Ripple-Free Deadbeat Tracking Problem*

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This paper deals with the synthesis of a single-input/single-output ripple-free deadbeat tracking control system using a two-degree-of-freedom compensator scheme. The proposed system can simultaneously follow a reference input and eliminate the effects of a disturbance with no ripple in a finite time. Both the reference input and the disturbance are general types of those generated by linear time-invariant systems. Moreover, the ripple-free deadbeat system with optimal transient performance is proposed, which can make as moderate as possible the transient responses for both the reference input and the disturbance. The algorithm is based on a simple iterative operation with no matrix inversion.

Key Words: Analogue Control, Digital Control, Transient Response, Deadbeat Tracking Control, Optimal Control, Ripple-Free Response

1. Introduction

Recently the demand for a control system which can achieve excellent control performance at high speed has been increasing. This paper is concerned with a tracking control problem in which the controlled output can perfectly follow a reference input in a finite time. Such a control problem is called the deadbeat tracking control problem. The deadbeat tracking control system is said to have the following problems from the practical point of view. (P1) The system is not robust for parameter perturbation of a plant. (P2) The system usually needs excessive control energy, and an undesirable overshoot may occur in the transient responses. (P3) In attenuating a disturbance, excessive energy may be required and the controlled output may be highly violated.

Considerable attention has been paid to these problems during the last years. Zhao and Kimura⁽¹⁾

and Funahashi and Kato⁽²⁾ have considered problem (P1). The proposed systems optimized performance indices reflecting the robustness of the system under the minimum settling steps. Horiguchi et al.⁽³⁾ have treated both problems (P2) and (P3). The proposed system can make the control energy as small as possible and the transient controlled output response as moderate as possible for both step-type reference input and step-type disturbance. However, the above deadbeat systems, at only the sampling points, compensate the deadbeat performance, and an undesirable oscillation called ripple may occur between the points.

Ripple-free deadbeat tracking control systems⁽⁴⁾⁻⁽⁸⁾ which can provide deadbeat performance not only at, but also between, the sampling points have been proposed. Funahashi et al.⁽⁸⁾ considered the ripple-free system taking both problems (P2) and (P3) into account. Two introduced performance indices consist of the control input and the controlled output error sequences for a reference input and a disturbance. The proposed system is designed by optimizing the performance indices under given settling steps. Since the design problem is reduced to an unconstrained optimization problem, the design procedure is

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straightforward. However, the algorithm involves the calculation of matrix inversions. Both the reference input and disturbance are restricted to step-type ones.

In this paper, we extend the above result. Our advantages are as follows. (1) Both the available reference input and disturbance are general types of those generated by linear time-invariant systems. (2) The proposed system can be designed easily through an iterative procedure without matrix inversion. Since the algorithm is iterative in terms of the settling step, we can calculate sequentially the controllers and the value of the performance indices for the various settling steps.

2. Ripple-Free Deadbeat Tracking Control System

2.1 Preliminaries

Consider the single-input/single-output discrete-time system shown in Fig. 1, where r , u_d , y and v are the reference input, the output of the discrete-time compensator, the controlled output and the disturbance, respectively. Such a control system is called a two-degree-of-freedom compensator system. λ denotes a one-step delay operator. $P(\lambda)$ is composed of a zero-order hold, a proper continuous-time precompensator $\bar{C}(s)$, a proper continuous-time plant $\bar{P}(s)$ and a sampler in series. The block diagram is shown in Fig. 2, where the superscript “-” denotes the continuous-time signal. $\bar{u}$ and u are the continuous-time control input and the discrete-time one, respectively. We define

$$P(\lambda) = n_p(\lambda)/d_p(\lambda), \quad (1)$$

where $d_p(\lambda)$ and $n_p(\lambda)$ are coprime polynomials with $d_p(0) \neq 0$. The two-degree-of-freedom compensator scheme in Fig. 1 is then given by

$$d_c(\lambda)u_d(kT) = n_{cr}(\lambda)r(kT) - n_{cy}(\lambda)y(kT), \quad (2)$$

where T is the sampling period and k is an integer. $d_c(\lambda)$, $n_{cr}(\lambda)$ and $n_{cy}(\lambda)$ are polynomials with $d_c(0) \neq 0$. The closed-loop characteristic polynomial $\alpha(\lambda)$ becomes

$$\alpha(\lambda) = d_p(\lambda)d_c(\lambda) + n_p(\lambda)n_{cy}(\lambda). \quad (3)$$

From the deadbeat condition, $\alpha(\lambda) = 1$. From the coprime condition of $d_p(\lambda)$ and $n_p(\lambda)$, there always exist $d_c(\lambda)$ and $n_{cy}(\lambda)$ such that $\alpha(\lambda) = 1$, i.e.,

$$d_p(\lambda)d_c(\lambda) + n_p(\lambda)n_{cy}(\lambda) = 1. \quad (4)$$

Then it follows that

$$u_d(\lambda)/r(\lambda) = d_p(\lambda)n_{cr}(\lambda) \quad (5)$$

$$y(\lambda)/r(\lambda) = n_p(\lambda)n_{cr}(\lambda) \quad (6)$$

$$u_d(\lambda)/v(\lambda) = -d_p(\lambda)n_{cy}(\lambda) \quad (7)$$

$$y(\lambda)/v(\lambda) = d_p(\lambda)d_c(\lambda). \quad (8)$$

We can design $n_{cr}(\lambda)$ in terms of the control input and the controlled output response for a reference input. $d_c(\lambda)$ and $n_{cy}(\lambda)$ can be designed in terms of a disturbance.

2.2 Ripple-free deadbeat control system for reference input

The controlled output error $e_y(\cdot)$ for a reference input is defined as

$$e_y(\cdot) = r(\cdot) - y(\cdot). \quad (9)$$

Consider a proper continuous-time reference input $\bar{r}(s)$ which is generated by a linear time-invariant system:

$$\bar{r}(s) = \bar{n}_r(s)/\bar{d}_r(s), \quad (10)$$

where $\bar{d}_r(s)$ and $\bar{n}_r(s)$ are coprime polynomials. Assume that $\bar{d}_r(s)$ and the numerator polynomial of $\bar{P}(s)$ are coprime. This assumption is necessary in designing $\bar{C}(s)$ later. Define the discrete-time representation of $\bar{r}(s)$ as

$$r(\lambda) = n_r(\lambda)/d_r(\lambda), \quad (11)$$

where $d_r(\lambda)$ and $n_r(\lambda)$ are coprime polynomials with $d_r(0) \neq 0$. A ripple-free deadbeat tracking control system is a system which eliminates the output error not only at, but also between, the sampling points in a finite time. Franklin and Emai-Naeini⁽⁶⁾ showed the condition of the controller to achieve such a system under a one-degree-of-freedom compensator scheme. For a two-degree-of-freedom compensator scheme, the following lemma is given.

◀Lemma 1▶

In Figs. 1 and 2, a ripple-free deadbeat control system for the reference input $\bar{r}(s)$ is obtained if the following conditions hold.

(I) The denominator polynomial of $\bar{P}(s)\bar{C}(s)/s$ possesses all factors of $\bar{d}_r(s)$.

(II) The output sequence of the discrete-time compensator for the reference input reaches a constant in a finite number of control steps, i.e., $(1 - \lambda)u_d(\lambda)$ is a polynomial.

(III) The error sequence vanishes in a finite number of control steps, i.e., $e_y(\lambda)$ is a polynomial.

(Proof)

The proof is similar to that in Ref. (8), and is omitted here. ◻

From now on, assume that we have already

Fig. 1 Two-degree-of-freedom control system

Fig. 2 Continuous-time control system

designed a continuous-time precompensator $\bar{C}(s)$ satisfying both the condition (I) of Lemma 1 and the following condition.

(IV) There is no cancellation between zeros of $\bar{P}(s)$ and poles of $\bar{C}(s)$.

From the coprimeness of $\bar{d}_r(s)$ and the numerator polynomial of $\bar{P}(s)$, there always exists such a $\bar{C}(s)$ e.g., $\bar{C}(s)=1/\bar{d}_r(s)$. Condition (IV) is necessary for a formulation of the control input error sequence. The following theorem provides a class of controller satisfying the above ripple-free deadbeat tracking condition.

《Theorem 1》

In Figs. 1 and 2, a ripple-free deadbeat control system for the reference input $\bar{r}(s)$ is obtained by choosing

$$n_{cr}(\lambda) = n_{cr}^*(\lambda) + d_r(\lambda)q(\lambda), \quad (12)$$

where $q(\lambda)$ is an arbitrary polynomial and $n_{cr}^*(\lambda)$ is the minimum-degree solution of $n_{cr}(\lambda)$ such that

$$n_p(\lambda)n_{cr}(\lambda) + d_r(\lambda)f_y(\lambda) = 1. \quad (13)$$

There exists such an $n_{cr}(\lambda)$ if and only if $d_r(\lambda)$ and $n_p(\lambda)$ are coprime.

(Proof)

From Lemma 1, we now determine $n_{cr}(\lambda)$ satisfying both conditions (II) and (III). From Eqs. (5) and (11),

$$(1-\lambda)u_d(\lambda) = (1-\lambda)d_p(\lambda)n_{cr}(\lambda)n_r(\lambda)/d_r(\lambda). \quad (14)$$

Note that $(1-\lambda)d_p(\lambda)$ is the denominator of the discrete-time representation of $\bar{P}(s)\bar{C}(s)/s$. Since condition (I) holds, $(1-\lambda)d_p(\lambda)/d_r(\lambda)$ is a polynomial. Condition (II) is always satisfied. From Eqs. (6), (9) and (11), the output error sequence is given by

$$e_y(\lambda) = \{1 - n_p(\lambda)n_{cr}(\lambda)\}n_r(\lambda)/d_r(\lambda). \quad (15)$$

Define

$$f_y(\lambda) = \{1 - n_p(\lambda)n_{cr}(\lambda)\}/d_r(\lambda). \quad (16)$$

From the coprimeness of $d_r(\lambda)$ and $n_r(\lambda)$, condition (III) means that $f_y(\lambda)$ is a polynomial. This holds if and only if $d_r(\lambda)$ and $n_p(\lambda)$ are coprime from Eq. (13). Then $n_{cr}(\lambda)$ is given by Eq. (12)⁽⁹⁾. $\square$

We will investigate the class of the input and the output error sequences and the settling time under Theorem 1. Let $C(\lambda)$ be the discrete-time representation of the system which is composed of a zero-order hold, $\bar{C}(s)$ and a sampler in series, as shown in Fig. 2. We define

$$C(\lambda) = n_{cc}(\lambda)/d_{cc}(\lambda), \quad (17)$$

where $d_{cc}(\lambda)$ and $n_{cc}(\lambda)$ are coprime polynomials. From Eqs. (12) and (14), the control input sequence becomes

$$\begin{aligned} u(\lambda) &= C(\lambda)u_d(\lambda) \\ &= d_p(\lambda)n_{cc}(\lambda)n_{cr}(\lambda)n_r(\lambda)/d_{cc}(\lambda)d_r(\lambda) \\ &= d_p(\lambda)n_{cc}(\lambda)n_r(\lambda)\{n_{cr}^*(\lambda)/d_r(\lambda) + q(\lambda)\}/d_{cc}(\lambda). \end{aligned} \quad (18)$$

Let

$$d_{pc}(\lambda) = d_p(\lambda)/d_{cc}(\lambda). \quad (19)$$

Denoting the z -transform by the symbol $Z\{\cdot\}$, $d_p(\lambda)$ is equal to the denominator of $(1-\lambda)Z\{\bar{P}(s)\bar{C}(s)/s\}$, and $d_{cc}(\lambda)$, to that of $(1-\lambda)Z\{\bar{C}(s)/s\}$. Since condition (IV) holds, $d_{pc}(\lambda)$ is a polynomial. Denote the quotient and the remainder by $f_u(\lambda)$ and $g_u(\lambda)$, respectively, when the numerator of the first set of terms of $u(\lambda)$ in Eq. (18) is divided by $d_r(\lambda)$, i.e.,

$$d_{pc}(\lambda)n_{cc}(\lambda)n_r(\lambda)n_{cr}^*(\lambda) = d_r(\lambda)f_u(\lambda) + g_u(\lambda), \quad (20)$$

and Eq. (18) becomes

$$u(\lambda) = g_u(\lambda)/d_r(\lambda) + f_u(\lambda) + d_{pc}(\lambda)n_{cc}(\lambda)n_r(\lambda)q(\lambda). \quad (21)$$

Equation (21) can be interpreted as follows. The control input sequence is composed of $g_u(\lambda)/d_r(\lambda)$ and $[f_u(\lambda) + d_{pc}(\lambda)n_{cc}(\lambda)n_r(\lambda)q(\lambda)]$. The former is the input sequence consisting of the modes of the reference input, and never decays, that is, it is the steady-state input sequence. The latter is the part of the input sequence which vanishes in a finite number of control steps, that is, it is the transient input sequence. Define the control input error sequence $e_u(\lambda)$ for the reference input as follows.

$$e_u(\lambda) = g_u(\lambda)/d_r(\lambda) - u(\lambda) \quad (22)$$

《Theorem 2》

When Theorem 1 holds, the input and output error sequences for the reference input $\bar{r}(s)$ are obtained by

$$e_u(\lambda) = -f_u(\lambda) - d_{pc}(\lambda)n_{cc}(\lambda)n_r(\lambda)q(\lambda), \quad (23)$$

$$e_y(\lambda) = \{f_y^*(\lambda) - n_p(\lambda)q(\lambda)\}n_r(\lambda), \quad (24)$$

respectively. Then the settling time are given by

$$e_u(kT) = 0, \quad \forall k \geq \deg[d_{pc}(\lambda)n_{cc}(\lambda)n_r(\lambda)q(\lambda)] + 1$$

$$e_y(t) = 0, \quad \forall t \geq [\deg\{n_p(\lambda)n_r(\lambda)q(\lambda)\} + 1]T, \quad (25)$$

where $f_y^*(\lambda)$ is the minimum-degree solution of $f_y(\lambda)$ satisfying Eq. (13) and $\deg(\cdot)$ denotes the degree of a polynomial. t is a real number.

(Proof)

Substituting Eq. (21) into Eq. (22) leads to Eq. (23). From Eqs. (15) and (16),

$$e_y(\lambda) = f_y(\lambda)n_r(\lambda).$$

Since the general solution of Eq. (13) is given by

$$f_y(\lambda) = f_y^*(\lambda) - n_p(\lambda)q(\lambda),$$

Eq. (24) is obtained. From Eq. (13), it follows that⁽⁹⁾

$$\deg f_y^*(\lambda) \leq \deg n_p(\lambda) - 1,$$

$$\deg n_{cr}^*(\lambda) \leq \deg d_r(\lambda) - 1,$$

and, from Eq. (20),

$$\deg f_u(\lambda)$$

$$\leq \deg[d_{pc}(\lambda)n_{cc}(\lambda)n_r(\lambda)] + \deg n_{cr}^*(\lambda) - \deg d_r(\lambda)$$

$$\leq \deg[d_{pc}(\lambda)n_{cc}(\lambda)n_r(\lambda)] - 1.$$

Hence, it is straightforward that Eq. (25) holds. $\square$

《Remark 1》 Theorems 1 and 2 are a generalization of the result of Funahashi and Yamada⁽⁸⁾ in terms of the reference input.

《Remark 2》 When $q(\lambda)=0$, the minimum-time ripple-free deadbeat tracking control system for the refer-

ence input $\bar{r}(s)$ is obtained. Then the input and output error sequences are given by

$$\begin{aligned} e_u(\lambda) &= -f_u(\lambda), \\ e_y(\lambda) &= f_y^*(\lambda)n_r(\lambda), \end{aligned}$$

respectively. When $q(\lambda) \neq 0$, the settling steps can be assigned arbitrarily by the degree of $q(\lambda)$. The input and output error sequences can be modified by the coefficients of $q(\lambda)$.

2.3. Ripple-free deadbeat control system for a disturbance

We define the controlled output error $\tilde{e}_y(\cdot)$ for a disturbance.

$$\tilde{e}_y(\cdot) = y(\cdot) \quad (26)$$

Consider a proper continuous-time disturbance $\bar{v}(s)$ which is generated by a linear time-invariant system:

$$\bar{v}(s) = \bar{n}_v(s)/\bar{d}_v(s) \quad (27)$$

where $\bar{d}_v(s)$ and $\bar{n}_v(s)$ are coprime polynomials. Assume that $\bar{d}_v(s)$ and the numerator polynomial of $\bar{P}(s)$ are coprime. This assumption is necessary in designing $\bar{C}(s)$ later. We define the discrete-time representation of $\bar{v}(s)$ as

$$v(\lambda) = n_v(\lambda)/d_v(\lambda), \quad (28)$$

where $d_v(\lambda)$ and $n_v(\lambda)$ are coprime polynomials with $d_v(0) \neq 0$. From Eq.(26), the ripple-free deadbeat tracking control system for the disturbance is a system which eliminates the effect of the disturbance not only at, but also between, the sampling points in a finite time.

«Lemma 2»

In Figs. 1 and 2, a ripple-free deadbeat control system for the disturbance $\bar{v}(s)$ is obtained, if the following conditions hold.

(I) The denominator polynomial of $\bar{P}(s)\bar{C}(s)/s$ possesses all factors of $\bar{d}_v(s)$.

(II) The output sequence of the discrete-time compensator for the disturbance reaches a constant in a finite number of control steps, i.e., $(1-\lambda)u_d(\lambda)$ is a polynomial.

(III) The error sequence vanishes in a finite number control steps, i.e., $\tilde{e}_y(\lambda)$ is a polynomial.

(Proof)

The proof is similar to that in Ref. (8), and is omitted here. $\square$

From now on, assume that we have already designed a continuous-time precompensator $\bar{C}(s)$ satisfying both condition (I) of Lemma 2 and condition (IV). From the coprimeness of $\bar{d}_v(s)$ and the numerator polynomial of $\bar{P}(s)$, there always exist such a $\bar{C}(s)$, e.g., $\bar{C}(s) = 1/\bar{d}_v(s)$. We define the coprime polynomials $d'_p(\lambda)$ and $d'_v(\lambda)$ by

$$d'_p(\lambda)/d'_v(\lambda) = d_p(\lambda)/d_v(\lambda). \quad (29)$$

Letting

$$\tilde{f}_y(\lambda) = d_c(\lambda)/d'_v(\lambda), \quad (30)$$

Eq. (4) becomes

$$d_p(\lambda)d'_v(\lambda)\tilde{f}_y(\lambda) + n_p(\lambda)n_{cy}(\lambda) = 1. \quad (31)$$

The following theorem provides a class of controllers satisfying Lemma 2.

«Theorem 3»

In Figs. 1 and 2, a ripple-free deadbeat control system for the disturbance $\bar{v}(s)$ is obtained by choosing

$$d_c(\lambda) = d'_v(\lambda)\{\tilde{f}_y^*(\lambda) + n_p(\lambda)\bar{q}(\lambda)\}, \quad (32)$$

$$n_{cy}(\lambda) = n_{cy}^*(\lambda) - d_p(\lambda)d'_v(\lambda)\bar{q}(\lambda), \quad (33)$$

where $\bar{q}(\lambda)$ is an arbitrary polynomial. $\tilde{f}_y^*(\lambda)$ and $n_{cy}^*(\lambda)$ are the minimum-degree solutions of $\tilde{f}_y(\lambda)$ and $n_{cy}(\lambda)$ satisfying Eq.(31), respectively. There exist such $d_c(\lambda)$ and $n_{cy}(\lambda)$ if and only if $d'_v(\lambda)$ and $n_p(\lambda)$ are coprime.

(Proof)

From Lemma 2, we now determine $d_c(\lambda)$ and $n_{cy}(\lambda)$ satisfying both conditions (II) and (III). From Eqs.(7) and (28),

$$(1-\lambda)u_d(\lambda) = -(1-\lambda)d_p(\lambda)n_{cy}(\lambda)n_v(\lambda)/d_v(\lambda). \quad (34)$$

Since condition (I) holds, $(1-\lambda)d_p(\lambda)/d_v(\lambda)$ is a polynomial and condition (II) is always satisfied. From Eqs.(8), (26) and (28),

$$\tilde{e}_y(\lambda) = d_p(\lambda)d_c(\lambda)n_v(\lambda)/d_v(\lambda), \quad (35)$$

and from Eqs.(29) and (30), it follows that

$$\tilde{e}_y(\lambda) = d'_p(\lambda)\tilde{f}_y(\lambda)n_v(\lambda). \quad (36)$$

From the coprimeness of $d'_v(\lambda)$ and $\{d'_p(\lambda), n_v(\lambda)\}$, condition (III) means $\tilde{f}_y(\lambda)$ is a polynomial. This holds if and only if $d'_v(\lambda)$ and $n_p(\lambda)$ are coprime from Eq. (31). Then $d_c(\lambda)$ and $n_{cy}(\lambda)$ are given by Eqs.(32) and (33), respectively. $\square$

We will investigate class of the input and output error sequences and the settling time under Theorem 3. By a similar statement as in section 2.2, $d_{pc}(\lambda)$ is a polynomial. Equations (17), (29) and (33) lead to

$$\begin{aligned} u(\lambda) &= C(\lambda)u_d(\lambda) \\ &= -d_p(\lambda)n_{cc}(\lambda)n_{cy}(\lambda)n_v(\lambda)/\{d_{cc}(\lambda)d_v(\lambda)\} \\ &= -d_{pc}(\lambda)n_{cc}(\lambda)n_v(\lambda)\{n_{cy}^*(\lambda)/d'_v(\lambda) - d'_p(\lambda)\bar{q}(\lambda)\}. \end{aligned} \quad (37)$$

Denote the quotient and the remainder by $\tilde{f}_u(\lambda)$ and $\tilde{g}_u(\lambda)$, respectively, when the numerator of the first set of terms of $u(\lambda)$ is divided by $d_v(\lambda)$, i.e.,

$$-d_{pc}(\lambda)n_{cc}(\lambda)n_v(\lambda)n_{cy}^*(\lambda) = d_v(\lambda)\tilde{f}_u(\lambda) + \tilde{g}_u(\lambda). \quad (38)$$

Then Eq.(37) becomes

$$\begin{aligned} u(\lambda) &= \tilde{g}_u(\lambda)/d_v(\lambda) + \tilde{f}_u(\lambda) \\ &\quad + d_{pc}(\lambda)d'_p(\lambda)n_{cc}(\lambda)n_v(\lambda)\bar{q}(\lambda). \end{aligned} \quad (39)$$

By a similar statement as in section 2.2, we define the control input error sequence $\tilde{e}_u(\lambda)$ for the disturbance as follows.

$$\tilde{e}_u(\lambda) = \tilde{g}_u(\lambda)/d_v(\lambda) - u(\lambda) \quad (40)$$

«Theorem 4»

When Theorem 3 holds, the input and output error

sequences for the disturbance $\tilde{v}(s)$ are obtained by

$$\tilde{e}_u(\lambda) = -\tilde{f}_u(\lambda) - d_{pc}(\lambda)d'_p(\lambda)n_{cc}(\lambda)n_v(\lambda)\tilde{q}(\lambda), \quad (41)$$

$$\tilde{e}_y(\lambda) = d'_p(\lambda)n_v(\lambda)\{\tilde{f}_y^*(\lambda) + n_p(\lambda)\tilde{q}(\lambda)\}, \quad (42)$$

respectively. Then the settling times are given by

$$\begin{aligned} \tilde{e}_u(kT) &= 0, \\ \forall k &\geq \deg[d_{pc}(\lambda)d'_p(\lambda)n_{cc}(\lambda)n_v(\lambda)\tilde{q}(\lambda)] + 1, \\ \tilde{e}_y(t) &= 0, \forall t \geq [\deg\{d'_p(\lambda)n_p(\lambda)n_v(\lambda)\tilde{q}(\lambda)\} + 1]T. \end{aligned} \quad (43)$$

(Proof)

Substitution of Eq.(39) into Eq.(40) leads to Eq.(41). Since the general solution of Eq.(31) is given by $\tilde{f}_y(\lambda) = \tilde{f}_y^*(\lambda) + n_p(\lambda)\tilde{q}(\lambda)$, substitution into Eq.(36) leads to Eq.(42). From Eqs.(29) and (31), it follows that⁽⁹⁾

$$\begin{aligned} \deg \tilde{f}_y^*(\lambda) &\leq \deg n_p(\lambda) - 1, \\ \deg n_{cy}^*(\lambda) &\leq \deg[d_p(\lambda)d'_v(\lambda)] - 1 \\ &= \deg[d'_p(\lambda)d_v(\lambda)] - 1, \end{aligned}$$

and, from Eq.(38),

$$\begin{aligned} \deg \tilde{f}_u(\lambda) &\leq \deg[d_{pc}(\lambda)n_{cc}(\lambda)n_v(\lambda)] \\ &\quad + \deg n_{cy}^*(\lambda) - \deg d_v(\lambda) \\ &\leq \deg[d_{pc}(\lambda)d'_p(\lambda)n_{cc}(\lambda)n_v(\lambda)] - 1. \end{aligned}$$

Hence, it is straightforward that Eq.(43) holds. $\square$

<Remark>

The same statement as Remark 2 can be made in this case.

3. The Optimization of Transient Responses

Let N and $\tilde{N}$ be the degrees of $q(\lambda)$ and $\tilde{q}(\lambda)$, respectively. The following performance indices are introduced.

$$J(N) = \sum_{k=0}^K \{e_u(kT)^2 + \rho^2 e_y(kT)^2\} \quad (44)$$

$$\tilde{J}(\tilde{N}) = \sum_{k=0}^{\tilde{K}} \{\tilde{e}_u(kT)^2 + \tilde{\rho}^2 \tilde{e}_y(kT)^2\} \quad (45)$$

$J(N)$ is the performance index consisting of the transient error sequences of the control input and the controlled output for the reference input. $\tilde{J}(\tilde{N})$ is that for the disturbance. K and $\tilde{K}$ denote the numbers of settling steps for the reference input and the disturbance, respectively. ρ and $\tilde{\rho}$ are real numbers, which can be given arbitrarily.

Our aim is to design a ripple-free deadbeat control system which makes the transient responses as moderate as possible under specified settling steps. Consider the following problem. Find $q(\lambda)$ and $\tilde{q}(\lambda)$ such that $J(N)$ and $\tilde{J}(\tilde{N})$ are minimized for given real numbers N and $\tilde{N}$, respectively.

3.1 Optimal solutions

We now consider $q(\lambda)$ which minimizes $J(N)$. We define the following polynomials.

$$\begin{aligned} a_u(\lambda) &= -f_u(\lambda), \quad b_u(\lambda) = -d_{pc}(\lambda)n_{cc}(\lambda)n_r(\lambda) \\ a_y(\lambda) &= f_y^*(\lambda)n_r(\lambda), \quad b_y(\lambda) = -n_p(\lambda)n_r(\lambda) \end{aligned} \quad (46)$$

The coefficients are denoted as

$$\begin{aligned} a_u(\lambda) &= \sum_{i=0}^{L-1} a_{ui}\lambda^i, \quad b_u(\lambda) = \sum_{i=0}^L b_{ui}\lambda^i \\ a_y(\lambda) &= \sum_{i=0}^{M-1} a_{yi}\lambda^i, \quad b_y(\lambda) = \sum_{i=0}^M b_{yi}\lambda^i \\ e_u(\lambda) &= \sum_{i=0}^{L+N} e_{ui}\lambda^i, \quad e_y(\lambda) = \sum_{i=0}^{M+N} e_{yi}\lambda^i, \\ q(\lambda) &= \sum_{i=0}^N q_i\lambda^i, \end{aligned} \quad (47)$$

Comparison of the coefficients on both side of Eqs.(23) and (24), respectively, leads to

$$\begin{aligned} e_u &= e_u^* + B_u q_N, \\ e_y &= e_y^* + B_y q_N, \end{aligned} \quad (48)$$

where

$$\begin{aligned} e_u &= [e_{u0} \cdots e_{u(L+N)}]^T \in R^{(L+N+1)} \\ e_u^* &= [a_{u0} \cdots a_{u(L-1)}, 0 \cdots 0]^T \in R^{(L+N+1)} \\ e_y &= [e_{y0} \cdots e_{y(M+N)}]^T \in R^{(M+N+1)} \\ e_y^* &= [a_{y0} \cdots a_{y(M-1)}, 0 \cdots 0]^T \in R^{(M+N+1)} \\ q_N &= [q_0, q_1 \cdots q_N]^T \in R^{N+1} \\ B_u &= \begin{bmatrix} b_{u0} & \cdots & 0 \\ b_{u1} & \cdots & b_{u0} \\ \vdots & & b_{u1} \\ b_{uL} & \cdots & \vdots \\ 0 & \cdots & b_{uL} \end{bmatrix} \in R^{(L+N+1) \times (N+1)} \\ B_y &= \begin{bmatrix} b_{y0} & \cdots & 0 \\ b_{y1} & \cdots & b_{y0} \\ \vdots & & b_{y1} \\ b_{yM} & \cdots & \vdots \\ 0 & \cdots & b_{yM} \end{bmatrix} \in R^{(M+N+1) \times (N+1)} \end{aligned} \quad (49)$$

Then the performance index $J(N)$ is given by

$$\begin{aligned} J(N) &= \left\| \begin{bmatrix} e_u \\ \rho e_y \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} B_u \\ \rho B_y \end{bmatrix} q_N + \begin{bmatrix} e_u^* \\ \rho e_y^* \end{bmatrix} \right\|^2, \end{aligned} \quad (50)$$

where the symbol $\|\cdot\|$ denotes the Euclidean norm. Since B_u and B_y are full rank matrix, $\{B_u^T B_u + \rho^2 B_y^T B_y\}$ is nonsingular. The optimal solution q_N which minimizes Eq.(50) is given by

$$q_N = -\{B_u^T B_u + \rho^2 B_y^T B_y\}^{-1} \{B_u^T e_u^* + \rho^2 B_y^T e_y^*\}. \quad (51)$$

We define

$$\begin{aligned} W_N &= B_u^T B_u + \rho^2 B_y^T B_y, \\ h_N &= B_u^T e_u^* + \rho^2 B_y^T e_y^*. \end{aligned} \quad (52)$$

Their elements are obtained by

$$\begin{aligned} W_N &= \begin{bmatrix} w_0 & w_1 & \cdots & w_N \\ w_1 & w_0 & & w_{N-1} \\ \vdots & & \ddots & \vdots \\ w_N & w_{N-1} & \cdots & w_0 \end{bmatrix} \in R^{(N+1) \times (N+1)} \\ h_N &= [h_0, h_1 \cdots h_N]^T \in R^{(N+1)}, \end{aligned} \quad (53)$$

where

$$\begin{aligned} w_i &= \sum_{j=0}^{L-i} b_{uj} b_{u(i+j)} + \rho^2 \sum_{j=0}^{M-i} b_{yj} b_{y(i+j)}, \\ i &= 0 \cdots \max\{L, M\} \end{aligned}$$

$$\begin{aligned}
w_k &= 0 \quad \forall k > \max\{L, M\} \\
h_i &= \sum_{j=0}^{L-i-1} b_{uj} a_{u(i+j)} + \rho^2 \sum_{j=0}^{M-i-1} b_{yj} a_{y(i+j)}, \\
i &= 0 \cdots \max\{L, M\} - 1 \\
h_k &= 0 \quad \forall k > \max\{L, M\} - 1.
\end{aligned}$$

Substituting Eq.(51) into Eq.(50) leads to the following theorem.

«Theorem 5»

For a given N , q_N which minimizes $J(N)$ is obtained by

$$q_N = -W_N^{-1} h_N. \quad (54)$$

Then the minimum value of $J(N)$ is given by

$$J(N)_{\min} = J(-1)_{\min} - h_N^T W_N^{-1} h_N, \quad (55)$$

where

$$J(-1)_{\min} = \sum_{i=0}^{L-1} a_{ui}^2 + \rho^2 \sum_{i=0}^{M-1} a_{yi}^2. \quad \square$$

«Remark»

The optimal solution $\tilde{q}(\lambda)$, which minimizes $\tilde{J}(\tilde{N})$ for a given $\tilde{N}$, and the minimum value of $\tilde{J}(\tilde{N})$ can be obtained by Eqs.(54) and (55), respectively, if Eq.(46) is redefined as

$$\begin{aligned}
a_u(\lambda) &= -\tilde{f}_u(\lambda), \\
b_u(\lambda) &= -d_{pc}(\lambda) d_p'(\lambda) n_{cc}(\lambda) n_v(\lambda) \\
a_y(\lambda) &= d_p'(\lambda) \tilde{f}_y^*(\lambda) n_v(\lambda), \\
b_y(\lambda) &= d_p'(\lambda) n_p(\lambda) n_v(\lambda).
\end{aligned} \quad (56)$$

3.2 Iterative procedures for the optimal solutions

W_N is a symmetric Toeplitz matrix⁽¹⁰⁾. The calculations of $q(\lambda)$ and $J(N)_{\min}$ obtained in Theorem 5 involve the inversion of this matrix. This section provides the simple iterative design procedures, with no matrix inversion, of $q(\lambda)$ and $J(N)_{\min}$.

W_i and h_i ($i=0 \cdots N$) can be represented by

$$\begin{aligned}
W_i &= \begin{bmatrix} W_{i-1} & w_{i-1} \\ w_{i-1}^T & w_0 \end{bmatrix} \in R^{(i+1) \times (i+1)}, \\
h_i &= \begin{bmatrix} h_{i-1} \\ h_i \end{bmatrix} \in R^{i+1},
\end{aligned} \quad (57)$$

where

$$w_i^T = [w_{i+1}, w_i \cdots w_1] \in R^{i+1}.$$

Let B_{ui} and B_{yi} be B_u and B_y rewritten N into i , respectively. Since B_{ui} and B_{yi} are full rank and

$$W_i = B_{ui}^T B_{ui} + \rho^2 B_{yi}^T B_{yi},$$

W_i ($i=0 \cdots N$) are nonsingular. We define

$$L_i = \begin{bmatrix} 0 & 1 \\ & \ddots \\ 1 & 0 \end{bmatrix} \in R^{(i+1) \times (i+1)}, \quad (58)$$

$$g_i = W_i^{-1} w_i. \quad (59)$$

From Eq. (57), it follows that⁽⁹⁾

$$\det W_i = (w_0 - w_{i-1}^T g_{i-1}) \det W_{i-1},$$

and we have

$$w_0 - w_{i-1}^T g_{i-1} \neq 0, \quad i=1 \cdots N.$$

«Theorem 6»

The optimal solution q_N can be obtained through the following steps.

Step 1: Calculate g_i ($i=0 \cdots N-1$).

$$\begin{aligned}
g_0 &= w_1/w_0 \\
g_i &= \begin{bmatrix} 0 \\ g_{i-1} \end{bmatrix} + \frac{w_{i+1} - g_{i-1}^T L_{i-1} w_{i-1}}{w_0 - w_{i-1}^T g_{i-1}} \begin{bmatrix} 1 \\ -L_{i-1} g_{i-1} \end{bmatrix}, \\
i &= 1 \cdots N-1
\end{aligned} \quad (60)$$

Step 2: Calculate q_i ($i=0 \cdots N$) iteratively.

$$\begin{aligned}
q_0 &= -h_0/w_0 \\
q_i &= \begin{bmatrix} q_{i-1} \\ 0 \end{bmatrix} - \frac{h_i - g_{i-1}^T h_{i-1}}{w_0 - w_{i-1}^T g_{i-1}} \begin{bmatrix} -g_{i-1} \\ 1 \end{bmatrix}, \quad i=1 \cdots N
\end{aligned} \quad (61)$$

Then the minimum value of $J(N)$ is obtained iteratively by

$$\begin{aligned}
J(0)_{\min} &= J(-1)_{\min} - h_0^2/w_0 \\
J(i)_{\min} &= J(i-1)_{\min} - \frac{(h_i - g_{i-1}^T h_{i-1})^2}{w_0 - w_{i-1}^T g_{i-1}}, \quad i=1 \cdots N.
\end{aligned} \quad (62)$$

(Proof)

It can be shown easily that⁽⁹⁾

$$\begin{aligned}
W_i^{-1} &= \begin{bmatrix} W_{i-1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\beta_{i-1}} \begin{bmatrix} -g_{i-1} \\ 1 \end{bmatrix} \begin{bmatrix} -g_{i-1}^T & 1 \end{bmatrix}, \\
\beta_{i-1} &= w_0 - w_{i-1}^T g_{i-1}.
\end{aligned} \quad (63)$$

Multiplying by h_i of Eq.(57) leads to Eqs.(61) and (62) from Eqs.(54) and (55), respectively. We now prove Eq.(60). It can be easily checked that

$$W_i = L_i W_i L_i$$

and

$$L_i^{-1} = L_i.$$

From Eq.(59), it follows that

$$g_i = L_i W_i^{-1} L_i w_i.$$

L_i and w_i are expressed as

$$\begin{aligned}
L_i &= \begin{bmatrix} 0 & 1 \\ L_{i-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & L_{i-1} \\ 1 & 0 \end{bmatrix}, \\
w_i^T &= [w_{i+1} \quad w_{i-1}^T].
\end{aligned}$$

Hence we have

$$g_i = \begin{bmatrix} 0 & 1 \\ L_{i-1} & 0 \end{bmatrix} W_i^{-1} \begin{bmatrix} L_{i-1} w_{i-1} \\ w_{i+1} \end{bmatrix}$$

Substitution of Eq.(63) gives Eq.(60). $\square$

«Remark»

For iterative design procedures of the optimal solution $\tilde{q}(\lambda)$ and the minimum value of $\tilde{J}(\tilde{N})$, a similar assertion holds. $\square$

3.3 Summary of controller design

The design procedure for achieving a ripple-free deadbeat tracking control system with optimal transient performance can be outlined as below.

Step 1: Choose $\bar{C}(s)$ such that the conditions of Lemma 1 (I), Lemma 2 (I) and (IV) hold.

Step 2: Determine ρ and $\tilde{\rho}$. Calculate $J(N)_{\min}$ recursively for various N 's by Eqs. (60) and (62), and determine one desirable N in terms of a trade-off between the settling step and the value of $J(N)_{\min}$. For $\tilde{N}$, a similar assertion holds.

Step 3: Calculate the optimal solutions q_N and $\tilde{q}_N$ by

Eq. (61).

Step 4: Calculate $d_c(\lambda)$, $n_{cr}(\lambda)$ and $n_{cy}(\lambda)$ by Eqs. (12), (32) and (33), respectively.

Step 5: Investigate the transient responses of the control input and the controlled output for the reference input and the disturbance. If the desired transient performances are obtained, the design is completed. Otherwise, go back to Step 2 and reselect ρ , $\bar{\rho}$, N and $\tilde{N}$.

4. Numerical Example

Consider a plant described by the following transfer function.

$$\bar{P}(s) = 1/(s(5s+1))$$

The reference input and the disturbance are, as shown in Fig. 3, ramp type and sinusoidal type, respectively, i.e.,

$$\bar{r}(s) = 1/s^2$$

$$\bar{v}(s) = \pi e^{-15s}/[6\{s^2 + (\pi/3)^2\}].$$

From Lemma 1 (I), Lemma 2 (I) and (IV), we choose

$$\bar{C}(s) = \pi/[6\{s^2 + (\pi/3)^2\}].$$

Given $T=0.3$ s and $\bar{\rho}=\rho=1$, Figs. 4 and 5 show the values of $J(N)_{\min}$ and $\tilde{J}(\tilde{N})_{\min}$ for various N 's and $\tilde{N}$'s, respectively. In this figure, we can see how much the transient input and output responses can be improved by increasing N and $\tilde{N}$. Figures 6 and 7 show the continuous-time responses of the control input and the controlled output, respectively, in the cases of $q(\lambda) = \tilde{q}(\lambda)=0$ and $(N, \tilde{N})=(2, 2)$ and $(5, 5)$. These figures show that the output tracks the reference input and eliminates the effect of the disturbance with no ripple in finite number of control steps, and the increase of N and $\tilde{N}$ makes the transient performance more moder-

Fig. 3 Reference input and disturbance

Fig. 4 Relationship between $J(N)_{\min}$ and N

Fig. 5 Relationship between $\tilde{J}(\tilde{N})_{\min}$ and $\tilde{N}$

(a) $q(z) = \tilde{q}(z) = 0$

(b) $N = \tilde{N} = 2$

(c) $N = \tilde{N} = 5$

Fig. 6 Control input responses

(a) $q(z) = \tilde{q}(z) = 0$

(b) $N = \tilde{N} = 2$

(c) $N = \tilde{N} = 5$

Fig. 7 Controlled output responses

ate.

5. Conclusions

We have presented a design procedure of the ripple-free deadbeat tracking control system with the optimal transient performance. The key observation is to make explicit the class of compensators which achieve a ripple-free deadbeat control system for the general type of reference input and disturbance, and to reduce the design problem to an unconstrained optimization problem. A simple iterative design algorithm is developed, and the proposed design procedure allows easy computation of the transient performance for the various settling steps.

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