig. 3 is the mechanism showing the substantial part of the prismatic pair formed at a desirable position from the standpoint of the selflocking.

However, there exists a position wherein we cannot rotate the driving link in a particular direction. This fact indicates that the motion-transmission characteristics of the mechanism need to be reevaluated to include other dynamic characteristics besides selflocking.

4.4 Experiment and considerations

The experimental apparatus of the six-link dwell-motion mechanism synthesized by K. Hain is shown in...
Fig. 9, whose kinematic constants are as follows.

\[ Z_0 = 100, \quad Z_1 = 45.84, \quad Z_2 = 71.81 \]
\[ Z_3 = 83.33, \quad Z_4 = 52.58, \quad Z_5 = 0 \]
\[ \alpha = 110.56^\circ, \quad x_c = 87.53, \quad y_c = 80 \]
where the unit of length is mm. The prismatic pair is realized by the linear motion bearing (THK Co. Ltd., RSR 9 XUU).

The curves of the average of the absolute values of the sensitivities \( S \), the average of the forces acting on the pairs, and the index of motion transmission characteristics \( r \) are shown in Fig. 10. From these figures, it is confirmed that both \( S \) and \( F \) show strong correlations to the reciprocal of \( r \). For example, the value 2.84 of \( S \) and the value 2.02 of \( F \) as the input angle \( \theta_i \) becomes 30° are included, respectively, in the intervals [0.79, 3.24] and [1.15, 3.68] obtained from Eqs. (23) and (24) with the value 0.186 of \( r \) at this input angle. Therefore, it is clear that these equations are useful for the estimation of the sensitivities and the forces acting on the pairs.

For reference, the difference between the index \( r \) presented in this paper and the sine of the transmission angle \( \mu \) is shown in Fig. 11.

The experimental apparatus is designed so that the driving link may be fixed to the stationary link by means of the disk brake at an arbitrary value of the input angle.

In the experimental condition where the driving link ranges from 0° to 360° in the increments of 10°, and a torque of about 2 Nm is applied to the driven link in both clockwise and counterclockwise directions, we measure the difference between the maximum and minimum values of the output angle as the mechanical error \( \varepsilon \) using the rotary encoder (Nikon, RH4-36000). The variation of \( \varepsilon \) with the input angle \( \theta_i \) is shown in Fig. 12, together with the curve of \( 1/r \).

It is difficult to formulate the relation between the average of the absolute values of sensitivities \( S \) and the mechanical error \( \varepsilon \); however, they seem to have a similar variation in form. Therefore, it is possible to determine the position by means of \( r \), where the mechanical error \( r \) becomes markedly large.

It was impossible to rotate the driving link of the experimental apparatus in one direction at the position where \( \theta_i \) takes about 350°. This is due to the fact that the frictional torque of the revolute pair of the output link GF is greater than its driving torque due to the force acting through the link EF, as the distance \( s \)
between the pairing point G and the center of the slider F becomes smaller.

This phenomenon is considered to occur in cases where the mechanical errors due to the clearance of the elements of pairs, the deformations of the links, and so on, are large.

Consequently, the index r presented in this paper is useful for the evaluation of the motion-transmission characteristics of the planar six-link mechanism with a prismatic pair.

5. Conclusions

We undertook an evaluation of the motion-transmission characteristics of the planar six-link mechanism, wherein the two-link chain connected by the prismatic pair is driven through the coupler point of the four-link mechanism. The results obtained are summarized as follows.

(1) The relation between the index of motion-transmission characteristics r, which is an expanded form of Alt's transmission angle, and the average of the absolute values of sensitivities of the angular and linear displacements S is approximated to the region bounded by two rectangular hyperbolas of Eq. (23).

(2) The relation between r and the average of the forces acting at the pairs F is approximated to the region bounded by two rectangular hyperbolas of Eq. (24).

(3) Both S and F decrease rapidly in the interval [0, 0.3], and slowly in [0.3, 1] as r increases.

(4) The index r is useful for the determination of the position in cases where the mechanical errors, namely, the sensitivities and the forces acting on the pairs, become markedly large, that is, when the motion-transmission characteristics are inferior.

References


