Rotational Vibration of a Helical Gear Pair with Modified Tooth Surfaces*

(Rotational Motion of a Gear Pair and Its Dynamic Increment of Tooth Load)

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The rotational motion of a gear pair with modified tooth surfaces under load is analyzed with their equivalent tooth profiles. The path of contact consists of two different curves connected discontinuously. On its continuous region, equations of motion are solved and the fluctuating tooth load is calculated. At the point where it is discontinuous, the impact load is calculated from the difference in angular momentum. The gear pair rotates steadily because the kinetic energy which is transferred from one member to the other in the mesh region is returned through the impact. The dynamic increment of tooth load during one period, i.e., the external force acting on the gear vibration system, is given by a periodical function composed of the fluctuating and the impact loads, which is expressed as a Fourier series. The newly developed method is applied to a gear pair with symmetrical convex tooth surfaces, by which most tooth surfaces modified for practical use can be approximated. An analytical equation for the amplitude of the dynamic increment of tooth load is derived and its characteristics are shown through a numerical example.

Key Words: Gear, Vibration, Gear Geometry, Transmission Error, Dynamic Load, Gear Noise

1. Introduction

In our first report⁽¹⁾, the engagement between modified tooth surfaces of a helical gear pair was replaced by that between equivalent tooth profiles of a spur gear pair with the same single flank error. The characteristics of the equivalent profile and its tooth actions causing the gear pair to transmit inconstant rotational motion were analyzed. The equivalent tooth profile under load was analyzed in the region of both single and multiple tooth-pair mesh. The tooth deflection and the tooth-pair spring constant in the region of multiple tooth-pair mesh were clarified and the fundamental requirements for gear-tooth contact under load were also obtained.

In this report, the rotational motion of a gear pair with modified tooth surfaces under load is analyzed with their equivalent profiles and the differential equations of motion are solved to obtain the dynamic

increment of tooth load. The newly developed method is applied to a gear pair with symmetrical convex tooth surfaces by which most tooth surfaces modified for practical use can be approximated and an analytical solution and an example of the dynamic increment of tooth load are presented.

2. Gear Pair and Its Equivalent Tooth Profiles

Figure 1 shows the engagement of a gear pair in the reference plane (the transverse plane at the middle of the facewidth). Gear I is a rigid gear with a correct involute helicoid I whose radius of base circle is R_{b10} , while gear II is an elastic gear with the modified tooth surface II. Therefore, in Fig. 1, tooth surface I traces a correct involute profile with radius of base circle R_{b10} (not indicated), while tooth surface II traces an equivalent profile. The zone of contact intersects the center line O_1O_2 at P_0 ; for this case the equivalent profile is defined as follows by the tangential polar coordinates whose base point ($\theta_2=0$) is P_0 ⁽¹⁾.

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$$\begin{aligned}
q_2(\theta_2) &= q_2(0) + R_{b20}\theta_2 + \Delta p(\theta_2) \\
\Delta p(\theta_2) &= -a_0\theta_2 - a_1\theta_2^2/2 \\
R_{b2}(\theta_2) &= R_{b20} + d\Delta p(\theta_2)/d\theta_2 \\
\Delta R_{b2}(\theta_2) &= -d\Delta p(\theta_2)/d\theta_2, \}
\end{aligned} \quad (1)$$

where

- θ_2 : angle between the tooth tangent at a point on the equivalent profile and that at P_0 ,
 $q_2(\theta_2)$: distance from O_2 to the tooth tangent ($q_2(0) = \overline{P_0H_0}$),
 $\Delta p(\theta_2)$: amount of composite tooth modification, which is assumed to be given at most by a quadratic expression of θ_2 ,
 a_i : base circle variation coefficient,
 R_{b20} : radius of the theoretical base circle of gear II,
 $R_{b2}(\theta_2)$: radius of the instantaneous base circle of gear II, and
 $\Delta R_{b2}(\theta_2)$: variation of $R_{b2}(\theta_2)$ ($\Delta R_{b2}(\theta_2) \ll R_{b2}(\theta_2)$).

The equivalent profile defined by Eq.(1) is assumed to be in contact at P'_i under load. The static equivalent profile under load through P'_i corresponding to P'_i is thus composed of two different curves denoted as $q_x(\theta_2)$ and $q_y(\theta_2)$, which are given in the same coordinates as Eq.(1) as follows⁽¹⁾.

$$\left. \begin{aligned}
q_x(\theta_2) &= q_2(0) + R_{b20}\theta_2 + \Delta p_x(\theta_2) \\
\Delta p_x(\theta_2) &= -a_{0x}\theta_2 - a_{1x}\theta_2^2/2 - x_s(0) \\
R_{bx}(\theta_2) &= R_{b20} + d\Delta p_x(\theta_2)/d\theta_2 \\
\Delta R_{bx}(\theta_2) &= -d\Delta p_x(\theta_2)/d\theta_2, \}
\end{aligned} \right\} \quad (2)$$

where

- $x_s(\theta_2)$: static component of deflection,
 $\Delta p_x(\theta_2)$: amount of composite modification including x_s , which is assumed to be given at most by a quadratic expression of θ_2 , and
 $\Delta R_{bx}(\theta_2)$: variation of $R_{bx}(\theta_2)$ ($\Delta R_{bx}(\theta_2) \ll R_{bx}(\theta_2)$).
The values of $q_y(\theta_2)$ and $R_{by}(\theta_2)$ are obtained in the

Fig. 1 Engagements of a gear pair with equivalent profiles

same manner. The static equivalent profile (q_x and q_y) is assumed to be continuous and to have no pitch error.

3. Path of Contact and Rotational Motion of Gear Pair

3.1 Path of contact

For the static equivalent profile under load given by Eq.(2), the path of contact $h_x(u, v_x)$ is expressed as follows⁽¹⁾.

$$\begin{aligned}
u_x &= q_x(\theta_2) \sin \Theta + R_{bx}(\theta_2) \cos \Theta \\
v_x &= -q_x(\theta_2) \cos \Theta + R_{bx}(\theta_2) \sin \Theta,
\end{aligned}$$

where

$$\Theta = \phi + \{R_{b20} - R_{bx}(\theta_2)\}/C \sin \phi,$$

$$C \cos \phi = R_{b10} + R_{b20}, \text{ and } C: \text{ center distance.}$$

The value of $h_y(u_y, v_y)$ is given in the same way. Figure 2 shows the path of contact corresponding to the static equivalent profile shown in Fig. 1. h_x and h_y are discontinuous at points E_A , S_c and E_B .

3.2 Discontinuous normal velocity and generation of impact

At point S_c , the normal displacement of the static equivalent profile is continuous, but the normal velocity becomes discontinuous. Therefore the relative normal velocity is not zero at S_c . These relationships are shown in Fig. 3. The profile q_x is in contact at S_c ; for this case the normal velocities V_{n1x} and V_{n2x} are obtained as follows.

$$V_{n1x} = R_{b10} \dot{\theta}_{1x}(t_s), \quad V_{n2x} = R_{bx}(\theta_{2s}) \dot{\theta}_{2x}(t_s)$$

$$V_{n1x} = V_{n2x} (\because \text{they are in contact}).$$

where

θ_{2s}, t_s : angular displacement of gear II and time at S_c ,

$R_{b10}, R_{bx}(\theta_{2s})$: base-circle radius of gears I and II (q_x) at S_c , and

Fig. 2 Path of contact of equivalent profiles

$\dot{\theta}_{1x}(t_s), \dot{\theta}_{2x}(t_s)$: angular velocity of gears I and II at S_c .

For the case in which the profile q_y is just about to come into contact at S_c , the normal velocities V_{n1y} and V_{n2y} and the difference ΔV_n are obtained as follows.

$$\begin{aligned} V_{n1y} &= R_{b10} \dot{\theta}_{1x}(t_s), \quad V_{n2y} = R_{by}(\theta_{2s}) \dot{\theta}_{2x}(t_s) \\ \Delta V_n &= V_{n1y} - V_{n2y} = V_{n2x} - V_{n2y} \\ &= \{R_{bx}(\theta_{2s}) - R_{by}(\theta_{2s})\} \dot{\theta}_{2x}(t_s) \neq 0, \end{aligned}$$

where $R_{by}(\theta_{2s})$: base-circle radius of gear II (q_y) at S_c . Although the normal displacement is continuous, the normal velocity becomes discontinuous due to the discontinuity of the instantaneous base circle, so that impact is generated.

3.3 Rotational motion of the gear pair

The gear pair whose path of contact is shown in Fig. 2 is considered to engage as follows.

(1) From E_A to S_c (region of one-pair mesh): the profile q_x engages continuously along h_x .

(2) At S_c (region of impact): from q_x to q_y , the point of contact shifts in the direction of the tooth tangent.

(3) From S_c to E_B (region of two-pair mesh): the profile q_y engages continuously along h_y .

(4) At E_B (region of impact): from q_y to the approaching q_x , the point of contact shifts back to the

initial point E_A .

Under the assumption that the gear pair repeats the above-mentioned engagements from (1) to (4), the rotational motion in the mesh region and the impact are analyzed to obtain the dynamic increment of tooth load. Time t and angle of tooth tangent θ at each point of contact are indicated in Table 1, where τ is the time of impact (infinitesimal). Pitch of time ($2t_p$) and angular pitches ($2\theta_{1p}, 2\theta_{2p}$) of gears I and II are given from Table 1 as follows.

$$2t_p = t_e - t_r, \quad 2\theta_{1p} = \theta_{1e} - \theta_{1r}, \quad 2\theta_{2p} = \theta_{2e} - \theta_{2r}$$

4. Equations of Motion and Their Solution

4.1 Differential equations of motion

The tooth pair engages in the region h_x and the radius of instantaneous base circle approximates $R_{bx}(\theta_2)$, because the dynamic deflection is negligibly small compared to the static deflection. Input and output torque do not fluctuate and friction force is negligible because of sufficient lubrication. Under these assumptions, differential equations of motion are given as follows.

$$J_1 \ddot{\chi}_1 = -T_1 + FR_{b10}$$

$$J_2 \ddot{\chi}_2 = -FR_{bx}(\theta_2) + T_2,$$

where

J_1, J_2 : moment of inertia of gears I and II,

χ_1, χ_2 : angle of rotation of gears I and II,

F : normal tooth force, and

T_1, T_2 : transmitted torque of gears I and II.

The direction of the angle of rotation and the transmitted torque are shown by arrows in Fig. 1.

The angle of rotation and the angle of tooth tangent have the following relationships⁽¹⁾.

$$\chi_1 = \theta_1 + \{R_{b2}(0) - R_{bx}(\theta_2)\}/C \sin \phi + \Delta p_x(0)/R_{b10}$$

$$\chi_2 = \theta_2 - \{R_{b2}(0) - R_{bx}(\theta_2)\}/C \sin \phi$$

Differentiating the above equations with respect to time and simplifying them because $R'_{bx}(\theta_2) = 0$, we obtain

$$\ddot{\chi}_1 = \ddot{\theta}_1 - R'_{bx}(\theta_2) \ddot{\theta}_2 / C \sin \phi$$

$$\ddot{\chi}_2 = \ddot{\theta}_2 + R'_{bx}(\theta_2) \ddot{\theta}_1 / C \sin \phi.$$

When the variation in base circle is assumed to be very small, $\ddot{\theta}_2$ must be small. Then $R'_{bx}(\theta_2) \ddot{\theta}_2 / C \sin \phi$ is neglected because it is of infinitesimal second order. Therefore, the differential equations of motion are transformed as follows by eliminating χ_1 and χ_2 from the preceding equations.

$$J_1 \ddot{\theta}_1 = -T_1 + FR_{b10} \quad (3)$$

$$J_2 \ddot{\theta}_2 = -FR_{bx}(\theta_2) + T_2 \quad (4)$$

4.2 Fundamental requirements for gear-tooth contact

When the variation in base circle is small, it is given as follows⁽¹⁾.

$$R_{b10} \dot{\theta}_1 = R_{b20} \dot{\theta}_2 + \Delta \dot{p}_x(\theta_2) - \dot{x}, \quad (5)$$

where x : dynamic component of deflection.

Table 1 Time and angle of tooth tangent

Point	Time t	θ_1	θ_2
E_A	t_r	θ_{1r}	θ_{2r}
P_0	—	0	0
S_c	t_s	θ_{1s}	θ_{2s}
S_c	$t_s + \tau$	θ_{1s}	θ_{2s}
E_B	t_e	θ_{1e}	θ_{2e}
E_A	$t_e + \tau$	θ_{1e}	θ_{2e}

Fig. 3 Normal velocities at point of impact

4.3 Normal tooth force

The normal tooth force is assumed to consist of the static component F_s and the dynamic one F_{dx} . Then we obtain

$$\left. \begin{aligned} F &= F_s + F_{dx} \\ F_{dx} &= K_x(\theta_2) \cdot x, \end{aligned} \right\} \quad (6)$$

where $K_x(\theta_2)$ is the tooth-pair spring constant of q_x and F_{dx} is assumed to be sufficiently small compared to F_s .

Equations (3)~(6) are simultaneous equations with unknown quantities θ_1 , θ_2 , F and x .

4.4 Solution of the simultaneous equations

The solution of θ_2 is expected to be given by a linear combination of the angular displacement with the mean velocity ω_{20} and its deviation $\Delta\theta_2$. Then we obtain

$$\left. \begin{aligned} \theta_2 &= \omega_{20}t + \Delta\theta_2 \\ \dot{\theta}_2 &= \omega_{20} + \Delta\dot{\theta}_2 \\ \ddot{\theta}_2 &= \Delta\ddot{\theta}_2, \end{aligned} \right\} \quad (7)$$

where $\Delta\theta_2$, $\Delta\dot{\theta}_2$ and $\Delta\ddot{\theta}_2$ are assumed to be infinitesimal. θ_1 is also derived in the same way.

From Eq. (2), the following equation is obtained.

$$d\{\Delta p_x(\theta_2)\} = -\Delta R_{bx}(\theta_2)d\theta_2 \quad (8)$$

Differentiating Eq. (8) and substituting into it Eq. (7), we obtain

$$\Delta\ddot{p}_x(\theta_2) = -\Delta R'_{bx}(\theta_2)\omega_{20}^2, \quad (9)$$

where $\Delta R_{bx}\Delta\ddot{\theta}_2$ is neglected because it is of infinitesimal second order.

Differentiating Eq. (5) and substituting into it Eqs. (3), (4) and (9), we obtain

$$\begin{aligned} R_{b10}(-T_1 + FR_{b10})/J_1 \\ = R_{b20}\{-FR_{bx}(\theta_2) + T_2\}/J_2 - \Delta R'_{bx}(\theta_2)\omega_{20}^2 - \ddot{x}. \end{aligned} \quad (10)$$

When input and output torque are in static equilibrium,

$$T_1 = R_{b10}F_s, \quad T_2 = R_{b20}F_s.$$

Substituting the above equations and Eq. (6) into Eq. (10), we obtain

$$\begin{aligned} \{R_{b10}^2/J_1 + R_{b20}^2/J_2\}K_x(\theta_2)x \\ = R_{b20}F_s\Delta R_{bx}(\theta_2)/J_2 - \Delta R'_{bx}(\theta_2)\omega_{20}^2 - \ddot{x}, \end{aligned} \quad (11)$$

where $K_x(\theta_2)x \cdot \Delta R_{bx}(\theta_2)$ is neglected because it is of infinitesimal second order.

Equation (11) is rewritten as follows.

Fig. 4 Equivalent model of motion in the mesh region

$$M\ddot{x} + K_x(\theta_2)x = Mf_x(\theta_2), \quad (12)$$

where

$$\begin{aligned} M &= J_1J_2/\{R_{b10}^2(i^2J_1 + J_2)\}, \\ i &= R_{b20}/R_{b10} \text{ and} \\ f_x(\theta_2) &= T_2\Delta R_{bx}(\theta_2)/J_2 - \Delta R'_{bx}(\theta_2)\omega_{20}^2. \end{aligned}$$

The meaning of Eq. (12) is shown in Fig. 4, where the mass-spring system ($M-K_x$) being in the coordinates x which moves by displacement $\Delta p(\theta_2)-x_s$ to the static coordinates z is forced to move by both the force caused by the variation in base-circle (the 1st item of $Mf_x(\theta_2)$) and the inertial resistance caused by the relative motion of the coordinates x to the coordinates z (the 2nd item).

As the gear pair rotates in the steady state, the dynamic component of deflection x can be obtained as the particular solution of Eq. (12). When Eq. (12) is solved only in the vicinity of a certain point of contact, $K_x(\theta_2)$ is assumed to be K_x (constant) and Eq. (12) becomes linear. $\Delta R_{bx}(\theta_2)$ is linear with regard to θ_2 under the assumption over Eq. (2), so that $f_x(\theta_2)$ becomes at most a linear equation of θ_2 , approximately $\omega_{20}t$. Therefore, the particular solution x_d can be given as a linear expression of t : for this case

$$M\ddot{x}_d(t) = 0,$$

$$F_{dx} = K_x x_d(t) = Mf_x(\theta_2).$$

Since the above relationship is valid at any point of contact, we have

$$F_{dx} = Mf_x(\theta_2). \quad (13)$$

Substituting Eq. (13) into Eq. (3) and simplifying, we obtain the angular acceleration, the angular velocity and the angular displacement of gear I as follows, where subscript x indicates that contact occurs in the region of h_x and ω_{10} denotes the mean angular velocity of gear I.

$$\left. \begin{aligned} \ddot{\theta}_{1x} &= \Delta\ddot{\theta}_{1x} = MR_{b10}f_x(\theta_2)/J_1 \\ \Delta\dot{\theta}_{1x} &= (MR_{b10}/J_1) \int^t f_x(\theta_2)dt \\ \dot{\theta}_{1x} &= \omega_{10} + \Delta\dot{\theta}_{1x} + C_{1x} \\ \theta_{1x} &= (\omega_{10} + C_{1x})t + \int^t \Delta\dot{\theta}_{1x}dt + C_{2x} \end{aligned} \right\} \quad (14)$$

In the region of two tooth-pair mesh h_y , these values are obtained in the same way.

$$\left. \begin{aligned} F_{dy} &= Mf_y(\theta_2) \\ \ddot{\theta}_{1y} &= \Delta\ddot{\theta}_{1y} = MR_{b10}f_y(\theta_2)/J_1 \\ \Delta\dot{\theta}_{1y} &= (MR_{b10}/J_1) \int^t f_y(\theta_2)dt \\ \dot{\theta}_{1y} &= \omega_{10} + \Delta\dot{\theta}_{1y} + C_{1y} \\ \theta_{1y} &= (\omega_{10} + C_{1y})t + \int^t \Delta\dot{\theta}_{1y}dt + C_{2y} \end{aligned} \right\} \quad (15)$$

They are also obtained in the same way with regard to θ_2 .

4.5 Boundary conditions

Integration constants C_{1x} , C_{2x} , C_{1y} and C_{2y} and angles of rotation θ_{1s} at time t_s and θ_{1e} at t_e are

determined as follows.

(1) Since the variation in angular velocity is sufficiently small compared to the mean angular velocity, times t_s and t_e can be approximated as follows.

$$\begin{aligned} t_s &= t_r + (\theta_{2s} - \theta_{2r})/\omega_{20} \\ t_e &= t_s + (\theta_{2e} - \theta_{2s})/\omega_{20} \\ &= t_r + 2t_p \quad (\because \theta_{2p} = \omega_{20}t_p) \end{aligned}$$

(2) Since the variation in base-circle radius due to the dynamic deflection is negligibly small, the angle of rotation θ_{1s} at time t_s is calculated as follows.

$$\theta_{1s} - \theta_{1r} = \int_{\theta_{2r}}^{\theta_{2s}} R_{bx}(\theta_2) d\theta_2 / R_{b10} = \int_{t_r}^{t_s} \dot{\theta}_{1x} dt$$

From condition (1), $R_{b20}(\theta_{2s} - \theta_{2r}) = R_{b10}\omega_{10}(t_s - t_r)$; then

$$C_{1x} = - \left\{ \int_{\theta_{2r}}^{\theta_{2s}} \Delta R_{bx}(\theta_2) d\theta_2 / R_{b10} + \int_{t_r}^{t_s} \Delta \dot{\theta}_{1x} dt \right\} / (t_s - t_r).$$

At time t_e , these values are obtained in the same way.

$$\begin{aligned} \theta_{1e} - \theta_{1s} &= \int_{\theta_{2s}}^{\theta_{2e}} R_{by}(\theta_2) d\theta_2 / R_{b10} = \int_{t_s}^{t_e} \dot{\theta}_{1y} dt \\ C_{1y} &= - \left\{ \int_{\theta_{2s}}^{\theta_{2e}} \Delta R_{by}(\theta_2) d\theta_2 / R_{b10} + \int_{t_s}^{t_e} \Delta \dot{\theta}_{1y} dt \right\} / (t_e - t_s) \end{aligned}$$

(3) When $t=0$,

if the point of contact is on h_x , then
 $\theta_{1x}=0$ and $C_{2x}=0$;

if it is on h_y , then $\theta_{1y}=0$ and $C_{2y}=0$.

(4) Angular displacement is continuous at $t=t_s$; in this case

$\theta_{1x}(t_s) = \theta_{1y}(t_s)$.
 $\dot{\theta}_{1x}(t)$, $\theta_{1x}(t)$, $\dot{\theta}_{1y}(t)$ and $\theta_{1y}(t)$ are functions with period $2t_p$.

5. Impact Load and Rotational Motion in Steady State

5.1 Impact load

At points S_c and E_b , where the path of contact is discontinuous, the angular velocity also becomes discontinuous, so that impact load is generated. Since the angular velocities before and after the time of impact are already given by Eqs. (14) and (15), the impact load is obtained as follows from the difference in angular momentum.

$$\left. \begin{aligned} S_c: F_{ds} &= \Delta I_s / (R_{b10} \cdot \tau) \\ \Delta I_s &= J_1 \{ \dot{\theta}_{1y}(t_s) - \dot{\theta}_{1x}(t_s) \} \\ E_b: F_{de} &= \Delta I_e / (R_{b10} \cdot \tau) \\ \Delta I_e &= J_1 \{ \dot{\theta}_{1x}(t_r) - \dot{\theta}_{1y}(t_e) \} \end{aligned} \right\} \quad (16)$$

5.2 Rotational motion during one period in steady state

The increment of kinetic energy of gear I during one period ΔE_1 is given by Eqs. (14), (15) and (16).

$$\Delta E_1 = \int_{\theta_{1r}}^{\theta_{1e}} J_1 \Delta \ddot{\theta}_1(t) d\theta_1$$

$$\begin{aligned} &= J_1 \omega_{10} \int_{t_r}^{t_s} \Delta \ddot{\theta}_{1x}(t) dt + \omega_{10} \Delta I_s \\ &\quad + J_1 \omega_{10} \int_{t_s}^{t_e} \Delta \ddot{\theta}_{1y}(t) dt + \omega_{10} \Delta I_e = 0 \end{aligned} \quad (17)$$

Equation (17) indicates that the energy determined by Eqs. (14) and (15) is supplied (or lost) in the mesh region and is then returned through the impact defined by Eq. (16), such that gear I continues to rotate steadily.

The situation is exactly the same for gear II.

6. Dynamic Increment of Tooth Load Given by Fourier Series

Since the gear pair rotates steadily as indicated by Eq. (17), the dynamic increment of tooth load during one period F_d is given as follows.

$$F_d = F_{dx} = Mf_x(\theta_2) \quad (t_r < t < t_s) \\ = F_{ds} \quad (t_s \leq t \leq t_s + \tau) \\ = F_{dy} = Mf_y(\theta_2) \quad (t_s < t < t_e) \\ = F_{de} \quad (t_e \leq t \leq t_e + \tau) \quad (18)$$

Substituting $t - (t_r + t_e)/2$ into t , we express F_d by the following Fourier series, because it is a periodic function defined during $[-t_p, t_p]$. Then

$$F_d = \sum_{n=1}^{\infty} \{ A_n \cos(n\pi/t_p)t + B_n \sin(n\pi/t_p)t \} \quad (19)$$

$$A_n = \int_{-t_p}^{t_p} F_d \cos(n\pi/t_p)\xi d\xi/t_p$$

$$B_n = \int_{-t_p}^{t_p} F_d \sin(n\pi/t_p)\xi d\xi/t_p,$$

where $n=1, 2, \dots, \infty$.

The amplitude of the dynamic increment of tooth load of the n -th order is denoted by F_{dn} ; then we obtain

$$F_d = \sum_{n=1}^{\infty} F_{dn} \sin(n\omega t + \alpha), \quad (20)$$

where

$$F_{dn} = \sqrt{A_n^2 + B_n^2}, \quad \omega = \pi/t_p \text{ and } \alpha = \tan^{-1}(A_n/B_n).$$

F_d given by Eq. (20) denotes the external force of the forced vibration system of the gear pair.

7. Force Excited by Vibration of the Gear Pair

The equation of peripheral vibration of the gear pair with the external force given by Eq. (20) is given approximately as follows, where the phase angle α is assumed to be 0.

$$M\ddot{x} + 2\gamma\sqrt{MK_m}\dot{x} + K_mx = F_{dn} \sin(n\omega t), \quad (21)$$

where K_m : mean tooth-pair spring constant during one period,

$$K_m = \left(\int_{\theta_{2r}}^{\theta_{2s}} K_x d\theta_2 + \int_{\theta_{2s}}^{\theta_{2e}} K_y d\theta_2 \right) / 2\theta_{2p}, \text{ and}$$

$$2\gamma\sqrt{MK_m}\dot{x} : \text{damping force.}$$

The amplitude F_{vn} of the force excited by vibration of the gear pair, which previous theories of dynamic load have sought, is obtained from the particular solution of Eq. (21) as follows.

$$\left. \begin{aligned} F_{vn} &= G(\omega) F_{dn} \\ G(\omega) &= 1 / \sqrt{\{1 - (n\omega/\omega_m)^2\}^2 + 4\gamma^2(n\omega/\omega_m)^2} \end{aligned} \right\} \quad (22)$$

where $G(\omega)$: transfer ratio and $\omega_m^2 = K_m/M$.

When F_{Rn} is defined as the maximum value of F_{vn} , the approximate value of F_{Rn} is obtained simply as follows.

$$F_{Rn} = F_{dn} / 2\gamma$$

F_{vn} represents a transverse component of the force normal to the tooth surface F_{Nvn} : therefore

$$F_{Nvn} = F_{vn} / \cos \psi_b, \quad (23)$$

where ψ_b : helix angle on the plane of action.

8. Index of Dynamic Performance of a Gear Pair

A gear unit is forced to vibrate by the force F_d ,

Table 2 Dimensions of gear pair

	Gear I	Gear II
Number of teeth	52	55
Center distance (mm)	122	
Normal module	1.953	
Normal pressure angle (deg.)	16.35	
Helix angle (deg.)	31.06	
Pitch diameter (mm)	118.58	125.42
Base diameter (mm)	112.19	118.66
Outside diameter (mm)	124.0	131.2
Addendum (mm)	2.71	2.89
Dedendum (mm)	3.51	3.33
Normal tooth thickness	2.97 (mm)	3.08
Face width (mm)	20	
Transverse contact ratio	1.985	
Overlap contact ratio	1.597	

Fig. 5 Transformation of static equivalent profile

which means that the amplitude F_{dn} can be used as an index of the dynamic performance of a gear pair itself. In other words, the gear pair which rotates more quietly and smoothly is considered to have a smaller value of F_{dn} . Present principles of gear design include indexes of strength, but there is no index of dynamic performance; design of the dimensions and the tooth surface for better noise and vibration has been based on engineers' experience and trial and error. F_{dn} based on this new theory enables design optimizing the gear noise and the dynamic load.

9. Dynamic Increment of Tooth Load Raked by a Gear Pair with Symmetrical Convex Tooth Surfaces

In this section, the newly developed method is applied to a gear pair with symmetrical convex tooth surfaces, by which most tooth surfaces modified for practical use can be approximated, and the analytical solution and a numerical example of the dynamic increment of tooth load are presented.

9.1 Equivalent profile for symmetrical convex tooth surfaces

The equivalent profile corresponding to symmetrical convex tooth surfaces under no load is given as follows from Eq. (1).

$$\left. \begin{aligned} q_2(\theta_2) &= R_{b20} \tan \phi + R_{b20} \theta_2 + \Delta p(\theta_2) \\ \Delta p(\theta_2) &= -a_1 \theta_2^2 / 2 \\ R_{b2}(\theta_2) &= R_{b20} - \Delta R_{b2}(\theta_2) \\ \Delta R_{b2}(\theta_2) &= a_1 \theta_2 \end{aligned} \right\} \quad (24)$$

where ϕ : angle between the tooth tangent at $\theta_2=0$ and the center line (the transverse pressure angle).

Table 2 shows the gear dimensions for numerical calculation. Figure 5 indicates the measured single flank error and its corresponding equivalent profile (Profile 0) given by Eq. (24), where a_1 is estimated at $1334 \mu\text{m}/\text{rad}$.

9.2 Transformation of the equivalent profile due to load

9.2.1 Spring constant and deflection of tooth pair The tooth-pair spring constant is given as follows⁽²⁾.

$$K(\theta_2) = K_p \cdot \exp(C_p |\theta_2^3|),$$

where K_p , C_p : constants determined by the gear dimensions.

Figure 6 shows the tooth-pair spring constant assumed in this example. The static deflection of tooth pair $x_s(\theta_2)$ is obtained as follows.

$$x_s(\theta_2) = F_s / K(\theta_2)$$

9.2.2 Static equivalent profile in the region of one-pair mesh This is given from Eq. (2) as follows.

$$q_x(\theta_2) = R_{b20} \tan \phi + R_{b20} \theta_2 + \Delta p_x(\theta_2)$$

$$R_{bx}(\theta_2) = R_{b20} + d\Delta p_x(\theta_2)/d\theta_2$$

$$\Delta p_x(\theta_2) = \Delta p(\theta_2) - x_s(\theta_2)$$

Because $d\Delta p_x(0)/d\theta_2 = 0$, $\Delta p_x(\theta_2)$ can be approximated by a parabola with axis $\theta_2 = 0$. Then the following equations are obtained.

$$\left. \begin{aligned} \Delta p_x(\theta_2) &= -a_{1x}\theta_2^2/2 - x_s(0) \\ \Delta R_{bx}(\theta_2) &= a_{1x}\theta_2 \end{aligned} \right\} \quad (25)$$

9.2.3 Static equivalent profile in the region of two-pair mesh This is given as follows.

$$q_y(\theta_2) = R_{b20} \tan \phi + R_{b20}\theta_2 + \Delta p_y(\theta_2)$$

$$R_{by}(\theta_2) = R_{b20} + d\Delta p_y(\theta_2)/d\theta_2$$

Because $d\Delta p_y(\theta_{2p})/d\theta_2 = 0$, $\Delta p_y(\theta_2)$ can be approximated by a parabola with axis $\theta_2 = \theta_{2p}$. Then

$$\left. \begin{aligned} \Delta p_y(\theta_2) &= -a_{1y}(\theta_2 - \theta_{2p})^2/2 + \Delta p(\theta_{2p}) - x_s(\theta_{2p})/2 \\ \Delta R_{by}(\theta_2) &= a_{1y}(\theta_2 - \theta_{2p}). \end{aligned} \right\} \quad (26)$$

9.2.4 Static equivalent profile in the region of three-pair mesh This is given in the same way.

$$q_z(\theta_2) = R_{b20} \tan \phi + R_{b20}\theta_2 + \Delta p_z(\theta_2)$$

$$R_{bz}(\theta_2) = R_{b20} + d\Delta p_z(\theta_2)/d\theta_2$$

Because $d\Delta p_z(0)/d\theta_2 = 0$, $\Delta p_z(\theta_2)$ can be approximated by a parabola with axis $\theta_2 = 0$. Then

$$\left. \begin{aligned} \Delta p_z(\theta_2) &= -a_{1z}\theta_2^2/2 + \{2K(2\theta_{2p})\Delta p(2\theta_{2p}) - F_s\} / \{2K(2\theta_{2p}) + K(0)\} \\ \Delta R_{bz}(\theta_2) &= a_{1z}\theta_2. \end{aligned} \right\} \quad (27)$$

$\Delta p_w(\theta_2)$ and $\Delta R_{bw}(\theta_2)$ in the region of four-pair mesh and those for more than four-pair mesh are obtained in the same way.

Figure 7 shows the approximated parabolas $\Delta p_x(\theta_2)$ and $\Delta p_y(\theta_2)$ given by Eqs. (26) and (27) compared to the theoretical values, which indicates that the approximation is sufficiently accurate.

Figure 5 shows the transformation of the equivalent profile due to load given by Eqs. (25), (26) and

Fig. 6 Tooth-pair spring constant

Fig. 7 Approximation of static equivalent profile (Profile III)

(27). The increasing load makes the static equivalent profile smoother as shown here (from Profiles I to IV).

9.2.5 Transformation of the instantaneous base circle due to load Figure 8 shows the transformation of the instantaneous base circle corresponding to that of the static equivalent profile shown in Fig. 5. Under light load, the instantaneous base circles are discontinuous at the intersection of the two curves because of the tooth modification (from Profiles 0 to III). However, increasing load results in them becoming smoother and actually continuous (Profile IV). The variation in instantaneous base circle causes fluctuating and impact tooth loads of the gear pair.

9.2.6 Transformation of the equivalent tooth-pair spring constant and the equivalent deflection under load Figure 9 shows the transformation of the equivalent tooth-pair spring constant under load, in which the variation decreases according to the increase of the actual contact ratio under load.

Figure 10 shows the transformation of the equivalent deflection corresponding to that of the equivalent tooth-pair spring constant and the amount of tooth modification under no load as a broken line. The equivalent deflection varies corresponding to the variation of the equivalent tooth-pair spring constant and the load.

9.3 Rotational motion of the gear pair

Under the assumption that the gear pair is of Profile I under a given static load F_s , the path of contact is shown in Fig. 11, which is composed of two curves, h_x and h_y , connected discontinuously at E_A and

Fig. 8 Transformation of instantaneous base circle

S_c . h_x is a distance of $x_s(0)=5\text{ }\mu\text{m}$ away from P_0 in this example. Since Profile I is continuous and its Δp_x and Δp_y are symmetrical about $\theta_2=0$, the symbols which indicate time and inclination angle of the tooth tangent at each point of contact are given as shown in Table 3. Profiles II, III and IV are treated in the same way.

9.4 Solution of the differential equations of motion

9.4.1 Fluctuating tooth load

The fluctuating

Fig. 9 Transformation of equivalent tooth-pair spring constant

Fig. 10 Transformation of equivalent deflection

tooth load F_{dx} generated by $\Delta p_x(\theta_2)$ of Profile I in the region of h_x is obtained as follows from Eq.(13).

$$F_{dx} = M \{ T_2 \Delta R_{bx}(\theta_2) / J_2 - \Delta R'_{bx}(\theta_2) \omega_{20}^2 \} \quad (28)$$

F_{dy} is obtained in the same way. Therefore, the fluctuating tooth load during one period is determined as follows.

$$\left. \begin{aligned} F_{dye} &= M a_{1y} \{ T_2 (\omega_{20} t + \theta_{2p}) / J_2 - \omega_{20}^2 \} & (-2t_p + t_q < t < -t_q) \\ F_{dx} &= M a_{1x} \{ T_2 \omega_{20} t / J_2 - \omega_{20}^2 \} & (-t_q < t < t_q) \\ F_{dys} &= M a_{1y} \{ T_2 (\omega_{20} t - \theta_{2p}) / J_2 - \omega_{20}^2 \} & (t_q < t < 2t_p - t_q) \end{aligned} \right\} \quad (29)$$

where $F_{dye}(t) = F_{dys}(t + 2t_p)$.

Figure 12 shows the result of calculation based on Eq.(29) for Profile I under the operating condition of $\omega_{20} = 288\text{ rad/s}$, which indicates that the fluctuating tooth load depends mainly on both the angular velocity (ω_{20}) and the base circle variation coefficient under load (a_{1x}, a_{1y}) in this example.

9.4.2 Angular velocity Angular velocity is obtained from Eqs.(14) and (15) as follows.

Table 3 Time and angle of tooth tangent

Point	Time t	θ_1	θ_2
D'	$-t_p$	$-\theta_{1p}$	$-\theta_{2p}$
E _a	$-t_q$	$-\theta_{1q}$	$-\theta_{2q}$
P ₀	—	0	0
S _c	t_q	θ_{1q}	θ_{2q}
D	t_p	θ_{1p}	θ_{2p}

Fig. 11 Path of contact (Profile I)

Fig. 12 Fluctuating tooth load (Profile I)

$$\left. \begin{aligned} \dot{\theta}_{1ye}(t) &= \omega_{10} + \Delta \dot{\theta}_{1ys}(t + 2t_p) + C_{1y} \\ &\quad (-2t_p + t_q < t < -t_q) \\ \dot{\theta}_{1x}(t) &= \omega_{10} + \Delta \dot{\theta}_{1x}(t) + C_{1x} \quad (-t_q < t < t_q) \\ \dot{\theta}_{1ys}(t) &= \omega_{10} + \Delta \dot{\theta}_{1ys}(t) + C_{1y} \\ &\quad (t_q < t < 2t_p - t_q) \end{aligned} \right\} \quad (30)$$

where

$$\begin{aligned} \Delta \dot{\theta}_{1x}(t) &= MR_{b10} a_{1x} t (T_2 \omega_{20} t / 2J_2 - \omega_{20}^2) / J_1 \\ \Delta \dot{\theta}_{1ys}(t) &= MR_{b10} a_{1y} t (T_2 \omega_{20} t / 2J_2 - \omega_{20}^2 - T_2 \theta_{2p} / J_2) / J_1 \\ C_{1x} &= -MR_{b10} a_{1x} T_2 \omega_{20} t_q^2 / (6J_1 J_2) \\ C_{1y} &= MR_{b10} a_{1y} \{ T_2 \omega_{20} (2t_p^2 + 2t_p t_q - t_q^2) / (6J_2) \\ &\quad + \omega_{20}^2 t_p \} / J_1. \end{aligned}$$

Integration constants and times are determined as follows.

$$\begin{aligned} t_q &= \theta_{2q} / \omega_{20}, \quad t_p = \theta_{2p} / \omega_{20} \\ C_{1x} &= - \left\{ \int_{-\theta_{2q}}^{\theta_{2q}} a_{1x} \theta_2 d\theta_2 / R_{b10} \right. \\ &\quad \left. + \int_{-t_q}^{t_q} \Delta \dot{\theta}_{1x} dt \right\} / 2t_q \\ C_{1y} &= - \left\{ \int_{-\theta_{2q}}^{2\theta_{2p} - \theta_{2q}} a_{1y} \theta_2 d\theta_2 / R_{b10} \right. \\ &\quad \left. + \int_{t_q}^{2t_p - t_q} \Delta \dot{\theta}_{1ys} dt \right\} / 2(t_p - t_q) \end{aligned}$$

Figure 13 shows the deviation in angular velocity of Profile I from ω_{10} , which indicates that $\dot{\theta}_{1x}(0)$ and $\dot{\theta}_{1ys}(t_p)$ are nearly equal to ω_{10} in this example.

9.4.3 Impact load and impact energy The impact load and the impact energy are obtained from Eqs.(16) and (17) as follows, where τ is the time of impact (infinitesimal).

$$\begin{aligned} F_{de} &= J_1 \{ \dot{\theta}_{1x}(-t_q) - \dot{\theta}_{1ye}(-t_q) \} / (R_{b10} \cdot \tau) \\ &= M [\omega_{20}^2 \{ a_{1y} t_p + (a_{1x} - a_{1y}) t_q \} \\ &\quad - T_2 \omega_{20} \{ t_q^2 (a_{1y} - a_{1x}) + a_{1y} t_p (t_p - 2t_q) \} / 3J_2] / \tau \end{aligned}$$

Fig. 13 Variation in angular velocity (Profile I)

Fig. 14 Energy of impact

$$\begin{aligned} F_{ds} &= J_1 \{ \dot{\theta}_{1ys}(t_q) - \dot{\theta}_{1x}(t_q) \} / (R_{b10} \cdot \tau) \\ &= M [\omega_{20}^2 \{ a_{1y} t_p + (a_{1x} - a_{1y}) t_q \} \\ &\quad + T_2 \omega_{20} \{ t_q^2 (a_{1y} - a_{1x}) + a_{1y} t_p (t_p - 2t_q) \} / 3J_2] / \tau \\ \Delta E_{1e} &= J_1 \omega_{10} \{ \dot{\theta}_{1x}(-t_q) - \dot{\theta}_{1ye}(-t_q) \} \\ \Delta E_{1s} &= J_1 \omega_{10} \{ \dot{\theta}_{1ys}(t_q) - \dot{\theta}_{1x}(t_q) \} \end{aligned}$$

Figure 14 shows the variation in impact energy ΔE_{1e} according to the load. This example indicates that an increase in load makes the static equivalent profile smoother and the energy smaller, so that the impact load has much more influence in light loads.

9.4.4 Dynamic increment of tooth load during one period Dynamic increment of tooth load during one period F_d is obtained from Eq.(18) as follows.

$$\left. \begin{aligned} F_d &= F_{dye} \quad (-t_p < t < -t_q) \\ &= F_{de} \quad (-t_q - \tau \leq t \leq -t_q) \\ &= F_{dx} \quad (-t_q < t < t_q) \\ &= F_{ds} \quad (t_q \leq t \leq t_q + \tau) \\ &= F_{dys} \quad (t_q < t < t_p) \end{aligned} \right\} \quad (31)$$

Dynamic increment of tooth load F_d is a periodic function given by Eq.(31).

9.5 Fourier expansion of the dynamic increment of tooth load

F_d is expressed by the following Fourier series because of a periodic function defined during $[-t_p, t_p]$.

$$\begin{aligned} F_d &= \sum_{n=1}^{\infty} \{ A_n \cos(n\pi/t_p)t + B_n \sin(n\pi/t_p)t \} \\ A_n &= \int_{-t_p}^{t_p} F_d \cos(n\pi/t_p)\xi d\xi / t_p \\ &= 2M\omega_{20}^2 [\{ (a_{1x} - a_{1y}) t_q / t_p + a_{1y} \} \cos(n\pi t_q / t_p) \\ &\quad - \{ (a_{1x} - a_{1y}) / n\pi \} \sin(n\pi t_q / t_p)] \\ B_n &= \int_{-t_p}^{t_p} F_d \sin(n\pi/t_p)\xi d\xi / t_p \\ &= 2M(T_2 \omega_{20} / J_2) [\{ (a_{1y} - a_{1x}) t_q - a_{1y} t_p \} \\ &\quad \times (1/n\pi) \cos(n\pi t_q / t_p) + \{ (a_{1y} - a_{1x}) (t_q^2 / 3t_p \\ &\quad - t_p / n^2 \pi^2) + a_{1y} (t_p - 2t_q) / 3 \} \sin(n\pi t_q / t_p)] \end{aligned}$$

The amplitude of the n -th order F_{dn} is given as follows.

$$\begin{aligned} F_{dn} &= 2M \sqrt{(\omega_{20}^2 U_n)^2 + \{(T_2 \theta_{2p} / J_2) V_n\}^2} \quad (32) \\ U_n &= \{ (a_{1x} - a_{1y}) \zeta + a_{1y} \} \cos(n\pi \zeta) \\ &\quad - \{ (a_{1x} - a_{1y}) / n\pi \} \sin(n\pi \zeta) \\ V_n &= (1/n\pi) \{ (a_{1y} - a_{1x}) \zeta - a_{1y} \} \cos(n\pi \zeta) \\ &\quad + \{ (a_{1y} - a_{1x}) (\zeta^2 / 3 - 1/n^2 \pi^2) \\ &\quad + a_{1y} (1 - 2\zeta) / 3 \} \sin(n\pi \zeta) \\ \zeta &= t_q / t_p = 2 - m_p \quad (1 \leq m_p \leq 2) \\ &= m_p - 2 \quad (2 \leq m_p \leq 3) \\ &= 4 - m_p \quad (3 \leq m_p \leq 4), \end{aligned}$$

when $m_p \geq m_{p0}$, then $m_p = m_{p0}$,

where m_p : actual contact ratio,

m_{p0} : theoretical contact ratio.

ζ indicates the position of the point of intersection of Δp_x and Δp_y , and U_n and V_n represent a sort of shape of the static equivalent profile under load, which are given by the base-circle variation coefficient (a_1), the tooth-pair spring constant ($K(\theta_2)$), the theoretical

contact ratio (only when $m_p \geq m_{p0}$) and the load (T_2). Therefore, F_{dn} is a function composed of the equivalent mass (M), the angular pitch (θ_{2p}), the static equivalent profile under load $\{a_1, K(\theta_2), (m_{p0}), T_2\}$ and the angular velocity (ω_{20}).

9.6 Characteristics of the amplitude F_{dn}

Figure 15 shows calculated results of the base-circle variation coefficient, the actual contact ratio, ζ , U_n and V_n . According to the increase of load, each base-circle variation coefficient varies monotonously (Fig. 15(a)), but the increase in actual contact ratio changes ζ periodically (Fig. 15(b)) and results in complicated load characteristics of U_n and V_n (Figs. 15(c) and (d)). When the actual contact ratio reaches the theoretical value, ζ becomes constant, so that U_n and V_n become monotonic functions of the base-circle variation coefficients under load only.

Figure 16 shows calculated results of F_{dn} based on Fig. 15, which is a two-variable function of T_2 and ω_{20}

Fig. 15 Calculated variables of F_{dn}

because M , $K(\theta_2)$ and a_1 are given. With regard to the load T_2 , F_{dn} varies complexly until the actual contact ratio reaches the theoretical value, while it is a monotonic function with respect to ω_{20} .

These calculated results will be verified by experiment in the next report.

9.7 Force excited by vibration of the gear pair

Figure 17 shows the amplitude F_{vn} of the force excited by F_d , where the damping coefficient γ is

Fig. 16 Calculated results of F_{dn}

Fig. 17 Relationships between F_{d1} and F_{v1} (calculated)

assumed to be 0.07. This figure indicates that tooth separation happens easily under light load. F_{vn} is invalid when the tooth separation occurs, hence F_{vn} must be smaller than F_s when designing gear pairs.

10. Conclusions

The rotational motion of a gear pair with modified tooth surfaces under load has been analyzed with the equivalent tooth profiles and the following results are obtained.

(1) The path of contact during one period is composed of two different curves which connect discontinuously at the intersection of the successive tooth profiles.

(2) In the continuous region of the path of contact, the equations of motion have been solved to obtain the fluctuating tooth load analytically.

(3) At a discontinuous point, impact load caused by the discontinuous base circle is generated, which has been calculated from the difference in angular momentum.

(4) The gear pair rotates steadily because the kinetic energy which is transferred from one member to the other in the mesh region is returned through the impact.

(5) The dynamic increment of tooth load during one period has been given by a periodic function composed of the fluctuating and the impact tooth loads, which is expressed through a Fourier series. This is the external force acting on the gear vibration system and can be used as an index of the dynamic

performance of the gear pair, such as noise, vibration and dynamic load.

(6) The newly developed method has been applied to a gear pair with symmetrical convex tooth surfaces, by which most tooth surfaces modified for practical use can be approximated. An analytical equation for the amplitude of the dynamic increment of tooth load has been derived, which is given by a comparatively simple expression relating the moment of inertia, the tooth-pair spring constant, the amount of modification and the operating conditions (r /min and torque). The characteristics are shown by a numerical example.

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