Vibration Analysis of Rubber Vibration Isolators of Vehicle Using the Restoring Force Model of Power Function Type*  
(Analysis of Nonlinear Vibration Using Frequency Characteristics Determined by the Hysteresis Loop) 

Kazuhito MISAJI**, Shigeo HIROSE*** and Koichi SHIBATA****

We have already proposed a method for analyzing the nonlinear vibration response properties of rubber in relation to frequency and amplitude of displacement (Ref.(3)). That method determined the predominant frequency of rubber itself by using the FFT (Fast Fourier Transform). In this paper, we describe a new method for analyzing nonlinear vibration response. This method determines the predominant frequency of relative displacement of rubber itself by using a period on each one of the hysteresis loops. We were able to verify the appropriateness and accuracy of this method by directly comparing the analytical results with the experimental results for the dynamic response of a system. This new method compares favorably with the method of Ref.(3) in terms of the accuracy of calculation. We consider that our method may be applicable to a multiple-degree-of-freedom system.

** Key Words**: Rubber, Nonlinear Vibration Response, Hysteresis Characteristics, Vehicle, Random Vibration

1. Introduction

The origin of road noise in a car is shown in Fig. 1. Vibration of the suspension influences dynamic behavior of the vibration system comprised of a suspension arm with mass and suspension rubber bushes with a spring. Consequently it is very important to reduce road noise. However, the behavior of the suspension is difficult to determine theoretically in advance, because dynamic characteristics of the suspension bush with rubber are changed by the influence of the frequency and amplitude of displacement by external factors. We have already proposed a method for analyzing the nonlinear vibration characteristics of rubber, using experimental results from dynamic hysteresis characteristics tests. These loading tests involve stepwise in velocity (frequency) achieved by the use of a practical suspension bush. This method was developed based on an equivalent linear system using a restoring force model of power function type. As a result of our studies, we were able to determine the vibration characteristics of rubber (dynamic spring constant and damping factor) in relation to frequency and amplitude of displacement (Refs.(1) and (2)). We have also proposed a method (3) for analyzing vibration response properties of a system with one degree of freedom and modeling the rubber as a spring, using the result obtained by the above method. In the method proposed in Ref.(3) we determined the predominant frequency of the relative displacement response of rubber by using the FFT

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** Honda R&D Co., Ltd., Tochigi R&D Center, 4630 Shimotakanazawa, Haga-machi, Hagagun, Tochigi 321-33, Japan
*** Graduate Student, Department of Mathematical Engineering, College of Industrial Technology, Nihon University, 1-2-1 Izumi-cho, Narashino-shi, Chiba 275, Japan
**** Department of Mathematical Engineering, College of Industrial Technology, Nihon University, 1-2-1 Izumi-cho, Narashino-shi, Chiba 275, Japan

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(Fast Fourier Transform). Consequently, for an existing system with many predominant frequencies or a multiple-degree-of-freedom system, this method has difficulty in obtaining sufficient calculation accuracy. In this paper, we describe a new method for analyzing nonlinear vibration response. The new method does not use FFT, but determines the predominant frequency of the relative displacement of rubber by using the hysteresis loop. We verify the appropriateness and accuracy of this new method by directly comparing its results with those of experiment. For the vibration system with the hysteretic restoring force characteristics, there have been studies on nonlinear vibration analysis of structure. However, our studies are the first ones analyzing the vibration response characteristics of rubber vibration isolators of a vehicle considering the nonlinear vibration characteristics of rubber in relation to both frequency and amplitude of displacement.

2. Analytical Method

2.1 Analytical method for equivalent linear system using the restoring force model of power function type (This analysis method is first described in Ref. (1).)

When the one-degree-of-freedom system is acted upon by an external force $p_0 \cos \omega t$, the equation of motion can be expressed as follows:

$$m \ddot{x} + f(x, \omega) = p_0 \cos \omega t$$  \hspace{1cm} (1)

in which $m$ is mass; $x$ is amplitude of displacement; $p_0$ is amplitude of external force; $\omega$ is circular frequency of external force and $f(x, \omega)$ is the hysteresis characteristic of rubber as a function of $x$ and $\omega$.

We introduce the following dimensionless parameters:

- $X = x/\bar{x}$  \hspace{1cm} (2.a)
- $X_0 = x_0/\bar{x}$  \hspace{1cm} (2.b)
- $\omega^2 = F_0/(x_0 \cdot m)$  \hspace{1cm} (2.c)
- $\eta = \omega/\omega_s$  \hspace{1cm} (2.d)
- $\tau = \omega / \omega_d$  \hspace{1cm} (2.e)
- $P_0 = p_0/F_0$  \hspace{1cm} (2.f)

and

$$k_s = F_0/x_0$$  \hspace{1cm} (2.g)

in which $x_0$ is the displacement of the linear region, $F_0$ is the load of the linear region, $\omega_s$ is the circular frequency of the linear region and $x_0$ is the amplitude of displacement. Equation (1) can be transformed to

$$\frac{d^2X}{dt^2} + F(X, \eta) = P_0 \cos \eta t$$  \hspace{1cm} (3)

in which $F(X, \eta)$ is dimensionless restoring force. The following equation is obtained by replacing the hysteretic vibration system of Eq. (3) with the equivalent linear system.

$$\frac{d^2X}{dt^2} + 2\omega_0 \frac{dX}{dt} + K_{eq}X = F_0 \cos \eta t$$  \hspace{1cm} (4)

$H_{eq}$ and $K_{eq}$ are expressed as the following equations.

$$H_{eq} = \frac{1}{2 \pi \eta}\left(\frac{1}{X_0}\right)^2 \int_0^{X_0} F(X) R(X, \eta) dX$$  \hspace{1cm} (5)

$$K_{eq} = \frac{2}{\pi} \left(\frac{1}{X_0}\right)^2 \int_0^{X_0} F(X) (X/X_0) dX$$  \hspace{1cm} (6)

in which $G(X_0, \eta)$ is the area of the hysteresis loop expressed as a function of the amplitude of displacement $X_0$ and frequency $\eta$; $P(X) = X/\sqrt{X^2 - X_0^2}$; $R(X, \eta)$ is added in ascending and descending branches of the loop.

The fundamental model of the hysteresis loop for calculating $G(X_0, \eta)$ and $R(X_0, \eta)$ is applied to the restoring force model of power function type that is analogous to the form of the hysteresis loop of the rubber isolator. The bone curve is

$$F(X, \eta) = kX^\alpha$$  \hspace{1cm} (7)

and the ascending and descending branches of the hysteresis loop are described by

$$F(X, \eta) = \pm 2k \left[ \frac{1}{2} (X_0 \pm X) \right]^{\alpha} + kX_0^\alpha$$  \hspace{1cm} (8)

In the following equations $H_{eq}$ and $K_{eq}$ are obtained by using Eqs. (5), (6), (7) and (8).

$$H_{eq} = \frac{4k}{\pi} \left(\frac{1}{1 + \alpha}\right)^{1/\alpha} X_0$$  \hspace{1cm} (9)

$$K_{eq} = \frac{2k}{\pi} \frac{1}{1 + \alpha} X_0^{-\alpha-1}$$  \hspace{1cm} (10)

$a$ and $k$ are given by

$$a(X_0, \eta) = \frac{4F_0(X_0, \eta) - G_0(X_0, \eta)}{4F_0(X_0, \eta) + G_0(X_0, \eta)}$$  \hspace{1cm} (11)

$$k(X_0, \eta) = \frac{F_0(X_0, \eta)}{X_0}$$  \hspace{1cm} (12)

in which $G_0(X_0, \eta)$, $F_0(X_0, \eta)$ were expressed as functions of the amplitude $X_0$ and frequency $\eta$, using the hysteresis loop obtained from the result of the loading test. $a$ and $k$ vary with depending on the frequency $\eta$ and amplitude of displacement $X_0$. Thus, the damping and spring characteristics also vary. $G_0$ and $F_0$ are expressed as functions of $X_0$ and $\eta$, and the hysteresis loop is replaced by the restoring force model of power function type (Eqs. (7) and (8)). $H_{eq}$ and $K_{eq}$ are determined from Eqs. (9) and (10). We are able to calculate $K$ and $C$ in terms of frequency and amplitude of displacement using the following equations.

$$K(x_0, \omega) = k_s k_{eq}$$  \hspace{1cm} (13)

$$C(x_0, \omega) = \frac{2F_0}{\omega_0 x_0} H_{eq}$$  \hspace{1cm} (14)

2.2 Analytical response

Using $K$ and $C$ given by Eqs. (13) and (14), the equation of motion of the one-degree-of-freedom system in response to external force is

$$m \ddot{x} + K(x_0, \omega_0)x + C(x_0, \omega_0) \dot{x} = -m \ddot{y}$$  \hspace{1cm} (15)

in which $\ddot{y}$ is the acceleration due to external force, $x$
Table 1 K and C corresponding to $x_0$ and $\omega_0$

<table>
<thead>
<tr>
<th></th>
<th>Time $t$</th>
<th>$K$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \rightarrow B$ t_i &lt; $t$</td>
<td>$K(x_{n_{0}}, \omega_{0_{n_{0}}})$</td>
<td>$C(x_{n_{0}}, \omega_{0_{n_{0}}})$</td>
<td></td>
</tr>
<tr>
<td>$B \rightarrow C$ $t_i &lt; t &lt; t_{i_{1}}$</td>
<td>$K(x_{n_{0_{1}}}, \omega_{0_{n_{0_{1}}}})$</td>
<td>$C(x_{n_{0_{1}}}, \omega_{0_{n_{0_{1}}}})$</td>
<td></td>
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Fig. 2 Determination of $x_0$ and $\omega_0$

is the relative displacement of the mass, $m$, and $\omega_0$ is the predominant frequency of the relative displacement of the rubber itself. We are able to calculate the response of the system in relation to the amplitude and frequency of rubber by using the solution of Eq. (15). We carried out the calculation of response as follows.

(1) Definition of $x_0$ and $\omega_0$

We recognized a value of the abscissa when the direction of relative displacement (or the sign of relative velocity) changed as the amplitude of displacement $x_{n_{0}}$ (B in Fig. 2 and Table 1). Frequency $\omega_{n_{0}}$ is determined from the following equation using the time $\Delta t_i$ between point A and point B (see Fig. 2 and Table 1).

$$\omega_{n_{0}} = \frac{\pi}{\Delta t_i}$$

(16)

Even if there are many rubbers as spring element in a multiple-degree-of-freedom system, the frequency of relative displacement for individual rubber can decide to each that the period of the hysteresis loops of individual rubber turns down by using this method. Consequently, we consider this method is applicable to a multiple-degree-of-freedom system.

(2) Determination of the equivalent stiffness $K$ and damping factor $C$, and calculation of response

$K$ and $C$ are determined using $x_0$ and $\omega_0$ given in (1). Using $K$ and $C$, we carry out the calculation of response until the next turning point C (in Fig. 2 and Table 1), using the linear acceleration method. From the next point the same procedure is repeated.

3. Experiments

3.1 Experimental methods (Experimental methods are first described in Refs. (1) and (3)).

The structure and properties of the suspension bush used in this experiment are shown in Fig. 3 and Table 2. The load-displacement curves are obtained by experiment using the testing conditions in Table 3. For example, the load-displacement curves for the frequency of 45 Hz and direction PA are shown in Fig. 4. The dynamic spring constant and damping coefficient correspond to the amplitude of displacement shown in Figs. 5 and 6, by applying Eqs. (7) ~ (14) and using the load-displacement curves. The values shown by ● and ▲ are the dynamic spring properties of rubber obtained using a measuring device. Refer to Refs. (1) and (3) for other frequencies. The vibration test was conducted using a model with one degree of freedom (mass, $m=13$ kg), which is illustrated in Fig. 7. Refer to Ref. (3) for details of the experimental system. We carried out an experiment on vertical vibration, when the vibration
system is acted upon by a sinusoidal or random external force. Amplitude of the sine wave is 500 gal and frequency is 10 Hz, maximum amplitude of the random wave is 4000 gal, and frequency is less than or equal to 200 Hz. We measured the time history and power spectrum of the absolute response acceleration, relative velocity and relative displacement of a mass using the FFT.

3.2 Experimental results
3.2.1 Experiment on sinusoidal vibration
The absolute response acceleration, relative velocity and relative displacement of a concentrated mass obtained by the experiment on sinusoidal vibration are shown by the broken lines in Figs. 8, 9 and 10.
3.2.2 Experiment on random vibration

The absolute response acceleration, relative velocity and relative displacement of a concentrated mass obtained by the experiment on random vibration (the input acceleration of a random wave is shown in Fig.11 (power spectrum shown in Fig.12). The transfer function of the system is shown in Fig.13. The natural frequency of the test piece is 38 Hz.) are shown by the broken lines in Figs. 14, 15 and 16. The corresponding power spectra are shown by broken lines in Figs. 17, 18 and 19.

4. Analytical Results

4.1 Comparison of the analytical results with the experimental results for time history response

We calculated the absolute response acceleration, relative velocity and relative displacement of a concentrated mass using the method described in section 2. We can verify the appropriateness and accuracy of this method by directly comparing its results with those of experiment.

4.1.1 Analytical results of response of the mass in the case of sinusoidal vibration

The absolute response acceleration, relative velocity and relative displacement of a concentrated mass obtained by this method are shown by the solid lines in Figs. 8, 9 and 10. For the absolute response acceleration, the analytical values agree well with the experimental values. There is little difference between the analytical values
and the experimental values for the amplitude and phase, and we consider that the accuracy of calculation is sufficient.

4.1.2 Analytical results of response of the mass in the case of random vibration The absolute response acceleration, relative velocity and relative displacement of a concentrated mass obtained by this method are shown by the solid lines in Figs. 14, 15 and 16. The analytical results of the absolute response acceleration, relative velocity and relative displacement agree very well with the experimental results.

Fig. 16 Comparison between analytical and experimental results for relative displacement

Fig. 17 Comparison between analytical and experimental results for power spectrum of absolute response acceleration

Fig. 18 Comparison between analytical and experimental results for power spectrum of relative velocity

Fig. 19 Comparison between analytical and experimental results for power spectrum of relative displacement

Fig. 20 Comparison between analytical and experimental results for relative displacement (using method of Ref. (3))

Fig. 21 Comparison between analytical and experimental results for power spectrum of relative displacement (using method of Ref. (3))
(the broken lines). The error in the power spectrum of relative displacement calculated by the analysis method of Ref. (3) with respect to the experimental values is about 13% when the predominant frequency is 32.5 Hz. On the other hand, the error in the present method is about 0.9%. Consequently, we can conclude that this method compares favorably with the method of Ref. (3) which is determined by the predominant frequency using the FFT in terms of the accuracy of calculation. (Comparison of the relative displacement (the solid lines) obtained using the method of Ref. (3) with the experimental results (the broken lines) is shown in Fig. 20.)

4.2 Comparison of the analytical results with the experimental results for characteristics of frequency

The power spectra of absolute response acceleration, relative velocity and relative displacement of concentrated mass for random vibration are shown in Figs. 17, 18 and 19. The analytical results are shown by solid lines, and the experimental results by broken lines. It is evident from Figs. 19 and 21 that this method compares favorably with the method of Ref. (3) in terms of the accuracy of calculation. (Comparison of the power spectrum of the relative displacement (the solid lines) calculated by the method of Ref. (3) with the experimental results (the broken lines) is shown in Fig. 21.)

5. Conclusions

We have already proposed an analytical method for analyzing the nonlinear vibration response properties of rubber in relation to frequency and amplitude of displacement (Ref. (3)). The above method determined the predominant frequency of the rubber itself by using the FFT. This method has difficulty when applied to a multiple-degree-of-freedom system. In this paper, we described a new method for analyzing nonlinear vibration response. We think that this new method can be applied to a multiple-degree-of-freedom system, because it determines the predominant frequency of the relative displacement of rubber itself by using a period on each one of the hysteresis loops in relation to frequency and amplitude of displacement. We were able to verify the appropriateness and accuracy of this new method by directly comparing experimental results with the analytical results for the dynamic response of the system. After examining the results we arrived at the following conclusions.

(1) We were able to calculate the nonlinear vibration response properties of a system with one degree of freedom and model the rubber as a spring with high accuracy.

(2) This new method compares favorably with the method of Ref. (3) in terms of the accuracy of calculation.

(3) Although the example analyzed in this paper is a one-degree-of-freedom system, we think that this analysis method is applicable to a multi-degree-of-freedom system. We will apply this analysis method to a multi-degree-of-freedom system in future and will investigate the appropriateness and accuracy of this method by directly comparing its results with those of experiment.

References


