Development of New Mode-Superposition Technique for Truncating Lower- and/or Higher-Frequency Modes* (Application to Eigenmode Sensitivity Analysis)

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A new mode-superposition technique with higher accuracy and efficiency was proposed by the authors in Ref. (1). This paper applies the technique to eigenmode sensitivity analysis for a structural system and a coupled acoustic-structural system. It will be shown that the use of a new mode-superposition technique greatly improves the convergency of a modal sensitivity analysis, and makes it possible to calculate the sensitivity with better efficiency than Fox and Kapoor's modal method and Wang's modified modal method, and also with better efficiency than Nelson's method. The theoretical results are verified through application of the method to a box structure and a practical vehicle interior noise problem.

Key Words: Mode Superposition Technique, Sensitivity Analysis, Modal Analysis, Structural Dynamics, Coupled Acoustic-Structural System

1. Introduction

In Ref. (1), we proposed a new mode-superposition technique in the frequency response analysis for a damped system. It was shown that the use of this new mode-superposition technique made it possible to truncate the lower modes as well as the higher modes and also to calculate more accurately and efficiently than conventional mode-superposition techniques, such as the mode-displacement method, mode-acceleration method⁽²⁾ and Hansteen and Bell's method⁽³⁾. In this report, this new mode-superposition technique will be applied to eigenmode sensitivity analysis of a structural system and a coupled acoustic-structural system. It will also be shown that use of the technique greatly improves the convergency of a modal sensitivity analysis and makes it possible to calculate sensitivity with better efficiency.

Many techniques have been proposed for modal sensitivity analysis of a structural system. Among these techniques the four major ones are the finite difference method^{**1}, Fox and Kapoor's modal method⁽⁴⁾, Nelson's method⁽⁵⁾ and Wang's modified modal method⁽⁶⁾. First, in 1968 Fox and Kapoor developed an eigenmode sensitivity method based on the modal-displacement method of the mode-superposition technique. With Fox and Kapoor's modal method, we can calculate the sensitivity coefficients efficiently, but the precision often decreases when the higher modes are truncated. Next, in 1976, Nelson proposed a method that solved the eigenmode sensitivity equation strictly and improved the precision.

Nelson's method has already been incorporated into MSC/NASTRAN⁽⁷⁾. In that method, it is necessary for us to solve a different system equation with every eigenmode. Thus the method is not efficient when sensitivity coefficients of several eigenmodes are to be calculated. Therefore Wang⁽⁶⁾ proposed a modified modal method in 1985. Wang's modal method was based on the acceleration-modal method of the mode-superposition technique, and it significantly improves the numerical accuracy with fewer eigen-

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**1 Due to the simplicity of the finite difference method, the examination of it is omitted in this report.

modes than Fox and Kapoor's modal method. Since it is sufficient to solve a static equation only once to calculate the sensitivity coefficients of a large number of eigenmodes, Wang's modal method is more efficient than Nelson's.

On the other hand, we have developed sensitivity analysis methods⁽⁸⁾⁻⁽¹⁰⁾ for the coupled acoustic-structural system to reduce vehicle interior noise. In the case of coupled acoustic-structural systems, the coefficient matrices of the dynamic equations are asymmetrical, which precludes the use of conventional mode-superposition technique in their original form. Therefore, the concept of left and right eigenvectors was introduced⁽⁸⁾ and four propositions were derived with respect to this concept, for developing a mode-superposition technique for coupled acoustic-structural systems. This developed mode-superposition technique was used to formulate a modal sensitivity method for application to coupled systems⁽⁸⁾.

Concurrently, we developed sensitivity analysis methods for the direct frequency response⁽⁹⁾ and the modal frequency response⁽¹⁰⁾, also for coupled systems. No other method exists for the coupled structural-acoustic system. But the modal method used for the coupled system developed in our previous papers⁽⁸⁾⁻⁽¹⁰⁾ is the conventional modal displacement which has been expanded to the coupled system. That is, it is an expansion of Fox and Kapoor's method for coupled acoustic-structural systems. When the modal sensitivity is calculated with the method in Ref. (8), the displacement sensitivity elements of the structure have good accuracy, but the sensitivity elements of the sound pressure have rather poor accuracy. This is essentially due to the inaccuracy of the modal displacement method.

Therefore we must improve the method developed in Ref. (8). In this paper a new modal sensitivity method based on the mode-superposition technique proposed in Ref. (1) will be developed, and it will be proven that this new modal sensitivity method represents a considerable improvement over conventional methods.

2. Conventional Sensitivity Analysis Methods

The following eigenvalue problem is used here to model a damping-free structural system for the sake of simplicity.

$$(K - \lambda_j M)\phi_j = 0, \quad (1)$$

where λ_j is the eigenvalue, ϕ_j is the corresponding eigenvector, and K and M are the stiffness and mass matrices of the system. For the sake of simplicity, we assume that Eq. (1) does not have any repeated eigenvalues.

For the structural eigenvector system, the follow-

ing orthogonality equation (2), and the normalization equation (3) hold:

$$\phi_i^T K \phi_j = 0 \text{ and } \phi_i^T M \phi_j = 0 \quad (\text{for } i \neq j), \quad (2)$$

$$\phi_i^T M \phi_i = 1. \quad (3)$$

Letting α_k ($k=1, 2, \dots$) denote a design variable and differentiation equation (3) with respect to α_k ,

$$-\lambda_j' M \phi_j + (K - \lambda_j M) \phi_j' = -(K' - \lambda_j' M') \phi_j, \quad (4)$$

$$\phi_j^T M \phi_j' = -\frac{1}{2} \phi_j^T M' \phi_j, \quad (5)$$

are calculated. Here, λ_j' is the sensitivity of eigenvalue λ_j in relation to α_k , and ϕ_j' is the sensitivity of eigenvector ϕ_j in relation to α_k . Premultiplying Eq. (4) by the transpose of ϕ_j , and using $\phi_j^T (K - \lambda_j M) = 0$,

$$\lambda_j' = E_{jj}, \quad (6)$$

where

$$E_{jj} = \phi_j^T (K' - \lambda_j' M') \phi_j. \quad (7)$$

Here $\phi_j^T M \phi_j = 1$ is assumed.

As for the sensitivity of eigenvectors, Eq. (4) is expressed in the following ordinary linear equation,

$$A_j \phi_j' = b_j, \quad (8)$$

where

$$A_j = K - \lambda_j M, \quad b_j = (\lambda_j' M - K' + \lambda_j' M') \phi_j. \quad (9)$$

Equation (8), a linear expression with respect to ϕ_j' , cannot be solved in its present form because the coefficient matrix A_j is singular. In order to solve Eq. (8), the following three methods have been proposed.

2.1 Fox and Kapoor's modal method

Using the modal displacement method of the mode-superposition technique, eigenmode sensitivity ϕ_j' can be expressed as follows:

$$\phi_j' = \sum_{i=1}^n \phi_i C_{ij}^0. \quad (10)$$

Substituting Eq. (10) into Eq. (4), multiplying Eq. (4) from the left by ϕ_j^T and using the orthogonality equation (2), we obtain

$$C_{ij}^0 = \frac{-1}{\lambda_i - \lambda_j} E_{ij} \quad (\text{for } i \neq j), \quad (11)$$

where

$$E_{ij} = \phi_i^T (K' - \lambda_j' M') \phi_j. \quad (12)$$

Using Eq. (5), the coefficient C_{ii}^0 can be expressed as

$$C_{ii}^0 = -\frac{1}{2} \phi_i^T M' \phi_i. \quad (13)$$

As mentioned above, the precision of the calculated eigenmode sensitivity often decreases when we use Fox and Kapoor's modal method with truncated higher eigenmodes.

Thus, with Fox and Kapoor's modal method we must calculate more eigenmodes and then use those eigenmodes in order to obtain a precise eigenmode sensitivity. This situation is essentially due to the modal-displacement method's defect, and we can assume that it is also due to the Fox and Kapoor's method limit. Hence, Nelson's method, proposed in

1976, is said to be a more efficient method by many researchers.

2.2 Nelson's method

The difference between Nelson's method and Fox and Kapoor's method is that one calculates eigenmode sensitivity directly from Eqs. (8) and (5), not using the mode expansion equation (10) with Nelson's method. Letting N be the total number of D.O.F. of the system, the coefficient matrix A_j is singular of rank $(N-1)$. In order to solve the singular equation with Nelson's method, the solution X_j^0 of the following nonsingular equation is calculated first.

$$\bar{A}_j X_j^0 = \bar{b}_j, \quad (14)$$

where $\bar{A}_j$ is reduced from the coefficient matrix A_j by letting all the elements of the k -th column and the k -th row equal zero and by letting the k -th diagonal element equal 1. Here, $\bar{b}_j$ is reduced from b_j by letting the k -th element equal zero. Furthermore, the number k equals the element number of the largest absolute value of ϕ_j . Then the eigenmode sensitivity can be expressed as

$$\phi_j' = X_j^0 + C_j \phi_j. \quad (15)$$

Substituting Eq. (15) into Eq. (8), C_j can be calculated, and is expressed as follows:

$$C_j = -\phi_j^T M X_j^0 - \frac{1}{2} \phi_j^T M' \phi_j. \quad (16)$$

When K' and M' can be sought strictly, the exact sensitivity can be calculated by Nelson's method. Therefore, Nelson's method has been incorporated into MSC/NASTRAN. But with Nelson's method, Eq. (14) must be solved for every eigenmode sensitivity ϕ_j' ($j=1, 2, \dots$). For this reason, Nelson's method is inefficient when the sensitivity coefficients of many eigenmodes must be calculated.

2.3 Wang's improved modal method

As mentioned in the previous report⁽¹⁾, in the field of mode-superposition techniques, the mode acceleration method has been proposed in order to improve the accuracy of the modal displacement method. By using the mode acceleration method, the convergence of the mode-superposition technique is improved significantly and the use of this method makes it possible to obtain a more accurate solution than by the mode displacement method, with fewer eigenmodes.

Wang proposed the modified modal method in 1985 by applying the mode acceleration method to the sensitivity analysis of a structural system. With Wang's modal method, eigenmode sensitivity can be expressed as follows.

$$\phi_j' = X_j^1 + \sum_{i=1}^n \phi_i C_{ij}, \quad (17)$$

where X_j^1 is the solution of the static equation:

$$K X_j^1 = b_j. \quad (18)$$

C_{ij}^1 is as follows.

$$C_{ij}^1 = \frac{\lambda_j}{\lambda_i} \frac{1}{\lambda_j - \lambda_i} E_{ij} \quad (\text{for } i \neq j). \quad (19)$$

Then $C_{ii}^1 = C_{ii}^0$.

Due to the advantage of the mode acceleration method investigated in the previous report, Wang's method is significantly superior to Fox and Kapoor's method. Namely we can obtain more accurate sensitivity coefficients using Wang's method with fewer eigenmodes than using Fox and Kapoor's method. Also, Wang's method is more efficient than Nelson's method when sensitivities of many eigenmodes are necessary because with Wang's method, only one static system equation is sufficient to obtain the solution. However, in the case of truncating the lower eigenmodes, the error norm of Wang's method is sometimes larger than that of Fox and Kapoor's method, as is the mode acceleration defect.

If the inverse matrix of K cannot be calculated due to the singularity of K , Eq. (18) also cannot be calculated directly.

3. Proposed Modal Sensitivity Analysis Method

3.1 New mode-superposition technique

In the previous report, we proposed a new mode superposition technique. In this section, one part of the results⁽¹⁾ is shown first. The following frequency response equation is used here.

$$(K - \Omega^2 M) U = F, \quad (20)$$

where Ω and F are the input frequency and amplitude of the force, respectively. The range of input frequency lies between $[w_a, w_b]$, i.e., $\Omega \in [w_a, w_b]$. M and n indicate lower and upper frequencies of the eigenvector which satisfies the conditions $w_m < w_a$ and $w_n > w_b$ for m and n , respectively. Applying the new modal response analysis⁽¹⁾ in a damping-free system in accordance with Eq. (20), the modal frequency response U will be

$$U = U_s + \sum_{i=m}^n \phi_i \phi_i, \quad (21)$$

where U_s is the solution of the pseudodynamic equation,

$$(K - \mu M) U_s = F, \quad (22)$$

and

$$p_i = \frac{\Omega^2 - \mu}{\lambda_i - \mu} \frac{\phi_i^T F}{\lambda_i - \Omega^2} \quad (23)$$

Here, $\mu (=w_c^2, \text{ in Ref. (1)})$ is the given constant.

As stated in Ref. (1), the new modal response proposed by the authors to be calculated using Eq. (21), significantly improves the solution obtained by the existing modal displacement and modal acceleration method. In other words, it enables response calculation by truncating the lower and higher modes, together with better convergency than the conventional modal methods. Furthermore, systems with

rigid body modes can be easily handled, and the present method can be reduced to the existing modal displacement and modal acceleration method by substituting different fixed values of $w_c^{(1)}$. In the next section (3.2), we will investigate the application of the new mode-superposition technique to the modal sensitivity analysis.

3.2 New eigenmode sensitivity analysis

Rewriting Eq. (8), we obtain

$$(K - \lambda_j M) \phi_j' = b_j. \quad (24)$$

It is clear from Eqs. (24) and (20) that they both have the same form where Ω^2 , U and F of Eq. (20) are replaced by λ_j , ϕ_j' and b_j , respectively, in Eq. (24). Hence, the solution of Eq. (24) can be written in the form of Eq. (21), as stated in the previous section (3.1), as,

$$\phi_j' = X_j + \sum_{i=m}^n \phi_i C_{ij}, \quad (25)$$

where X_j is the solution of a system of simultaneous linear equations

$$(K - \mu M) X_j = b_j \quad (26)$$

and

$$C_{ij} = \frac{\lambda_j - \mu}{\lambda_i - \mu} \frac{\phi_i^T b_j}{\lambda_i - \lambda_j}. \quad (27)$$

Substituting b_j of Eq. (9) into Eq. (27), we have

$$C_{ij} = \frac{\lambda_j - \mu}{\lambda_i - \mu} \frac{1}{\lambda_j - \lambda_i} E_{ij} \quad (\text{for } i \neq j). \quad (28)$$

From Eq. (27), $C_{ii} = 0/0^{**2}$, which is nondeterministic. However, as in the conventional methods, substitution of Eq. (25) into the normalization condition of Eq. (5) leads to

$$C_{ii} = -\phi_i^T M X_i - \frac{1}{2} \phi_i^T M' \phi_i. \quad (29)$$

In general, except for the case of $\mu = \lambda_j$, the following equation is obtained:

$$\phi_i^T M X_i = 0, \quad (30)$$

and $C_{ii} = C_{ii}^0$. For a particular value of μ , this present sensitivity analysis method will be reduced to the conventional methods. When $\mu \rightarrow -\infty$, from Eq. (26), $X_j \rightarrow 0$ and C_{ij} of Eq. (28) becomes

$$C_{ij} = \frac{1}{\lambda_j - \lambda_i} E_{ij} = C_{ij}^0 \quad (\text{for } i \neq j). \quad (31)$$

Thus the present method is equivalent to Fox and Kapoor's modal method. When $\mu = \lambda_j$, similarly from Eq. (28), $C_{jj} (i \neq j)$ becomes 0 and Eq. (25) takes the form

$$\phi_j' = X_j + C_{jj} \phi_j = X_j^0 + C_j \phi_j. \quad (32)$$

Hence, the present method is identical to that of Nelson. Furthermore $\mu = 0$ from Eq. (26), and Eq. (18), X_j becomes X_j^1 while C_{ij} of Eq. (28) becomes

$$C_{ij} = \frac{\lambda_j}{\lambda_i} \frac{1}{\lambda_j - \lambda_i} E_{ij} = C_{ij}^1 \quad (\text{for } i \neq j). \quad (33)$$

Therefore, we can arrive at Wang's modified modal

method from the present one. Although μ can take arbitrary values, the accuracy of the present method can be upgraded and a better solution than that obtained by Wang's modified modal method can be achieved by selecting those values which satisfy the given conditions, as stated in the next section, 3.3. Hence, it provides a significantly better solution than Fox and Kapoor's modal method.

Furthermore, compared to Wang's method, which does not allow truncation of the lower modes, it also enables us to calculate the sensitivity of higher eigenmodes with better accuracy. Moreover, since Wang's method is not applicable for rigid-body modes, the present one is a more versatile and generalized method of modal analysis. By selecting optimum values of μ , a better solution can be obtained⁽¹⁾ due to the advantages of the present modal analysis. The present method needs only one solution of Eq. (26), compared to Nelson's method which calculates sensitivity of a number of eigenmodes. It also avoids solving the singular matrix A_j in Eq. (8) and thus is more efficient and simpler than Nelson's method.

3.3 Error analysis

As in Ref. (1), error analysis will be performed for the present sensitivity analysis and the convergence, and the necessary conditions of the convergency will be discussed below.

First, when the i -th mode is truncated for the calculation of the j -th eigenvector sensitivity, the absolute error obtained from Eq. (10) in Fox and Kapoor's method may be written as $e_{ij}^0 = \|\phi_j C_{ij}\|^{**3}$. The absolute error of the present method is

$$e_{ij} = \left| \frac{\lambda_j - \mu}{\lambda_i - \mu} \right| e_{ij}^0. \quad (34)$$

Hence if the following condition,

$$\left| \frac{\lambda_j - \mu}{\lambda_i - \mu} \right| < 1, \quad (35)$$

is satisfied, the absolute error of the present method will be less than that of Fox and Kapoor's method. From Eq. (35), the following conditions are allowed: when higher modes are truncated ($\lambda_i > \lambda_j$),

$$-\infty < \mu < \frac{1}{2} (\lambda_i + \lambda_j); \quad (36)$$

when lower modes are truncated ($\lambda_i < \lambda_j$),

$$\frac{1}{2} (\lambda_i + \lambda_j) < \mu < +\infty. \quad (37)$$

Again from Eq. (17), the absolute error of Wang's method will become $e_{ij}^1 = (\lambda_j / \lambda_i) e_{ij}^0$, and if

$$\left| \frac{\lambda_j - \mu}{\lambda_i - \mu} \right| < \frac{\lambda_j}{\lambda_i} \quad (38)$$

is satisfied, the absolute error of the present method will be less than that of Wang's method. The following

^{**2} $\phi_i^T b_i = \lambda_i' - E_{ii} = 0$

^{**3} $\| \cdot \|$ means the norm of vector

condition can be obtained from Eq. (38) :
for truncation of higher modes ($\lambda_i > \lambda_j$),

$$0 < \mu < \frac{2\lambda_i\lambda_j}{\lambda_i + \lambda_j}; \quad (39)$$

for truncation of lower modes ($\lambda_i < \lambda_j$),

$$-\infty < \mu < \lambda_i \text{ or } \frac{2\lambda_i\lambda_j}{\lambda_i + \lambda_j} < \mu < +\infty. \quad (40)$$

Furthermore, it is clear from Eq. (34), that as $\mu \rightarrow \lambda_j$, the absolute error e_{ij} of the present method tends to zero. However, when $\mu = \lambda_j$, the error is mathematically zero, but the coefficient matrix ($K - \mu M$) of Eq. (26) is singular and the solution cannot be obtained. Hence in the present method, we assume $\mu \neq \lambda_j$. Actually, when solving with the help of Eq. (25) we must fix the lowest mode number, m , and the highest mode number, n , beforehand. Below we suggest a procedure for the selection of m and n using an approximate equation determined from Eq. (23).

Let ϕ_j^* be the accurate sensitivity of the j -th eigenvector. Then, the absolute error due to the truncation of modes lower than the m -th mode and higher than the n -th mode can be written as follows:

$$e_j = \|\phi_j^* - \phi_j'\| / \|\phi_j^*\|. \quad (41)$$

Substituting Eq. (25) into Eq. (41), we obtain

$$e_j = \left\| \sum_{i=1}^{m-1} \phi_i C_{ij} + \sum_{i=n+1}^N \phi_i C_{ij} \right\| / \|\phi_j^*\| \leq e_j^L + e_j^H, \quad (42)$$

$$e_j^L = \left\| \sum_{i=1}^{m-1} \phi_i C_{ij} \right\| / \|\phi_j^*\|, \quad e_j^H = \left\| \sum_{i=n+1}^N \phi_i C_{ij} \right\| / \|\phi_j^*\|, \quad (43)$$

where e_j^L and e_j^H are the absolute errors due to the truncation of the lower and higher modes.

Using Eq. (28) and Eq. (11), e_j^H becomes

$$e_j^H = \left\| \sum_{i=n+1}^N \frac{\lambda_j - \mu}{\lambda_i - \mu} C_{ij}^0 \phi_i \right\| / \|\phi_j^*\| \leq \left| \frac{\lambda_j - \mu}{\lambda_{n+1} - \mu} \right| \delta_j, \quad (44)$$

where

$$\delta_j = \max \{S_{n+1}^k; k = n+1, \dots, N\}, \quad (45)$$

$$S_{n+1}^k = \left\| \sum_{i=n+1}^k \phi_i C_{ij}^0 \right\| / \|\phi_j^*\|. \quad (46)$$

It is clear from Eq. (46) that S_{n+1}^k is the absolute error of Fox and Kapoor's method with truncated modes starting from the $(n+1)$ -th to the k -th modes. For simplification, if we neglect the strict mathematical analysis, it is permissible to assume $S_{n+1}^k < S_{n+1}^N$ (for $k = n+1, \dots, N$). Then $\delta_j (= S_{n+1}^N)$ will be the absolute error of Fox and Kapoor's method for truncating all the higher modes above the n -th one. Devoting the permissible absolute error, δ_H , due to the higher modes, from Eq. (44) we have

$$\lambda_{n+1} \geq \mu + \beta_H |\lambda_j - \mu|. \quad (47)$$

Hence $\beta_H = \delta_j / \delta_H$ is the factor of accuracy of Fox and Kapoor's method. For example, to obtain an accuracy level one decimal point less than (i.e., 1/10th of) Fox and Kapoor's method, set $\beta_H = 10$. Therefore, from Eq. (47) we can determine analytically the highest mode number n . Similarly, in truncating lower modes, the

following equation holds:

$$\lambda_{m-1} \leq \mu - \beta_L |\lambda_j - \mu|. \quad (48)$$

β_L is the factor of accuracy of Fox and Kapoor's method. Using Eq. (48) the lowest mode, m , can be determined analytically.

4. Sensitivity Analysis of Coupled Acoustic-Structural System

For a coupled acoustic-structural system, similar to Eq. (1), the eigenvalue problem can be written. However, coefficient matrices K and M are unsymmetrical as shown in Eq. (49):

$$K = \begin{bmatrix} K_{ss} & K_{sa} \\ 0 & K_{aa} \end{bmatrix}, \quad M = \begin{bmatrix} M_{ss} & 0 \\ M_{as} & M_{aa} \end{bmatrix}, \quad (49)$$

where M_{ss} and K_{ss} are the structural mass and stiffness matrices, M_{aa} and K_{aa} are the acoustic mass and stiffness matrices and $K_{sa} = -M_{as}^T$. Again, the coupled eigenvector $\phi_j^T = \{\phi_{sj}^T, \phi_{aj}^T\}$ consists of structural modal displacement vectors ϕ_{sj} and acoustic modal displacement vectors ϕ_{aj} .

Since Eq. (2) does not hold in general for the eigenvector ϕ_j of the above-mentioned coupled system, we have derived the following normalization, as stated in Ref. (8).

$$\bar{\phi}_i^T K \phi_j = 0 \text{ and } \bar{\phi}_i^T M \phi_j = 0 \quad (\text{for } i \neq j) \quad (50)$$

$\bar{\phi}_i$ is called the left eigenvector.

For eigenvalue $\lambda_i \neq 0$:

$$\bar{\phi}_i^T = \left\{ \phi_{si}^T, \frac{1}{\lambda_i} \phi_{ai}^T \right\}, \quad (51)$$

For eigenvalue $\lambda_i = 0$:

$$\bar{\phi}_i^T = \{0, \phi_{ai}^T\} \quad (\text{for } \phi_{ai} \neq 0),$$

$$\bar{\phi}_i = \begin{bmatrix} I \\ K_{aa}^{-1} M_{as} \end{bmatrix} \{\phi_{si}\} \quad (\text{for } \phi_{ai} = 0). \quad (52)$$

Furthermore, for a coupled eigenproblem, the normalization condition with mass matrix, M , is

$$\bar{\phi}_i^T M \phi_i = 1. \quad (53)$$

Using the above orthogonality and normalization conditions, the results of the previous section can be extended. The equations for determining the sensitivities of eigenvalue λ'_i and eigenvector ϕ'_i , Eqs. (6), (25) and (28), can be used in the same form where E_{ij} is replaced by

$$E_{ij} = \bar{\phi}_i^T (K' - \lambda_j M') \phi_j, \quad (54)$$

and C_{ii} of Eq. (29) is

$$C_{ii} = \frac{1}{2} \bar{\phi}_i^T M' \phi_i. \quad (55)$$

5. Numerical Examples

5.1 Structural system

As shown in Fig. 1, we consider a (200 cm $\times$ 160 cm $\times$ 150 cm) box made of steel plates of 0.4 cm thickness. Young's modulus is 2.1×10^5 Pa, density is 0.8×10^{-6} kg/cm³ and Poisson's ratio is 0.3. The finite-element model comprises 98 nodal points and 96 shell

elements. For simplification, we take the thickness of the top plate as a design variable, and the sensitivities of the outward normal component of the eigenvector are calculated at two points (later denoted as observation points 1 and 2) situated at the middle of the top plate and the left-hand-side plate. In Table 1, for different values of μ_f ($\mu_f = \sqrt{\mu}/2\pi$), the sensitivity of the 1st eigenvector ($f_1 = 8.6049$ Hz) with higher modes truncated, is shown. As stated earlier, for $\mu = -\infty$ and $\mu = 0$, the proposed method converges to Fox and Kapoor's modal method and the modified Wang modal method, respectively. The results of sensitivity analysis at observation point 1 (nodal point 13) are tabulated in Table 1(a), and when all the modes are considered, the exact solution is $\phi'_{1,13} = 10.401$. When only one mode is taken into account, the absolute errors of Fox and Kapoor's method, the modified Wang method and the present method (for $\mu_f = 8.0$ Hz and 8.5 Hz) are 92%, 65%, 26% and 6%, respectively. When the first 3 modes (0 Hz–14 Hz) are used, the error of Fox and Kapoor's method is 9%, compared to the 1.3% absolute error of Wang's method, and that of the proposed method become 0.2% ($\mu_f = 8.0$ Hz) and 0.04% ($\mu_f = 8.5$ Hz).

Hence, to attain an absolute error level of 0.05%, it is clear from Table 1 that Fox and Kapoor's method requires 47 modes and Wang's method needs 15 modes, in contrast to the first 3 modes of the proposed method. Similar results are obtained at observation point 2 tabulated in Table 1(b). Again, it can be concluded that the convergence rate accelerates when μ_f approaches the value of the 1st eigenfrequency.

Table (2) includes the results of the sensitivity analysis of the 29th eigenvector ($f_{29} = 77.734$ Hz) when the lower modes are truncated. To investigate the effect of lower modes, we set $n = 53$ and change the

Fig. 1 FEM model of uncoupled and/or coupled acoustic box structure system

value of m . Results corresponding to observation point 1 are shown in Table 2(a). The exact solution is $\phi'_{29,13} = -15.447$. As shown in Table 2(a), when all the modes below the 29th eigenmode are truncated, the absolute errors for Fox and Kapoor's method, Wang's method and the proposed method (for $\mu_f = 80$ Hz and 78 Hz) are 105%, 200%, 27% and 4%, respectively. Wang's method becomes less accurate than that of Fox and Kapoor's method when the lower 2 modes below the 29th mode are included, and the absolute error is reduced to 5% and 7% in Fox and Kapoor's method and Wang's method, respectively. On the other hand, the corresponding absolute error of the proposed method (for $\mu_f = 80$ Hz and 78 Hz) is 0.

Table 1 Comparison of convergency with higher modes truncated in the uncoupled system

(a) Sensitivity of 1st eigenvector at observation point 1

(Exact Solution = 10.401)				
m = 1 n =	F - Method ($\mu = -\infty$)	W - Method ($\mu = 0.0$)	New Method ($\mu_f = 8.0$ Hz)	New Method ($\mu_f = 8.5$ Hz)
1	0.80335	3.6273	7.7105	9.7592
3	9.4602	10.263	10.378	10.397
6	9.9755	10.388	10.400	10.401
15	10.020	10.390	10.400	10.401
36	10.424	10.401	10.401	10.401
47	10.409	10.401	10.401	10.401

(b) Sensitivity of 1st eigenvector at observation point 2

(Exact Solution = - 0.36969)				
m = 1 n =	F - Method ($\mu = -\infty$)	W - Method ($\mu = 0.0$)	New Method ($\mu_f = 8.0$ Hz)	New Method ($\mu_f = 8.5$ Hz)
1	0.17546	-0.23803	-0.34424	-0.36459
3	0.14771	-0.25931	-0.35279	-0.36664
6	-0.44838	-0.37093	-0.36987	-0.36972
15	-0.44653	-0.37086	-0.36985	-0.36972
36	-0.39899	-0.36982	-0.36971	-0.36970
47	-0.36700	-0.36969	-0.36969	-0.36969

Table 2 Comparison of convergency with lower modes truncated in the uncoupled system

(a) Sensitivity of 1st eigenvector at observation point 1

(Exact Solution = - 15.447)				
n = 53 m =	F - Method ($\mu = -\infty$)	W - Method ($\mu = 0.0$)	New Method ($\mu_f = 80$ Hz)	New Method ($\mu_f = 78$ Hz)
29	0.81901	15.399	-11.353	-14.834
27	-14.689	-3.2964	-15.349	-15.434
20	-15.216	-4.0473	-15.431	-15.445
9	-15.303	-4.3019	-15.438	-15.446
2	-15.371	-8.9655	-15.442	-15.446
1	-15.450	-15.447	-15.447	-15.447

(b) Sensitivity of 1st eigenvector at observation point 2

(Exact Solution = -0.79797)				
n = 53 m =	F - Method ($\mu = -\infty$)	W - Method ($\mu = 0.0$)	New Method ($\mu_f = 80$ Hz)	New Method ($\mu_f = 78$ Hz)
29	-0.12418	0.80707	0.48459	0.74909
27	0.14977	2.7622	0.90246	0.81182
20	0.7664	1.6329	0.79814	0.79799
9	0.76478	1.6280	0.79800	0.79797
2	0.78295	2.2120	0.79805	0.79810
1	0.76560	0.79641	0.79807	0.79798

Table 3 Comparison of convergency with higher modes truncated in the coupled system

(a) Sensitivity of 1st eigenvector at structural field observation point

(Exact Solution = 0.4810)

m=1 n=	F- Method ($\mu = -\infty$)	W-Method ($\mu = 0.0$)	New Method ($\mu_f = 9.20\text{Hz}$)	New Method ($\mu_f = 9.27\text{Hz}$)
3	0.5474	0.5054	0.4817	0.4810
8	0.4729	0.4762	0.4810	0.4810
22	0.5117	0.4813	0.4810	0.4810
34	0.5014	0.4811	0.4810	0.4810
70	0.4810	0.4810	0.4810	0.4810

(b) Sensitivity of 1st eigenvector at acoustic field observation point

(Exact Solution = 9.208E - 7)

m=1 n=	F- Method ($\mu = -\infty$)	W-Method ($\mu = 0.0$)	New Method ($\mu_f = 9.20\text{Hz}$)	New Method ($\mu_f = 9.27\text{Hz}$)
3	2.216E-7	8.037E-7	9.186E-7	9.205E-7
8	2.293E-7	7.982E-7	9.183E-7	9.205E-7
22	9.860E-7	9.244E-7	9.209E-7	9.208E-7
34	7.357E-7	9.206E-7	9.208E-7	9.208E-7
70	9.208E-7	9.208E-7	9.208E-7	9.208E-7

6% and 0.08%, respectively. In the case when only 1 mode is truncated, the present method significantly improves the accuracy in comparison with those of Fox and Kapoor's and Wang's methods, which have 0.5% and 42% error, respectively. In Table 2(b), as in Table 2(a), the results corresponding to observation point 2 are tabulated.

5.2 Sensitivity of eigenmode of coupled acoustic-structural problem

A box, as shown in Fig. 1, filled with air is considered in the study of the coupled acoustic-structural problem. The sound field is divided into 64 solid elements with 125 nodes. Transforming the physical coordinates of the structural and acoustical field to modal coordinates, coupled analysis is performed with 53 modal coordinates of the structural system and 17 modal coordinates of the acoustic system. Then, as shown in Fig. 1, the thicknesses of the plates of elements 1 to 10 are taken as design variables. The responses are calculated in the y -direction at node 40 of the structural system (observation point 1) and node 32 of the acoustical field (observation point 2).

In Table 3, for different values of μ_f , the convergence rate values of the 2nd eigenvector with respect to n are tabulated. Referring to Table 3(a) (results of structural response at node 40) and 3(b) (results of acoustical response at node 32), similar to the results of the sensitivity analysis of the structural system as stated in the previous section, it is observed that by choosing different values of μ_f within the range of the error analysis, the convergency rate is significantly improved with the proposed method.

6. Conclusions

Based on the new mode superposition technique, a new eigenmode sensitivity analysis is proposed. With the proposed method, lower and higher modes can be truncated, and better accuracy is achieved in comparison to the existing Fox and Kapoor's modal method and modified Wang's modal method. Unlike Nelson's method, which calculates sensitivities of a number of eigenvectors, the proposed method solves the quasi-static equation only once, and a singular matrix can be avoided. Hence, the present method is much simpler and more efficient than Nelson's.

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