Shape Optimization of the Wire Cross Section of Helical Springs*

(Study on Shapes of Inner and Outer Boundaries of Wire with a Hole)

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One of the effective ways for reducing the mass of a helical spring is to make a hole in the cross section of the spring wire. In this paper, we discuss the shape of outer and inner boundaries of the wire cross section of a helical spring that has minimum mass. Stress is calculated by means of the finite-element method (FEM) based on the stress function, and the optimization problem is solved by means of the steepest descent method used in the previous paper. In consequence, the converged solutions obtained from various initial shapes greatly differ in shape, but are almost equal in mass. The stress around the outer boundary of the converged cross sections can be made nearly uniform. It is also found that there are many solutions whose mass values are almost equal to those of the converged solutions. In comparison with a helical round wire spring, the weight of the present spring has been reduced by about 50%.

Key Words: Structural Analysis, Optimum Design, Nonlinear Problem, Finite-Element Method, Helical Spring, Cross-Sectional Shape

1. Introduction

Helical springs, which are important elements of various machine systems such as the valve train and the suspension of automobiles, are inevitably required to be lighter and smaller than ever before for material and fuel economy. Application of high-strength steel to helical springs has been an effective way of meeting this demand so far⁽¹⁾⁻⁽³⁾.

Since conventional helical springs of round wire do not perform efficiently, the mass of the helical springs will be reduced much more through stress equalization effected by improving the shape of the wire cross section.

With this idea in mind, we derived a shape for a cross section of solid wire which minimizes the mass of springs in previous research⁽⁴⁾⁻⁽⁷⁾. A more effective way of reducing the mass of helical springs is to make a hole in the wire cross section. Although Takahashi⁽⁸⁾ discussed the mass and stress of helical springs made

of steel tubing, no one has discussed the optimal shape of a wire cross section with a hole which minimizes the mass of helical springs. We present in this report the optimal shape of a wire cross section with a hole.

When the inner and outer boundaries of a wire cross section are arbitrary in shape, it is difficult to calculate the stress by means of the existing theories⁽⁹⁾⁻⁽¹²⁾. Thus, we use the originally developed finite-element method⁽⁶⁾ based on the stress function to calculate the stress. Shape optimization is performed step by step so as to minimize the mass of the helical spring under the constraints of maximum design stress, spring rate, coil diameter and applied load.

Since the optimized final shape may depend on the initial shape (that is, convergence to the optimum shape is not guaranteed by the present optimization method in the case of a helical spring of hollow wire), we first examine the effect of the initial shape on the converged solutions. Next, we discuss the optimality of the converged solutions. We also discuss the effect of the spring index on the optimum shape.

2. Formulation of the Problem

2.1 Design variables, object function and constraints

Our design problem is to vary the cross-sectional

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shape of the wire of a helical spring so as to minimize the mass of the spring.

In this problem, applied load, spring rate, coil diameter and maximum design stress are to be specified. A cross section of the wire is modeled using eight-noded isoparametric quadrilateral elements as shown in Fig. 1. In this model the hollow area of the wire cross section, hatched in Fig. 1, is assumed to be filled with material whose shearing modulus of elasticity is very small, 10^{-9} of the value for steel. To modify the cross-sectional shape, each nodal point on the outer and inner boundaries of the cross section is moved along the line drawn from the cross-sectional center O to each nodal point as shown in Fig. 2. Namely, the elements $(r_{P1}, r_{P2}, \dots, r_{Pn})$ of outer radius vector r_P and the elements $(r_{H1}, r_{H2}, \dots, r_{Hn})$ of inner radius vector r_H are the design variables where r_{Pi} and r_{Hi} are, respectively, the length measured from the cross-sectional center O to each node on the outer and inner boundary, as shown. The coordinates of the other nodes in the cross section are appropriately moved in the radial direction so that the finite elements are not excessively distorted in the shape optimization process in order to accurately calculate stress.

As a summary of the above, the design problem can be defined as a determination of vectors r_P and r_H , which minimize the mass of helical spring W subject to the following conditions.

Coil diameter	$D = \text{constant}$
Applied load	$P = \text{constant}$
Spring rate	$k = \text{constant}$
Maximum design stress	$\tau_a = \text{constant}$

2.2 Solution process

The shape optimization is carried out under the

Fig. 1 Finite-element mesh

Fig. 2 Schematic of the modification of cross-sectional shape (short-dashed line shows the modified shape)

constraints noted above by repeating the following steps (1) through (3) until the converged solution is obtained^{(4),(6)}.

Step 1: Move one of the nodal points on the outer or inner boundaries by Δr (or $-\Delta r$) along the line noted above, and calculate the stress by the method shown in the following section. Then, change the size of the cross section, defined by the vectors r_P and r_H , keeping the shape constant, until the calculated maximum stress coincides with the maximum design stress.

Step 2: Obtain the number of active coils satisfying the specifications for the spring rate, then calculate the mass W and the mass change ΔW with Δr . A negative value of ΔW indicates a decrease of the mass. Equations for the spring rate and the mass are shown in the following section.

Step 3: Apply the above process to every node on the boundaries and compare the values of ΔW . An improved cross-sectional shape can be obtained by movement of the particular node that was found to be the most effective in reducing the mass. Another improved shape may be obtained by simultaneously moving all the nodes that are effective in reducing the mass. If the latter modification of the shape gives better results, we choose the latter.

To save calculation time, the step size Δr is first set sufficiently large, under the condition that excessive irregularity does not occur on the boundaries. This step size is decreased when the mass can no longer be reduced in the optimization process. If the step becomes very small and a negative value of ΔW can no longer be obtained, stop the optimization process. Otherwise, return to step 1 with the decreased step size if necessary. The validity of this optimum design method has been confirmed through investigation of a bar problem in our previous research⁽⁶⁾.

3. Finite-Element Method for a Helical Spring of Arbitrary Cross Section

3.1 Derivation of stress⁽⁶⁾

Consider a close-coiled helical spring whose pitch angle is zero; thus we can assume that load P acts along the coil axis. This assumption is valid for nearly all practical springs. For this problem we use a stress function ϕ . Then the equilibrium equation in cylindrical coordinates (r, θ, z) , shown in Fig. 3, can be satisfied by

$$\left. \begin{aligned} \tau_{r\theta} &= \frac{GR^2}{r^2} \frac{\partial \phi}{\partial z} \\ \tau_{\theta z} &= -\frac{GR^2}{r^2} \frac{\partial \phi}{\partial r} \end{aligned} \right\}, \quad (1)$$

where G is the shear modulus of elasticity and R is the radius of the coil. Substituting Eq. (1) into com-

patibility equations derived from the theory of elasticity, we obtain the following equation^{(11),(12)}.

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{3\partial \phi}{r \partial r} + 2c = 0 \quad (2)$$

Since the resultant shear stress at the outer boundary of the cross section must be tangent to the boundary, the boundary condition becomes

$$\phi = 0 \quad (3)$$

on the outer boundary. The functional of Eq. (2) is as follows⁽⁶⁾.

$$\chi = \frac{GR^3}{2} \int_A \left[\left\{ \left(\frac{\partial \phi}{\partial r} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right\} / r^3 - \frac{4c\phi}{r^3} \right] dr dz \quad (4)$$

We can derive the simultaneous equations for the finite-element analysis from Eq. (4) in the same manner as described in the previous paper⁽⁶⁾. Since the stress function at nodes is obtained by solving this equation under the condition of Eq. (3), the stress can be obtained by substituting this stress function into Eq. (1).

3.2 Derivation of the spring rate and the mass^{(4),(6)}

The spring rate k is calculated with the following equation.

$$k = P / (2c\pi R^2 n), \quad (5)$$

where P and R are the applied load and the coil radius, respectively and n is the number of active coils. Furthermore, c , which is constant in Eqs. (2) and (4), represents the twisting angle of a wire of unit length at load P ⁽⁶⁾.

The mass of the helical spring is as follows.

$$W = 2\pi R_G n A \rho, \quad (6)$$

where R_G is the distance from the coil center to the gravity center of the cross section and A and ρ are the area of the cross section and the density of the material, respectively.

4. Effect of Initial Shape on the Converged Solution

According to our previous research⁽⁵⁾, convergence to the optimum by the present optimization method was guaranteed in a problem where one and only one optimum solution exists. However, in the

Fig. 3 Stress components in wire cross section

case of a helical spring of hollow wire, the optimized final shape may depend on the initial shape. Hence, we first discuss the effect of the initial shape on the converged solution.

4.1 Design conditions

We assume the design conditions of the helical spring as follows in this analysis.

Mean coil diameter	$D=24$ mm
Applied load	$P=850.52$ N
Spring rate	$k=9.8$ N/mm
Maximum design stress	$\tau_a=1\,000$ MPa
Shear modulus	$G=78.45$ GPa

In addition, the shear modulus of the imaginary material in the hollow area is assumed to be 78.45 Pa.

Under the same specifications, $D=24$ mm, $P=850.52$ N and $\tau_a=1\,000$ MPa, the wire diameter of the helical spring of solid round wire is 4.0 mm; its spring index is 6.

4.2 Initial shape

Figure 4 shows the initial shapes, which are a circle, a square and two ovals whose major and minor axes are parallel to the coil axis. The outer and inner boundaries of each initial cross section are exactly the same shape, and the inner size is one-half the outer size. Furthermore, the ratio of the lengths of the major to minor axes of the ovals is assumed to be 2.

4.3 Shape and mass of converged solutions

Figures 5 and 6 depict the converged shapes obtained. The solid and short-dashed lines in Fig. 5 show the results obtained from the circle and the square, respectively. The solid and the short-dashed lines in Fig. 6 show the results from the oval with the minor axis parallel to the coil axis and the oval with major axis parallel to the coil axis, respectively.

It can be seen that the converged shapes derived from various initial shapes greatly differ from one another, and that the aspect ratio—the height of the cross section divided by the width of the cross section

Fig. 4 Initial shapes

—of the converged cross section depends on the aspect ratio of the initial cross section. Namely, the aspect ratio of the converged cross section increases with increase of the initial aspect ratio.

Table 1 shows the dimensionless mass λ , that is, the mass of the converged-cross-section spring divided by that of the spring of solid round wire designed under the same specifications. Incidentally, smaller springs than ever before are also required for the miniaturization of machines. Thus, to evaluate the optimized springs for miniaturization, the solid height—the height of the fully compressed spring—of the optimized solutions is also shown in Table 1; the solid height shown in this table is divided by that of the spring of solid round wire designed under the same specifications.

From Table 1, it is clear that the resulting mass obtained from the initial square shape is the lightest of the four, while the resulting mass obtained from the

initial oval with the minor axis parallel to the coil axis is the heaviest of the four. However, the difference in mass between the heaviest and the lightest solutions is only 3.56%, and all four solutions are lighter than that for the spring of solid round wire by about 50%. In addition, although the resulting mass from the initial oval with the minor axis parallel to the coil axis is the heaviest of the four, as noted above, its solid height is the smallest of the four.

In summary, the converged solutions obtained from the four initial shapes are either greatly different or similar in shape, but are almost equal in mass. Therefore, it is considered that many other solutions which are almost equal in mass can exist in the case of a helical spring of hollow wire.

4.4 Stress distribution around the periphery of the converged cross section

Figures 7 and 8 show the shear stress $[=\sqrt{\tau_{\theta\theta}^2 + \tau_{\theta z}^2}]$ distributions along the inner and outer boundaries of the converged cross sections. The solid and short-dashed lines in Fig. 7 show the results for the converged cross sections from the circle and the square, respectively; the solid and short-dashed lines in Fig. 8 are the results from the oval with major axis parallel to the coil axis and the oval with the minor axis

Fig. 5 Converged shapes

Fig. 6 Converged shapes

Table 1 Mass ratio and solid height ratio of the optimized springs

Initial shape	Ratio of the mass λ	Ratio of the solid height
Circle	0.4687	0.9914
Square	0.4552	0.9914
Oval (Minor axis is parallel to the coil axis)	0.4908	0.8007
Oval (Major axis is parallel to the coil axis)	0.4675	1.2147

Fig. 7 Shear stress distributions along the boundaries of the converged cross section

parallel to the coil axis, respectively. Furthermore, the thick and thin lines in both figures show the results for the outer and inner boundaries, respectively.

From Fig. 7, it is found that the stress around the outer boundary of the converged cross sections from the circle and the square can be made nearly uniform, while the stress distributions along the inner boundaries are not uniform. The stress around the outer boundaries of the converged cross sections from the two ovals is also almost uniform as shown in Fig. 8, although these results are slightly less uniform than the results shown in Fig. 7.

In summary, the converged solutions obtained from the four initial shapes greatly differ in shape, but are almost equal in mass and in uniformity of stress distribution.

5. Evaluation of Optimality

5.1 Method of evaluation

Whether or not the multiple converged solutions obtained are local optimum solutions is studied here by determining the mass of the feasible solutions existing between the converged solutions in the design space, as shown below. When the obtained converged solution is a local minimum, the mass at feasible design points near the converged design point is heavier than that at the converged design point. Therefore, when the shape is transformed gradually from one converged sectional shape to another so as to satisfy the design specifications noted above, the mass is maximized at least in the course of transformation if the converged solution is the local minimum. Hence, the optimality of the four converged solutions can be evaluated by investigating the mass change in the course of gradual transformation from one conver-

Fig. 8 Shear stress distributions along the boundaries of the converged cross section

ged shape to another.

5.2 Mass variation between converged shapes

Figures 9 to 11 illustrate the variation of mass during linear movement from the converged design point—the converged solution from the square—to another in the design space. The ordinate shows the number of shape transformations and the abscissa shows the dimensionless mass λ . The positions of the converged design points are indicated by short-dashed lines in the figures. The mass variations in the course of gradual shape transformation from the other converged shapes are omitted here because of limited space, but the trends are almost the same. From Figs. 9 to 11, it is found that the converged solution for mass from the square is the lightest of all. None of the three other converged solutions are the local optimum, because further mass reduction is possible by shape transformation, as shown in Figs. 9 to 11.

As a result of shape optimization from the four initial shapes (which are considered to be separated in the design space), the mass ratios—the mass of the converged solutions divided by that of the spring of the solid round wire—range between 0.46 and 0.49,

Fig. 9 Change in mass in the course of gradual shape transformation from the converged shape obtained from the oval with minor axis parallel to the coil axis to the converged shape from the square

Fig. 10 Change in mass in the course of gradual shape transformation from the converged shape obtained from the circle to the converged shape from the square

while the shapes of the converged sections differ greatly. In addition, the feasible solutions for mass existing between these converged solutions in the design space are almost equal. Therefore, it may be said that the possibility that a solution with a smaller mass value exists is very low; thus, practically speaking, the shapes obtained here reflect the minimum mass values.

6. Effect of Spring Index on Optimum Shape

Since the stress distribution along the boundary of the cross section of the round wire spring becomes increasingly unequal as the spring index—the mean coil diameter divided by the typical wire diameter—decreases, it is apparent that the optimum shape varies with the spring index. Hence, we discuss the effect of the spring index on the optimum sectional shape and the spring mass. As the typical wire diameter for the spring index definition, we select the wire diameter of the spring with a solid round cross section under the same design conditions.

It is obvious that the optimum shape would be close to a hollow circle when the spring index is very large and when the results converge more rapidly as the initial cross section approaches the converged one in shape. Therefore, to save calculation time and to obtain satisfactory results, the calculations are carried out in the following manner for springs having the spring index ranging from 6.00 to 15.40.

Step 1: Apply the shape optimization to the hollow round wire spring whose spring index is 15.40.

Step 2: Carry out the shape optimization for the spring whose spring index is smaller by about 1 than that of the spring considered in the last calculation. In this step, we use the converged cross section obtained in the last calculation as the initial one.

Step 3: Repeat step 2 until the spring index becomes 6.

Fig. 11 Change in mass in the course of gradual shape transformation from the converged shape obtained from the oval with major axis parallel to the coil axis to the converged shape from the square

As a result of preparatory calculations with a hollow circle as the initial shape, it has been found that the solid height of the converged solution is almost equal to that of the coil spring with a solid round cross section under the same design conditions when the inner diameter of the cross section is 1/2 the outer diameter. Therefore, in order to implicitly restrict the solid height to the value of the solid round wire spring designed under the same design conditions, the inner diameter of the initial shape in step 1 is set at 1/2 its outer diameter.

The design specifications for the springs are listed in Table 2. The applied load and the coil diameter are changed according to the spring index so that the wire diameter of the round wire springs becomes 4.0 mm under these design specifications.

Figure 12 shows the optimized shapes of the cross section and their shear stress distributions along the inner and outer boundaries. Thick solid lines in the figure show the sectional shape, while short-dashed and thin solid lines show the stress distributions along the outer and inner boundaries, respectively. Figure 13 illustrates the optimized shapes for the springs having the spring indexes of 15.40, 12.33, 9.56 and 6.00 on a large scale. The solid and short-dashed lines in the figure show the results for the springs with the spring indexes of 15.40 and 12.33, respectively; the dot-dashed and double-dot-dashed lines represent results for the springs with the spring indexes of 9.56 and 6.00, respectively.

From Figs. 12 and 13, it is found that the outward shape of all the optimized cross sections is almost circular, and that the stress distribution along the outer boundary becomes almost uniform. It is also clear that the thickness—the dimension between the outer and inner radii of the cross section—at the outer side of the coil is smaller than that at the inner side of the coil, and that the thickness at the outer side of the

Table 2 Design specifications of the springs

Spring Index	Mean Coil Diameter (mm)	Applied Load (N)	Design Stress (MPa)	Spring Rate (N/mm)
15.40	61.60	376.83	1000	9.8
14.38	57.52	401.06	1000	9.8
13.36	53.44	406.35	1000	9.8
12.33	49.32	460.86	1000	9.8
11.31	45.24	497.92	1000	9.8
10.29	41.16	541.40	1000	9.8
9.56	38.24	577.22	1000	9.8
8.25	33.00	655.46	1000	9.8
7.22	28.88	732.15	1000	9.8
6.00	24.00	850.49	1000	9.8

Fig. 12 Converged shapes of the cross section and their stress distributions along the boundaries (spring indexes shown are for the corresponding round wire springs)

Fig. 13 Converged shapes of the cross section on a large scale

coil decreases as the spring index decreases.

Figure 14 shows how the mass of the optimized solution varies with the spring index. The abscissa of this figure represents the spring index of a solid round wire spring designed under the same specifications, while the ordinate represents the mass ratio (that is, the mass of the optimized spring divided by that of the solid round wire spring designed under the same specifications).

This figure shows that the mass ratio decreases as the spring index decreases, and that the mass reduction by shape optimization is greater as the spring index becomes smaller, resulting in mass reduction by 45 to 55% in a spring index range between 6 and 10.

Fig. 14 Effect of the spring index on the mass of the optimized springs

Figure 15 shows how the solid height of the optimized solution varies with the spring index. The abscissa of this figure represents the spring index of the solid round wire spring designed under the same specifications, while the ordinate represents the solid height of the optimized spring divided by that of the solid round wire spring designed under the same specifications.

From Fig. 15, it can be seen that the solid heights of the optimized springs are almost equal to that of the solid round wire spring under the same specifications, as we intended.

7. Conclusions

We discussed the optimal shape of the hollow

Fig. 15 Effect of the spring index on the solid height of the optimized springs

cross section of wire for a helical spring which minimizes the mass of the spring. The optimization problem is solved by means of the steepest descent method. In the present analysis, stress is determined by solving the function derived from the state equation for a curved beam by means of FEM.

First we studied the shapes, the stress distributions, the mass and the solid height of optimized springs obtained from the four initial shapes. Then we studied the feasible solutions for mass which were close to the optimized solutions.

As a result, it was found that there are many sectional shapes which result in a mass of 45 to 49% of that of the spring of solid round wire under the same design conditions in addition to the four optimized shapes, and that the spring with the converged cross section from the hollow square is the lightest of all the springs discussed here. Furthermore, the solid height of the spring with the converged cross section from the hollow square is almost equal to that of the solid round wire spring designed under the same specifications. Although some shapes of the converged sections differ greatly from one another, the stress distributions along their outer boundaries are almost uniform.

Finally, the effect of the spring index on the optimum shape and the mass was studied. Under the design conditions assumed here, the inner-side shape of the optimized cross section varies with the spring index, reducing the thickness at the outer side of the coil; the thickness at the outer side of the coil decreases as the spring index decreases. It has also been found that the mass of the spring can be further reduced as the spring index decreases by shape optimization of the wire cross section. For the spring having the spring index ranging from 6 to 10, the mass can be reduced by 45 to 55%.

Since obtaining the optimum shape of the hollow wire is a problem with high nonlinearity, the present shape optimization method cannot give the global

optimum solution exactly. However, it has been confirmed that this method is effective in reducing the mass of helical springs.

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