Vibration and Dynamic Stability of Anisotropic Plates*

(Free Vibration and Panel Flutter)

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The finite element method is used to study the effects of rotating the plate principal axes with respect to the geometric axes, orthotropy parameter, E_1/E_2 , and shear deformation on the vibration and flutter characteristics of anisotropic plates. The vibration and flutter of anisotropic square plates, under a variety of boundary conditions, are analyzed using the triangular finite element used in the previous paper. This element incorporates shear deformation according to Mindlin's theory.

Key Words: Free Vibration, Flutter, Finite Element Method, Anisotropic Plate, Eigenvalue Analysis

1. Introduction

The problem of deformation and strength of a plate is very important in the structural design of the fundamental elements in the structures. In recent years, application of composite materials has been spreading following the development of light structures, and it has become an important technological problem to clarify the properties of these materials. Plates of composite material are characterized by anisotropy and low out-of-plane shear rigidity.

The authors⁽¹⁾ proposed plate bending analysis of anisotropic plates using a finite element in which the shear deformation was taken into consideration. The formulation is based on the non-conforming element of Zienkiewicz⁽²⁾ and is obtained with a modified stiffness matrix in the condition in which the out-of-plane shear strain is constant in two directions within an element.

In the present paper, the finite element method is used to study the effects of the direction of the principal axis of elasticity and orthotropy parameter E_1/E_2 on the vibration and flutter characteristics of the anisotropic plates.

2. Theory

Figure 1 shows a typical anisotropic plate. The coordinate system xyz is taken such that the x - y plane coincides with the midplane of the plate. Dividing the plate into triangular elements, the element stiffness matrix is denoted as $[k_e]$ and the element mass matrix is denoted as $[m_e]$; then, the strain energy and the kinetic energy are represented by the nodal displacement $\{w_e\}$ as

$$U = \frac{1}{2} \{w_e\}^T [k_e] \{w_e\}, \quad T = \frac{1}{2} \{\dot{w}_e\}^T [m_e] \{\dot{w}_e\}. \quad (1)$$

The element stiffness matrix proposed by the authors in the previous paper in which the shear deformation was taken into consideration is used and the same interpolation function as that which was used when $[k_e]$ was calculated is used for calculating the element mass matrix $[m_e]$.

The virtual work done by the aerodynamic force will be considered. It is assumed that the aero-

Fig. 1 Coordinate system

* Received 19th October, 1992. Paper No. 91-1035B

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dynamic forces p acting on the surface of the plate is given by the two-dimensional first-order theory^{(3),(4)}:

$$p = -\frac{2q}{\sqrt{M^2-1}} \left(\frac{\partial w}{\partial x} + \frac{1}{V} \frac{M^2-2}{M^2-1} \frac{\partial w}{\partial t} \right) = -\left(\beta \frac{\partial w}{\partial x} + \mu \frac{\partial w}{\partial t} \right), \quad (2)$$

where $\beta = \frac{2q}{\sqrt{M^2-1}}$, $\mu = \beta \frac{1}{V} \frac{M^2-2}{M^2-1}$, and q , M and V are the dynamic pressure, Mach number and flow velocity, respectively.

The virtual work done by the aerodynamic forces $\bar{W}_p$ is

$$\bar{W}_p = \int_A p \bar{w} dA. \quad (3)$$

In the finite element method, w is usually assumed to be in the form

$$w = [N]\{w_e\}, \quad (4)$$

where $\{w_e\}$ is the vector of nodal displacements of element and $[N]$ is the interpolation function. Then, the virtual work becomes

$$\begin{aligned} \bar{W}_p &= \int_A p \bar{w} dA = -\{\bar{w}_e\}^T \left(\beta \int_A [N]^T [N_x] dA \{w_e\} + \mu \int_A [N]^T [N] dA \{\dot{w}_e\} \right) \\ &= -\{\bar{w}_e\}^T (\beta [A_e] \{w_e\} + \mu [D_e] \{\dot{w}_e\}) = -\{\bar{w}_e\}^T \{Q\}, \end{aligned} \quad (5)$$

where $[A_e]$ is the aerodynamic load matrix (nonsymmetric) and $[D_e]$ is the aerodynamic damping matrix (symmetric), which are given by the following formulas, respectively:

$$[A_e] = \int_A [N]^T [N_x] dA, [D_e] = \int_A [N]^T [N] dA, \{Q\} = \beta [A_e] \{w_e\} + \mu [D_e] \{\dot{w}_e\}, \quad (6)$$

where $[N_x]$ represents the differential of the interpolation function $[N]$ with respect to x .

If the density ρ and the thickness h are constant and if the rotatory inertia terms are neglected, $[D_e]$ and $[m_e]$ are related by

$$[D_e] = \frac{1}{\rho h} [m_e]. \quad (7)$$

Denoting the global nodal displacement of the plate by $\{w\}$ and the mass matrix, stiffness matrix and aerodynamic load matrix by $[M]$, $[K]$ and $[A]$ (matrices of size $(m \times m)$), respectively, the Lagrange equations are

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{w}_i} \right) + \frac{\partial U}{\partial w_i} = Q_i, \quad (8)$$

and the following equation is obtained:

$$[M]\{\ddot{w}\} + [K]\{\dot{w}\} + \beta[A]\{w\} + \frac{\mu}{\rho h}[M]\{\dot{w}\} = 0. \quad (9)$$

In this equation, if $\{w\} = \{W\}e^{i\omega t}$, the vibrations are harmonic; then Eq. (9) can be written as

$$([K] + \beta[A] - \lambda[M])\{W\} = 0, \quad (10)$$

where $\lambda = \omega^2 - i \frac{\mu\omega}{\rho h}$.

In the case of $\beta=0$, Eq. (10) gives the solution of free vibration. If Eq. (10) is solved by increasing the value of β from zero, the two eigenvalues will usually approach each other and coincide at $\beta = \beta_{cr}$, becoming complex conjugate pairs for β exceeding β_{cr} , which indicates the generation of flutter.

In calculating Eq. (10), it will take a longer time if $\beta[A]$ is calculated by varying the value of β . Thus, an approximation is made for $\{W\}$ by superposing the modes of free vibration as in the following representation⁽⁵⁾:

$$\{W\} = [\phi]\{\delta\}, \quad (11)$$

where $[\phi]$ is the normalized mode matrix of $m \times n$ obtained from the normal modes of free vibration analysis, and $\{\delta\}$ represents a vector of modal amplitudes.

Substitution of $\{W\}$ from Eq. (11) into Eq. (10) yields

$$([K^*] - \lambda[I])\{\delta\} = 0, [K^*] = [\omega^2] + \beta[\phi]^T[A][\phi]. \quad (12)$$

Finally, steps must be taken to calculate $[K^*]$ for each value of β and to find the critical value of β_{cr} by solving the above equation. However, time will be saved as compared with the case of solving Eq. (10) directly.

3. Numerical Examples

3.1 Isotropic plate

In order to evaluate the accuracy and usefulness of the study with regard to the analyses of free vibration and flutter, numerical computations are carried out for the analysis of isotropic plates. The example problem considered is a square plate of side length a , thickness h , bending rigidity D and modulus of transverse elasticity G . The domain is discretized into 10×10 elements. The element stiffness matrix and the mass matrix are computed from the average of the two dividing methods partitioned by a diagonal line for the quadrangle. The variation of the natural circular frequency of the 1st and 2nd orders, normalized by the classical theory solution, versus rigidity ratio D/Gha^2 is shown in Fig. 2. From this diagram, it is seen that as the rigidity ratio increases, the influence of the shear deformation becomes greater. For the purpose of comparison, the exact solutions⁽⁶⁾ obtained by the three-dimensional theory of elasticity are marked with ● and ■ in the diagram. It can be seen that the natural circular frequencies obtained by the present method conform with the exact solutions very well. Figure 3 shows the change of the flutter dynamic pressure parameter β_{cr} when the normal mode number n is changed. It is clear that the flutter dynamic pressure parameter almost converges at around the normal mode number $n=8$. Figure 4

shows the result of the calculation to obtain the flutter dynamic pressure parameter when the rigidity ratio D/Gha^2 is changed. Note that the influence of the shear deformation increases at around 10^{-3} of the value of the rigidity ratio.

3.2 Anisotropic plate

In order to investigate the influence of the shear deformation, the direction of the principal axis of elasticity and the orthotropy parameter E_1/E_2 , problems of vibration and flutter of the anisotropic square plates under various conditions of support were analyzed. The material constants used in the calculations were the following:

$$E_2 = 1 \times 10^{10} \text{ N/m}^2, G_{12} = G_{13} = 0.364 E_2, G_{23} = 2.5 G_{12}, \\ \nu_{12} = 0.24, \rho = 1.5 \times 10^3 \text{ kg/m}^3.$$

Orthogonal anisotropic principal axes of the material are denoted by 1, 2 and 3. Principal axes 1 and 2 are taken in the x - y plane and principal axis 3 is parallel to the coordinate axis z . Defining $D = \sqrt{D_1 D_2}$ and $Gh = \sqrt{(Gh)_{13}(Gh)_{23}}$, the calculations are performed for three values of $k = D/Gha^2$, namely $k = 10^{-6}$, 10^{-4} and 10^{-2} , and for various values of the direction of the principal axis of elasticity θ and the orthotropy parameter E_1/E_2 . Table 1 presents the natural circular frequencies of the 1st - 4th modes when the rigidity

Fig. 2 Influence of shear deformation on the natural frequency of simply supported isotropic square plate

Fig. 3 Convergence of flutter critical value of simply supported isotropic square plate ($\bar{\beta}_{cr} = \beta_{cr} \frac{a^3}{D}$)

ratio k and the direction of the principal axis of elasticity θ are varied. It is known that as rigidity ratio increases or the shear rigidity decreases, the influence of the shear deformation increases. Table 1 also shows that as the vibration shifts to higher orders, the influence of the shear deformation becomes larger. In Fig. 5 the natural circular frequency parameter $\bar{\omega}$ of 1st and 2nd modes is illustrated in terms of the direction of the principal axis of elasticity θ . This diagram also indicates that as the order of vibration becomes higher, the influence of the shear deformation increases. The very fine line shows the result of FEM analysis performed by Rossettos and Tong⁽⁷⁾ based on the hybrid stress method. Very little difference is observed compared with the result of the case of $k =$

Fig. 4 Influence of shear deformation on the panel flutter of simply supported isotropic square plate ($\bar{\beta}_{cr} = \beta_{cr} \frac{a^3}{D}$)

Fig. 5 Natural circular frequency of the cantilever anisotropic square plate ($\bar{\omega} = \omega \left(\frac{a^2}{h} \right) \sqrt{\frac{\rho}{E_2}}$)

Table 1 Natural frequencies of the simply supported and clamped anisotropic square plates ($\bar{\omega} = \omega \left(\frac{a^2}{h} \right) \sqrt{\frac{\rho}{E_2}}$)

θ (deg)	$k = D/Gha^2 (E_1/E_2)$	Simply supported				Clamped			
		1st mode	2nd mode	3rd mode	4th mode	1st mode	2nd mode	3rd mode	4th mode
0	10^{-6} (2)	6.254	13.967	17.513	23.557	11.988	21.227	26.256	32.882
	10^{-4} (2)	6.251	13.939	17.494	23.514	11.962	21.125	26.180	32.747
	10^{-2} (2)	5.835	11.776	15.111	19.387	9.673	15.066	19.206	22.581
	10^{-2} (5)	7.360	13.019	21.022	22.074	12.652	17.972	25.831	26.330
15	10^{-6} (2)	6.385	14.250	17.362	23.877	11.925	21.385	25.818	32.895
	10^{-4} (2)	6.381	14.223	17.342	23.828	11.898	21.287	25.739	32.751
	10^{-2} (2)	5.933	12.024	14.950	19.305	9.609	15.256	18.904	22.625
	10^{-2} (5)	7.511	13.585	20.570	22.431	12.457	18.200	25.232	26.683
30	10^{-6} (2)	6.655	14.851	17.067	24.467	11.799	21.829	24.823	32.932
	10^{-4} (2)	6.650	14.825	17.043	24.411	11.770	21.737	24.737	32.776
	10^{-2} (2)	6.143	12.557	14.609	19.452	9.486	15.753	18.186	22.657
	10^{-2} (5)	7.888	14.735	19.641	23.009	12.062	18.870	23.828	27.111
45	10^{-6} (2)	6.792	15.178	16.913	24.735	11.737	22.202	24.179	32.955
	10^{-4} (2)	6.787	15.153	16.886	24.676	11.707	22.115	24.086	32.796
	10^{-2} (2)	6.256	12.853	14.417	19.569	9.426	16.143	17.686	22.675
	10^{-2} (5)	8.109	15.278	19.188	23.286	11.864	19.426	22.910	27.216

Fig. 6 Modes of vibration of simply supported anisotropic square plate ($E_1/E_2=2$, $k=D/Gha^2=10^{-2}$)

10^{-6} , and it is seen that these results almost coincide. In Fig. 5, the frequency parameter $\bar{\omega}$ is plotted for various values of the orthotropy parameter E_1/E_2 . It is found that the influence of E_1/E_2 on the natural circular frequency at around $\theta=0$ deg is small, particularly up to around $\theta=30$ deg for the natural circular frequency of the vibration of the lowest order. Figures 6 and 7 illustrate the modes of vibration of the simply supported anisotropic square plates of the cantilever type (orthotropy parameter $E_1/E_2=2$). Although the 2nd and 3rd modes are degenerated in the case of the isotropic square plate, there is a

Fig. 7 Modes of vibration of the cantilever anisotropic square plate ($E_1/E_2=2$, $k=D/Gha^2=10^{-2}$)

considerable difference between the natural circular frequencies corresponding to these two modes, as can be seen in Table 1. Also, Figs. 6 and 7 present the behavior of the nodal line of the mode of vibration that rotates in the direction of θ as θ changes. In Table 2, the flutter dynamic pressure parameter $\bar{\beta}_{cr}$ and the flutter circular frequency $\bar{\lambda}_{cr}$ are presented for the anisotropic square plates both simply supported and clamped at the boundaries. It shows that as the rigidity ratio becomes larger or the shear rigidity becomes smaller, $\bar{\beta}_{cr}$ and $\bar{\lambda}_{cr}$ decrease. The flutter dynamic pressure parameter $\bar{\beta}_{cr}$ is little affected by the orthotropy parameter E_1/E_2 at $\theta=90$ deg. In

Table 2 Flutter critical values of the simply supported and clamped anisotropic square plates

θ (deg)	Simply supported ($E_1/E_2=2$)								Clamped ($E_1/E_2=2$)							
	$k=10^{-6}$		$k=10^{-4}$		$k=10^{-2}$		$k=10^{-2} (E_1/E_2=5)$		$k=10^{-6}$		$k=10^{-4}$		$k=10^{-2}$		$k=10^{-2} (E_1/E_2=5)$	
	$\bar{\beta}_{cr}$	$\bar{\lambda}_{cr}$	$\bar{\beta}_{cr}$	$\bar{\lambda}_{cr}$	$\bar{\beta}_{cr}$	$\bar{\lambda}_{cr}$	$\bar{\beta}_{cr}$	$\bar{\lambda}_{cr}$	$\bar{\beta}_{cr}$	$\bar{\lambda}_{cr}$	$\bar{\beta}_{cr}$	$\bar{\lambda}_{cr}$	$\bar{\beta}_{cr}$	$\bar{\lambda}_{cr}$	$\bar{\beta}_{cr}$	$\bar{\lambda}_{cr}$
0	66.851	229.55	66.745	229.26	54.020	196.12	112.48	397.05	111.94	547.69	111.37	545.06	64.014	323.31	122.48	596.90
30	60.360	217.20	60.235	216.85	48.147	184.92	72.976	265.25	97.234	491.81	96.620	466.93	54.980	288.29	83.748	426.86
45	53.935	219.31	53.807	201.81	42.508	171.31	57.002	238.67	84.479	443.36	83.898	440.63	47.283	260.73	66.137	380.20
60	47.309	181.96	47.187	181.65	37.078	154.37	45.669	207.36	73.935	406.54	73.383	403.95	40.265	236.33	52.027	338.49
90	40.050	154.49	39.940	154.23	31.666	133.24	33.226	155.52	64.304	375.86	63.753	373.25	33.463	214.19	38.582	302.30

$$k = \frac{D}{G h a^2}, \quad D = \sqrt{D_1 D_2}, \quad G h = \sqrt{(G h)_1 (G h)_2}, \quad \bar{\beta} = \beta \frac{a^3}{E_2 h^3}, \quad \bar{\lambda} = \lambda \frac{\rho a^4}{E_2 h^2}$$

Fig. 8 Flutter dynamic pressure parameter of the cantilever anisotropic square plate ($\bar{\beta}_{cr} = \beta_{cr} \frac{a^2}{E_2 h^3}$)

Fig. 8, the flutter dynamic pressure parameter $\bar{\beta}_{cr}$ is plotted versus the direction of the principal axis of elasticity θ , for the case of the cantilever anisotropic square plate. The result in Ref. (7) is shown in the same diagram. The two curves almost coincide, though there are some discrepancies at around the maximum value of $\bar{\beta}_{cr}$. The flutter dynamic pressure parameter $\bar{\beta}_{cr}$ takes its minimum value at around $\theta = 70$ deg. It can be seen that as the rigidity ratio increases from 10^{-6} to 10^{-2} , the flutter dynamic pressure parameter $\bar{\beta}_{cr}$ becomes smaller by about 20 - 30%.

4. Conclusions

Investigations were carried out on the free vibration characteristics and the flutter characteristics of the isotropic and anisotropic plates by using the plate bending elements of an anisotropic plate in which the

shear deformation is taken into consideration. The results obtained in this study are summarized as follows.

(1) The natural frequencies of the isotropic plates were calculated and compared with the strict solutions obtained from the three-dimensional theory of elasticity. The result was sufficiently accurate.

(2) The flutter critical value, for example, obtained by modal analysis through flutter analysis of the isotropic and anisotropic plates made in this study enabled the authors to obtain solutions accurate enough for practical use with normal modes of a small number.

(3) In the natural frequency analysis and the flutter analysis of the anisotropic plate, the influence of the direction of the principal axis of elasticity, orthotropy parameter E_1/E_2 and the shear deformation was clarified.

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