Vibration Analysis of Continuous Bodies Using Analytic Solutions Discretized on Boundaries*

(A Basic Study on Two-Dimensional Problems Described by Helmholtz's Equation)

Yoshihiko URATA** and Toshiaki NAKAGAWA***

We discuss a new method of analyzing two-dimensional eigenvalue problems described by Helmholtz's equation. A general solution is assumed as a linear combination of plane waves which have the same wave number and travel in different directions. Nodal points are located only on the boundary of a convex domain which has an arbitrary shape. A set of equations similar to finite-element equations can be derived from the general solutions. Eigenvalues obtained by this method are very close to the exact values. This method is effective in simplifying of the programming and in shortening computational time.

Key Words: Vibration of Continuous System, Helmholtz's Equation, Eigenvalue, Analytic Solution, Plane Wave, Nodal Point, Accuracy

1. Introduction

Although problems of the mechanics of continuous bodies are generally described by partial differential equations, it is difficult to find exact solutions for arbitrarily shaped domains of two and three dimensions. Thus the finite-element method (FEM) and the boundary element method (BEM) are widely used for practical computation. These methods are powerful and suitable for general use. However, they often yield results with appreciable errors and they require a long time for preparation and computation. Therefore some alternative methods which are complementary to the FEM and the BEM are desirable. In our previous studies⁽¹⁾⁻⁽⁴⁾, a method of direct combination of analytic solutions and the usual finite ele-

ments was proposed. In this method, the domain to be analyzed is divided into several subdomains. For simply shaped subdomains, analytic solutions that satisfy the governing equation and the boundary conditions, except the conditions of continuity on the subdomain borders, may be found easily. These solutions are transformed into discretized relations between the potential and the concentrated flux at nodal points located on the subdomain borders. The resulting relations are similar to the FEM equations. Therefore the equation of the whole domain can be constructed by a procedure similar to the FEM. This method is effective for improving the accuracy and shortening computational time. However, it is limited to a few problems, because it is based on solutions of the kinds of variable separation. Therefore, a new method for problems described by Helmholtz's differential equation of two dimensions is discussed in this paper. In the present method, it is supposed that a general solution of Helmholtz's equation may be a linear combination of particular solutions that express plane waves traveling in all directions. The general solution is discretized on boundaries in the same way as the previous studies. This method is

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** Department of Mechanical Engineering, Shizuoka Institute of Science and Technology, Toyosawa 2200-2, Fukuroi, Shizuoka 437, Japan

*** Toyota Central R&D Laboratories, Inc., Nagakute-cho, Aichi-gun, Aichi 480-11, Japan

applicable to convex domains with arbitrary shapes and it yields precise results.

It may be expected that methods similar to this can be applied to problems of various kinds of linear partial differential equations.

2. Method of Analysis

2.1 General solution of Helmholtz's equation for single-connected domains

Let Ω and Γ be a domain and its boundary, respectively. Suppose Helmholtz's equation (1) holds in the domain Ω .

$$\nabla^2 \varphi + k^2 \varphi = 0 \quad (1)$$

The problem is to find the eigenfunctions φ and the eigenvalues k under some homogeneous boundary conditions given on Γ . The eigenfunctions φ must be nonsingular and unique in Ω . Let us suppose a circle contains Ω within it. Moreover, let us assume that the solutions which satisfy the boundary conditions on Γ are included in a class of solutions that are nonsingular and unique in the circle. A function in the class can be generally expressed in polar coordinates whose origin coincides with the center of the circle.

$$\varphi = \sum_{n=0}^{\infty} C_n J_n(kr) e^{jn\theta} \quad (2)$$

This is a Fourier series expansion of φ in terms of θ . Bessel functions of only the first kind appear in the coefficients of the series, because φ must not be singular at the origin. One of the integral representations of the Bessel function gives⁽⁵⁾

$$\begin{aligned} J_n(kr) e^{jn\theta} &= \frac{1}{2\pi} \int_0^{2\pi} e^{j\{n(\gamma+\theta) - kr \sin \gamma\}} d\gamma \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{j\{n(\gamma+\pi/2) - kr \cos(\gamma-\theta)\}} d\gamma. \end{aligned} \quad (3)$$

Substituting Eq. (3) into Eq. (2), we obtain

$$\varphi = \int_0^{2\pi} \left\{ \frac{1}{2\pi} \sum_{n=0}^{\infty} C_n e^{jn(\gamma+\pi/2)} \right\} e^{-jkr \cos(\gamma-\theta)} d\gamma. \quad (4)$$

The expression in the braces in Eq. (4) is an unknown periodic function of γ with the fundamental period 2π . Let this function be denoted by $D(\gamma)$. Therefore Eq. (4) can be rewritten as

$$\varphi = \int_0^{2\pi} D(\gamma) e^{-jkr \cos(\gamma-\theta)} d\gamma. \quad (5)$$

Equations (2) and (5) are equivalent to each other.

Fig. 1 Simply connected domain

Both the equations are the most general expressions of the solutions of Eq. (1) in the neighborhood of the origin. These include variable separation solutions. Every term of the series in Eq. (2) can be derived directly by the method of separation of variables in the polar coordinates. On the other hand, it is not obvious that Eq. (5) includes variable separation solutions in Cartesian coordinates. For example, we take the weighting function $D(\gamma)$ as

$$\begin{aligned} D(\gamma) &= \frac{1}{4} \{ \delta(\gamma - \alpha) + \delta(\gamma - \alpha + \pi) + \delta(\gamma - \alpha - \pi) \\ &\quad + \delta(\gamma - \alpha + 2\pi) \}, \end{aligned}$$

where δ denotes Dirac's delta function. Substitution of this weighting function into Eq. (5) leads to

$$\varphi = \cos(kx \cos \alpha) \cos(ky \sin \alpha).$$

2.2 Meanings of the general solution

The meaning of the integrand in Eq. (5) is as follows. Figure 2 shows the xy -coordinates and the $\xi\eta$ -coordinates. The ξ -axis is directed to the positive angle γ from the x -axis. A plane wave traveling in the ξ -direction with the wave number k may be expressed as

$$\varphi = e^{-jk\xi}.$$

The phase of this wave at point A (r, θ) is the same as the phase at point B on the ξ -axis. Then we obtain

$$\varphi(r, \theta) = e^{-jkr \cos(\gamma-\theta)}.$$

This is contained in the integrand in Eq. (5). Thus Eq. (5) shows that φ at any point is a superposition of the plane waves traveling in all directions. As mentioned before, Eqs. (2) and (5) are equivalent. Therefore either of the equations can be used in approximate numerical computation. However, Eq. (5) yields a simpler scheme of computation if a polygonal approximation of the domain is adopted.

A circular wave emitted from the point source located at the origin may be written as

$$\varphi = AH_0^{(2)}(kr),$$

where the time factor $e^{j\omega t}$ is omitted. $H_0^{(2)}$ denotes Hankel's function of the second kind and order 0. If $kr \gg 1$, the above equation asymptotically becomes

$$\varphi = \left\{ A \sqrt{\frac{2}{\pi kr}} e^{-j(kR - \pi/4)} \right\} e^{-jk\xi}.$$

Fig. 2 Plane wave propagating in the ξ -direction

In derivation of this equation, the approximation $r = R + \xi \approx R$, where R is a constant much larger than ξ , is used. The above equation expresses a plane wave whose amplitude and phase are determined by the coefficient. The equation stated here means that a plane wave may be regarded as an approximation of the wave from a very far point source. Therefore, in the approximation method based on Eq. (5), the intensity and the phase of point sources distributed at infinity in all directions should be determined such that the solution satisfies the boundary condition given on Γ . This situation is different from the BEM in which point sources are put on the boundaries. Thus no singular integral is necessary in the present method.

2.3 Cases where Eq. (5) is not applicable

As stated above, Eq. (5) is a superposition of plane waves which express far fields. Therefore, Eq. (5) is restricted to the domains that are described as far fields. For a ring-shaped domain between two concentric circular boundaries, a solution of Eq. (1) must be written in the following form.

$$\varphi = \{AJ_n(kr) + BY_n(kr)\}e^{jn\theta}$$

The second term in the braces, the Bessel function of the second kind, is singular at the origin. This term is necessary to express reflected wave from the inner boundary. Therefore, the ring-shaped domain cannot be a far field. For the same reason, multiply connected domains can generally not be far fields. Thus only simply connected domains will be discussed in this paper.

For similar reasons, Eq. (5) cannot be applied to a simply connected but deeply concave domain. If the boundary condition on the linear edges of the domain shown in Fig. 3 is given as $\varphi=0$, the solution must be

$$\varphi = \{AJ_\nu(kr) + BY_\nu(kr)\}\sin\nu\theta, \quad \left. \begin{array}{l} \nu = \frac{n\pi}{\alpha} \end{array} \right\}$$

This is almost the same as the solutions for ring domains.

If numerical integral methods are available, approximation schemes similar to that proposed in this paper can be applied to multiply connected and

Fig. 3 Simply connected but deeply concave domain

concave domains. However, this will be discussed in another study.

2.4 Method of approximation

We deal with only simply connected and convex domains for the reason mentioned so far. First Eq. (5), represented by the integral, is replaced by the following discrete summation.

$$\varphi = \sum_{i=1}^N D_i \varphi_i \quad (6)$$

$$\varphi_i = e^{jkr \cos(\tau_i - \theta)} \quad (7)$$

This is an approximate general solution that is a discrete superposition of the plane waves possessing the same wave number and traveling in different directions. Equation (6) should include the plane waves traveling in all directions. Furthermore, Eq. (6) should contain the pair of waves that travel in opposite directions, because no energy flow in any direction exists in any eigenvalue problems. Therefore the number of terms of the series in Eq. (6), N , should be an even number. The various methods of approximation starting from Eq. (6) are possible, but a similar method to that in our previous studies is discussed.

The domain Ω is replaced by an approximate N -sided polygon. N nodal points are put at the vertices of the polygon. If some nodal points are put on the long side of the polygon, they may be regarded as the vertices angles are π . Let P_n denote the n -th nodal point at the position (x_n, y_n) . We assume that the normal component of the flux, namely $-\partial\varphi/\partial n$, can be concentrated at the nodal points as the volumetric flux Q_n . Then

$$-\sum_{i=1}^N \left[\frac{\partial \varphi_i}{\partial n} \right]_r = \sum_{n=1}^N Q_n \delta(x - x_n, y - y_n), \quad (8)$$

where δ is Dirac's delta function. Both sides of this equation are multiplied by φ_m ($m=1, 2, \dots, N$) and then integrated over the whole boundary Γ . This is a technique of the weighted residual method. Thus we obtain

$$[h_{mi}]\{D_i\} = [g_{mn}]\{Q_n\}, \quad (9)$$

where

$$\left. \begin{array}{l} h_{mi} = - \int_{\Gamma} \frac{\partial \varphi_i}{\partial n} \varphi_m d\Gamma, \\ g_{mn} = \varphi_m(x_n, y_n). \end{array} \right\} \quad (10)$$

The elements of these matrices are described in detail later.

Let Φ_l be the value of φ at the l -th nodal point. From Eq. (6), we obtain

$$\Phi_l = \sum_{i=1}^N D_i \varphi_i(x_l, y_l) = \sum_{i=1}^N D_i g_{il}.$$

This can be rewritten as

$$\{\Phi_l\} = [g_{il}]^T \{D_i\}. \quad (11)$$

The discrete representation corresponding to the boundary condition $\varphi=0$ over all Γ is $\{\Phi_l\}=\{0\}$.

Substitution of this into Eq. (11) leads to the following eigenvalue equation.

$$[g_{il}]^T = [g_{il}] = 0 \quad (12)$$

The same result as this can be derived by applying the method of collocation to Eq. (6). Although it is generally considered that the method of collocation often gives inaccurate results, there is no problem in this case, as shown later.

The discrete representation of the boundary condition $\partial\varphi/\partial n=0$ over all Γ is $\{Q_n\}=0$. When this condition is applied, Eq. (9) changes to

$$[g_{mn}]^{-1}[h_{mi}]\{D_i\}=\{0\}. \quad (13)$$

From this equation, the eigenvalue equation is obtained as follows.

$$|h_{mi}|/|g_{mn}|=0 \quad (14)$$

According to the definition of h_{mi} in Eq. (10), the equation $|h_{mi}|=0$ corresponds to the conditions of no energy flow across the boundary, namely $\varphi=0$ and $\partial\varphi/\partial n=0$. The factor $1/|g_{mn}|$ in Eq. (14) plays the role of removing k 's corresponding to $\varphi=0$ from the zeros of Eq. (14). In some problems, the same values of k become the eigenvalue under both boundary conditions $\varphi=0$ and $\partial\varphi/\partial n=0$. The determinant $|h_{mi}|$ behaves as zeros of the second order at such values of k , but the left-hand side of Eq. (14) behaves as zeros of the first order because of the presence of the factor $1/|g_{mn}|$.

For two special boundary conditions, i. e., $\varphi=0$ and $\partial\varphi/\partial n=0$, it is recommended to use the eigenvalue equations (12) and (14). For other cases, the unknown coefficient vector $\{D_i\}$ is eliminated from Eqs. (9) and (11). Then the following equation is obtained.

$$\{Q_n\}=[g_{mn}]^{-1}[h_{mi}][g_{il}]^{-1}\{\Phi_i\} \quad (15)$$

This equation has a similar form to a large finite element. Therefore this can be combined to both the usual finite elements and equations similar to Eq. (15) for the neighboring domains.

2.5 Details of elements of the matrices

The elements of the matrices $[g_{mn}]$ and $[h_{mi}]$ should be obtained. The positions of the nodal points is expressed in Cartesian coordinates. Then Eq. (7)

Fig. 4 Notations of nodal points on sides of a polygon

is rewritten as

$$\varphi_m = \exp\{-jk(x \cos \gamma_m + y \sin \gamma_m)\}.$$

By use this equation and the second equality of Eq. (10), the following is obtained.

$$g_{mn} = \exp\{-jk(x_n \cos \gamma_m + y_n \sin \gamma_m)\} \quad (16)$$

On the other hand, the procedure for obtaining h_{mn} is as follows. Figure 4 shows the configuration of the nodal points. Let the intersection of the side $P_n P_{n+1}$ (or its extension) and the normal line of this side passing through the origin be $R_n(\bar{x}_n, \bar{y}_n)$. The coordinates of the intersection R_n are the following.

$$\left. \begin{aligned} \bar{x}_n &= -\frac{(x_{n+1}y_n - y_{n+1}x_n)(y_{n+1} - y_n)}{(x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2} \\ \bar{y}_n &= \frac{(x_{n+1}y_n - y_{n+1}x_n)(x_{n+1} - x_n)}{(x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2} \end{aligned} \right\} \quad (17)$$

Then the angle β_n between the x -axis and the normal line can be expressed in terms of $\bar{x}_n$ and $\bar{y}_n$.

$$\beta_n = \cos^{-1} \frac{\bar{x}_n}{\sqrt{\bar{x}_n^2 + \bar{y}_n^2}} \quad (18)$$

The angle β_n should be converted to a value in the region from 0 to 2π according to the quadrant where the intersection R_n exists. $H_{mi}^{(n)}$ is defined by the integral over the side $P_n P_{n+1}$.

$$H_{mi}^{(n)} = - \int_{P_n}^{P_{n+1}} \frac{\partial \varphi_i}{\partial n} \varphi_m d\Gamma \quad (19)$$

By use of Eq. (16), this integral is represented as

$$H_{mi}^{(n)} = - \frac{\cos(\gamma_i - \beta_n)}{\sin(\gamma_i - \beta_n) + \sin(\gamma_m - \beta_n)} (g_{m(n+1)} g_{i(n+1)} - g_{mn} g_{in}). \quad (20)$$

However, if $\sin(\gamma_i - \beta_n) + \sin(\gamma_m - \beta_n) = 0$, the following must be used instead of Eq. (20).

$$H_{mi}^{(n)} = jk \{ -(x_{n+1} - x_n) \sin \beta_n + (y_{n+1} - y_n) \cos \beta_n \} \times \exp\{-jk(x_n \cos \beta_n + y_n \sin \beta_n)[\cos(\gamma_i - \beta_n) + \cos(\gamma_m - \beta_n)]\} \quad (21)$$

Then h_{mi} is obtained by the following summation.

$$h_{mi} = \sum_{n=1}^N H_{mi}^{(n)} \quad (22)$$

The matrix $[h_{mi}]$ is a symmetric one, namely,

$$h_{mi} = h_{im}. \quad (23)$$

This relation can be derived by use of Green's theorem and the fact that φ_m and φ_i satisfy Eq. (1). The product of the three matrices in Eq. (15) also becomes a symmetric matrix.

As stated in section 2.4, Eq. (6) should include the pair of plane waves traveling in opposite directions. This enables representations in terms of real numbers. For this purpose, the following is used instead of Eqs. (6) and (7):

$$\varphi = \sum_{i=1}^N \{A_i \cos[kr(\gamma_i - \theta)] + B_i \sin[kr(\gamma_i - \theta)]\}, \quad (24)$$

where $0 \leq \gamma_i \leq \pi$. The matrices $[g]$ and $[h]$ can be obtained in a similar procedure as explained so far. Programming for numerical computation in the real representations becomes slightly more complicated

than in the complex representation. If the numbers of cos terms and sin terms in Eq. (24) are taken as different from each other, an odd number of the nodal points becomes possible.

2.6 Notes for numerical computation

Equation (15) has a form similar to a finite element. Thus this can be used like a finite element. However, there are some exceptions. Eigenvalues of a rectangle under the condition $\varphi=0$ on the whole boundary coincide with those under the condition $\partial\varphi/\partial n=0$, except ones corresponding to the modes satisfying $\partial\varphi/\partial x=0$ or $\partial\varphi/\partial y=0$ under the latter boundary condition. Thus the eigenvalue equations derived by substituting $\{\Phi_i\}=\{0\}$ and $\{Q_n\}=\{0\}$ into Eq. (15) are meaningless, because it is required that the solutions of $|g_{mi}|^2/|h_{mi}|=0$ and of $|h_{mi}|/|g_{mi}|^2=0$ be the same. For such cases, Eq. (12) or (14) should be used as the eigenvalue equation. The phenomena stated here may occur not only in the present method, but also in every method which contains the relations of the potential and the concentrated flux at the nodal points located only on the boundary. Under the whole boundary condition $\varphi=0$ or $\partial\varphi/\partial n=0$, Eq. (12) or (14) should be used for not only rectangles but also other shapes for the sake of efficiency.

The method of determinant search must be used in obtaining eigenvalues in the present method. In a case of complex representation, the value of the determinant becomes a complex number. However, only one of its real or imaginary parts is meaningful, while the other appears as a result of a computational error. Which of the real and imaginary parts has meaning is determined by such factors as the number of nodal points and their configuration. Comparisons of the real and complex computations show that there is no significant difference in accuracy between the results of the two representations. However, it can be said that the real representation is slightly better in terms of the stability of numerical computation than the complex representation.

The eigenvalue equations other than Eq. (12) have not only zeros but also singular points. The bisection method is used in this study. If the absolute value of the determinant increases in the loop of the bisection, the determinant remains around a singular point.

The origin of the coordinates to produce the data of nodal points can be put outside of the domain as well as inside. However, putting the origin on the boundary should be avoided, because the matrices may become singular.

3. Numerical Results and Their Examination

All results shown in this section, except those

obtained by the FEM, are obtained by double-precision Fortran on a personal computer.

For the purpose of examining the fundamental properties of the proposed method, the eigenvalues of a rectangle and a half-circle are calculated and compared with the exact values. Figure 5 shows the domains and the configurations of the nodal points. Tables 1 to 5 show the calculated results. The names of the modes of the rectangle indicate the numbers of the nodal lines of φ parallel to the x - and y -axes. On the other hand, the mode names of the half-circle are the numbers of the nodal diameters and the nodal circles of the corresponding full-circle domain.

Tables 1 and 2 show the results of the rectangle subjected to the conditions $\varphi=0$ and $\partial\varphi/\partial n=0$ on the whole boundary, respectively. Degenerated eigenvalues are not obtained, because the method of simple bisection is used. The results obtained by the proposed method are very close to the exact values in spite of the small number of nodal points. In Tables 1 and 2, the results obtained by the FEM are also shown. In the FEM computation, rectangular elements are used. The division of the domain into the

Fig. 5 Domains and nodal point configuration that are analyzed

Table 1 Eigenvalues of the rectangle ($\varphi=0$)

Mode	Exact	Present Method (Mode Error)	FEM
(0, 0)	1. 88786	1. 88786 (7. 04E-07)	1. 91256
(1, 0)	2. 61799	2. 61799 (7. 41E-06)	2. 67152
(0, 1)	3. 31152	3. 31152 (3. 47E-04)	3. 58349
(2, 0)	3. 51240	3. 51241 (2. 23E-04)	3. 73463
(1, 1)	3. 77572	3. 77581 (1. 69E-03)	3. 94968
(2, 1)	4. 44288	4. 44196 (1. 59E-02)	4. 58257
(3, 0)	4. 47362	4. 47344 (2. 92E-03)	5. 02974
(0, 2)	4. 82734	4. 82802 (6. 21E-03)	5. 66307
(1, 2)	5. 15684	5. 16050 (7. 40E-02)	5. 77237
(3, 1)	5. 23598	5. 23536 (9. 47E-03)	5. 47723
(4, 0)	5. 46653	5. 46753 (4. 71E-03)	6. 32089
(2, 2)	5. 66358	5. 66810 (2. 84E-02)	5. 98053

Table 2 Eigenvalues of the rectangle ($\partial\varphi/\partial n=0$)

Mode	Exact	Present Method	FEM
(1, 0)	1. 04720	1. 04720	1. 05920
(0, 1)	1. 57080	1. 57080	1. 61142
(1, 1)	1. 88786	1. 88786	1. 91256
(2, 0)	2. 09440	2. 09440	2. 19089
(2, 1)	2. 61799	2. 61799	2. 67152
(0, 2)	3. 14159	-	3. 46410
(3, 0)	3. 14159	-	3. 46410
(1, 2)	3. 31153	3. 31153	3. 58349
(3, 1)	3. 51241	3. 51244	3. 73463
(2, 2)	3. 77572	3. 77580 (singular)	3. 94968
(4, 0)	4. 18879	4. 18879	4. 89898
(3, 2)	4. 44288	4. 44333	4. 58257

elements is shown by the dashed lines in Fig. 5 (a). The present method yields much more accurate results than the FEM. As mentioned before, the present method coincides with the simple collocation method if the whole boundary condition is $\varphi=0$. However, there is no problem in accuracy.

The values of the mode error are also shown in Table 1. The errors are estimated as follows. Let φ and φ_0 be the approximate solution by the present method and the exact solution, respectively. They are normalized as

$$\int_{\Omega} \varphi^2 d\Omega = 1, \quad \int_{\Omega} \varphi_0^2 d\Omega = 1.$$

Then the error is defined as

$$\varepsilon = \sqrt{\int_{\Omega} (\varphi - \varphi_0)^2 d\Omega}.$$

Even the errors of the higher modes remain very small values.

The eigenvalue of the mode (2, 2) in Table 2 is not obtained, but it is captured as a singular point. This is caused by disagreement of the second-order zero of $|h_{mi}|$ and the pole of $1/|g_{mn}|$. This suggests that there is a limit to the approximation. However, the limiting wave number becomes large with the number of nodal

Table 3 Eigenvalues of the rectangle (Mixed boundary condition)

Mode	Exact	Present
(0, 0)	0. 94393	0. 94414
(1, 0)	1. 75620	1. 75451
(0, 1)	2. 41367	2. 41333
(2, 0)	2. 73327	2. 73023
(1, 1)	2. 83179	2. 83002
(2, 1)	3. 52215	3. 51998
(3, 0)	3. 74840	3. 74632
	-	3. 77569 (pseudo)
(0, 2)	3. 96174	3. 96286
(1, 2)	4. 22950	4. 22653
(3, 1)	4. 35721	4. 35381

Table 4 Eigenvalues of the half-circle ($\varphi=0$)

Mode	Exact	Present
(1, 0)	3. 83171	3. 83171
(2, 0)	5. 13562	5. 13562
(3, 0)	6. 38016	6. 38016
(1, 1)	7. 01559	7. 01559
(4, 0)	7. 58834	7. 58835
(2, 1)	8. 41724	8. 41724
(5, 0)	8. 77148	8. 77147
(3, 1)	9. 76102	9. 76137
(6, 0)	9. 9361	9. 9429
(1, 3)	10. 1735	10. 1736
(4, 1)	11. 0647	11. 2350
(2, 3)	11. 6198	11. 6205

points. The eigenvalues below the limit in Table 2 are more accurate than those in Table 1, because of the effect of the integration.

Table 3 shows the results for the rectangle subjected to the mixed boundary condition. The boundary condition is shown in Fig. 5(b). The conditions $\varphi=0$ and $\partial\varphi/\partial n=0$ are given on the hatched sides and on the others, respectively. The eigenvalue equation can be derived from Eq. (15). Although the errors in this case are slightly larger than those in Tables 1 and 2, there is no problem for practical use. However, a pseudosolution is obtained in the higher-wave-number region. The cause of this pseudosolution is not clarified, but this gives another limit of the approximation.

Tables 4 and 5 show the results for the half-circle under the boundary conditions $\varphi=0$ and $\partial\varphi/\partial n=0$, respectively. The errors of the eigenvalues in Table 5 are slightly larger than in the other cases. These errors are brought about because the integration is carried out on the polygonal sides instead of the half-circle. On the other hand, the results in Table 4 have good accuracy. In the case of Table 4, the integral is not concerned with the derivation of the eigenvalue equation, and thus the influence of the replacement of the half-circle by the polygon does not

Table 5 Eigenvalues of the half-circle ($\partial\varphi/\partial n=0$)

Mode	Exact	Present
(1, 0)	1.84118	1.85643
(2, 0)	3.05424	3.07951
(3, 0)	4.20119	4.23594
(4, 0)	5.31755	5.36149
(1, 1)	5.33142	5.37562
(5, 0)	6.41562	6.43312
(2, 1)	6.70613	6.76169
(6, 0)	7.50127	7.65133
(3, 1)	8.01523	8.08163
(1, 2)	8.53631	8.84296

appear directly. If we use the expansion of plane waves by Bessel functions⁽⁵⁾, the integral can be carried out on circular edges. However, this makes the analysis more complicated. It is considered that the polygonal approximation is sufficient for the usual practical use, except cases in which very high accuracy is required.

As mentioned so far, the approximation method starting from Eq. (6) yields accurate results. The cause of this is considered to be that Eq. (6) is a partial sequence of the exact solutions for convex domains.

4. Conclusion

In this study, a new approximation method of analysis of the eigenvalue problems described by Helmholtz's differential equation for arbitrarily shaped and simply connected two-dimensional domains was examined. This method is based on an approximate general solution which is expressed as a superposition of plane waves traveling in every direction. The approximate general solution is discretized on boundaries. Because the eigenvalue equation includes the eigenvalue implicitly, the method of determinant search should be used in numerical computations. This method requires shorter computational time because only small matrices appear. Furthermore, the present method gives very accurate results, because it is based on the analytical solutions. Programming can be done easily. The

method is applicable for various problems merely by replacing the data of nodal point configuration. A subdomain discretized by this method can be combined to the neighboring subdomains discretized by either this method or the FEM. Thus it can be said that this method is flexible. However, further studies are necessary to clarify the details of the properties of the present method.

It should be noted that methods similar to the present one may be successful for problems described by other kinds of partial differential equations. This will be studied later.

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