A New Approach to the Design of Digital Model Reference Control Systems*

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A new approach is proposed for the design of digital model reference control systems which overcomes the problem of nonminimum phase sampling zeros. The design is based on the plant-input mapping method which was proposed recently as a digital redesign method and which guarantees the internal stability of the system for almost any sampling frequency. This method is applied, in the present work, to the direct digital design of a pole placement control system and, in particular, a model reference control system. The key to eliminating the effect of sampling zeros on the performance of a model reference control system is the use of a matched-pole-zero model of the plant in combination with the invariant model, rather than the truncated step invariant model which is used conventionally. Simulation results are presented to show that the proposed design provides superior results to the conventional model reference controller design.

Key Words: Digital Control, Model Reference Control, Pole Placement Control, Plant-Input Mapping Method

1. Introduction

Three major approaches exist for the design of digital control systems. The first approach is based on the sampled-data representation of a combined discrete-time (DT) and continuous-time (CT) system (a hybrid system), which (i) expresses the DT portion in a CT context, giving rise to a periodically time-varying system⁽¹⁾, (ii) utilizes the lifting technique to convert the hybrid system into an infinite dimensional DT time-invariant system⁽²⁾, or (iii) expresses both the CT and DT portions in a combined state-space form⁽³⁾. Although the sampled-data approach is gaining popularity, the techniques used are more intricate than the second and third approaches which are described next. The second approach is the direct digital design method, where a DT model of the plant is obtained and then the digital controller is designed based on this model (See Fig. 1(a)). The third

approach is the digital redesign method, where an analog controller is designed and subsequently discretized to obtain a digital controller. Although simpler than the first approach, these two approaches have some disadvantages. For instance, the direct digital method does not completely take into account the CT behavior of the plant which is being controlled by the DT controller. The digital redesign method, on the other hand, usually requires an excessively fast sampling frequency to attain good control performance and, if the sampling frequency is not sufficiently high, even the system stability is not guaranteed. Recently, we proposed a simple and very effective digital redesign method called the plant-input mapping (PIM) method⁽⁴⁾. The PIM design consists of three key stages: the first is to calculate the transfer function from the reference input to the plant input (called the plant-input transfer function (PITF)), the second is to take the matched-pole-zero model of the PITF, and the third is to realize this DT PITF in a suitable manner. This method provides a digital control system (i) which guarantees internal stability of the system for almost any sampling interval and (ii) whose performance approaches that of the original

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Fig. 1 Controller design

analog controller as the sampling interval approaches zero. Simulation studies confirm that the PIM method gives superior performance to other methods. The PIM method is very general so that any proper, linear, time-invariant analog control system can be discretized and implemented by the digital controller. In this paper, this method is applied to the direct digital design of pole placement controllers and model reference control systems.

In the design of a model reference control system, the controller is chosen so that there will be no unstable cancellation of the plant zeros by the controller poles. While this can be achieved in the analog implementation by assuming the minimum phase property of the CT plant, various problems can arise in the digital implementation. In fact, with the digital redesign method, the CT controller often must be discretized using a very high sampling rate to attain acceptable performance. With the direct digital design method, the step invariant model of the CT plant is used, resulting in a nonminimum phase model when the relative degree of the CT plant is greater than two and a high sampling rate is used⁽⁵⁾. Various solutions have been presented for the problem of eliminating nonminimum-phase zeros (see Ref. (6) and references cited therein). In the context of model reference control, the approaches proposed recently in Refs. (7) and (8) use the step invariant model of the CT plant and reference model, and then ignore the zeros which move out of the stable region as the sampling frequency is increased. To identify the zeros which are to be ignored, the delta or Euler operator form^{(8),(9)} is used. A truncation error is introduced by ignoring some of the zeros, but the resulting DT system is still a valid DT model in view of the definition presented in Ref. (10). In the present paper, a better approach is proposed for this problem, using the PIM concept, in which a matched-pole-zero model of the plant is used when the minimum phase property is required in the design and an invariant model is

used when an accurate representation of the actual setup of the D/A-plant-A/D combination is required. Computer simulations are carried out to compare the performance of the model reference control systems designed with the conventional and PIM methods.

This paper is organized as follows. The concept of the PIM method is explained in section 2 and then applied to the direct digital design of the pole placement controller and model reference control systems in section 3. Simulation results are presented in section 4 for the case of digital model reference control, followed by concluding remarks in section 5.

2. Concept of PIM

Let the CT plant be modeled as

$$\bar{G}(s) \equiv \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{\bar{n}(s)}{\bar{d}(s)}, \quad (1)$$

where $\bar{u}(s)$ is the plant input, $\bar{y}(s)$ is the plant output, and $\bar{n}(s)$ and $\bar{d}(s)$ are coprime polynomials. It is assumed that $\partial[\bar{d}(s)] = n \geq \partial[\bar{n}(s)] = m$, where ∂ denotes the degree of its polynomial argument. The CT PITF is defined as

$$\bar{M}(s) \equiv \frac{\bar{u}(s)}{\bar{r}(s)}, \quad (2)$$

where $\bar{r}(s)$ is the reference input. In order for the control input to be a stabilizing signal to the plant, the zeros of the control input must contain all the unstable poles, if any, of the plant⁽⁴⁾; that is, all the unstable plant poles must be cancelled by the zeros of the control signal. While this can be achieved easily in the CT domain, a problem can arise in the DT domain. This is because the manner in which the CT poles are mapped to the DT poles is usually different from that in which the CT zeros are mapped to the DT zeros. Therefore, the zeros of control input should be mapped in the same manner as the poles of the plant transfer function to assure stability in the DT domain. For this purpose, the matched-pole-zero discretization schemes are ideal⁽¹¹⁾. Let one of the matched-pole-zero models of $\bar{M}(s)$ be denoted as $M(\epsilon)$, where ϵ is

the Euler operator defined as $(z-1)/T$, z is the z operator, and T is the sampling interval. Also, let $r(\epsilon)$ be a discretization of $\bar{r}(s)$. Then, the corresponding output of $M(\epsilon)$ will be $u(\epsilon)$, the discretization of $\bar{u}(s)$. The control input to the CT plant is generated by applying $u(\epsilon)$ to a suitable hold element. It is shown in Ref. (4) that this control input guarantees the stability of the combined CT and DT system for any nonpathological sampling interval T with respect to the plant; that is, none of the points $\sigma + jw_s l$, where $l \neq 0$, $w_s = 2\pi/T$ and σ is a nonnegative real part of the CT plant pole, is the pole of the plant⁽¹⁾.

The next question is how to implement $M(\epsilon)$. The controller structure should be such that the PITF is equal to $M(\epsilon)$ and the system is internally stable. Figure 2 shows one such structure. The design steps can be summarized as follows.

1. Calculate the matched-pole-zero model, $M(\epsilon)$, of the CT PITF, $\bar{M}(s)$, of order p .
2. Calculate the invariant model $G(\epsilon) = n(\epsilon)/d(\epsilon)$ of the plant, using the same type of hold element as used in actual implementation of the control algorithm.
3. Set $\mu(\epsilon)$ as the numerator of $M(\epsilon)/d(\epsilon)$, where all common factors are cancelled.
4. Determine the polynomials $\alpha(\epsilon)$ and $\beta(\epsilon)$ where $\partial[\alpha(\epsilon)] = p - n$ and $\partial[\beta(\epsilon)] \leq p - n$, by solving the Diophantine equation given by

$$\alpha(\epsilon)d(\epsilon) + \beta(\epsilon)n(\epsilon) = \gamma(\epsilon), \quad (3)$$
 where $\gamma(\epsilon)$ is the denominator polynomial of $M(\epsilon)$.
5. Let $\lambda(\epsilon)$ and $\rho(\epsilon)$ be arbitrary Hurwitz polynomials where $\partial[\lambda(\epsilon)] = \partial[\rho(\epsilon)] = p - n$.
6. Implement the controller as shown in Fig. 2, where

$$A(\epsilon) = \frac{\mu(\epsilon)}{\rho(\epsilon)}, B(\epsilon) = \frac{\beta(\epsilon)}{\lambda(\epsilon)}, C(\epsilon) = \frac{\lambda(\epsilon)}{\alpha(\epsilon)}. \quad (4)$$

It should be emphasized that, as long as the desired $\bar{M}(s)$ is known, it is not necessary to know the CT controller design. This leads to the discussion in the next section.

3. Direct Digital Model Reference Control

In some cases, enough information of the plant and design specifications is available to the designer so that the CT PITF can be calculated without having

Fig. 2 Structure of the PIM controller

to design an analog model reference control system (See Fig. 1(b)). This is explained for pole placement control first, followed by model reference control.

The overall closed-loop characteristic polynomial of a pole placement control system has $(2n-1)$ poles, of which $(n-1)$ are those of the observer dynamics and are cancelled, resulting in the same order n as that of the plant⁽¹²⁾. In pole placement control, the desired n th-degree closed-loop characteristic polynomial, $\bar{h}(s)$, and a stable $(n-1)$ th-degree observer polynomial, $\bar{\lambda}(s)$, are specified. In this case, $\bar{M}(s)$ can be determined directly from the plant model and design specifications as

$$\bar{M}(s) = \frac{\bar{d}(s)\bar{\lambda}(s)}{\bar{h}(s)\bar{\lambda}(s)}, \quad (5)$$

Once the PITF is specified, the rest of the design is the same as the PIM design described in the previous section, where $A(\epsilon)$ can always be made unity by selecting $\rho(\epsilon)$ to be equal to a matched-pole-zero model of $\bar{\lambda}(s)$. The resulting system is stable for any nonpathological sampling interval.

As a variant of pole placement control, model reference control can be achieved when the CT plant model has no zeros in the unstable region. In model reference control, the desired characteristic polynomial $\bar{h}(s)$ is chosen to include the plant numerator polynomial $\bar{n}(s)$ as

$$\bar{h}(s) = \bar{h}^*(s)\bar{n}(s), \quad (6)$$

where $\partial[\bar{h}^*(s)] = n - m$. In this method, the reference model can be considered as $\bar{M}(s)$, which specifies the desired transfer function from the reference input to the desired plant input, not to the desired plant output as in the conventional model reference control. The PITF can be calculated directly from the plant model and design specifications as

$$\bar{M}(s) = \bar{G}_M(s)\bar{h}^*(s) \cdot \frac{\bar{d}(s)\bar{\lambda}(s)}{\bar{h}(s)\bar{\lambda}(s)} = \bar{M}_F(s)\bar{M}_B(s), \quad (7)$$

where $\bar{G}_M(s)$ is the reference model whose relative degree is larger than or equal to that of the plant,

$$\bar{M}_F(s) = \bar{G}_M(s)\bar{h}^*(s), \bar{M}_B(s) = \frac{\bar{d}(s)\bar{\lambda}(s)}{\bar{h}(s)\bar{\lambda}(s)}. \quad (8)$$

The feedforward portion of the PITF is $\bar{M}_F(s)$ and determines the $A(\epsilon)$ in Fig. 3, while the feedback portion of the PITF is $\bar{M}_B(s)$ and determines the $B(\epsilon)$ and $C(\epsilon)$ in Fig. 3. More specifically,

Fig. 3 Structure of the model reference PIM controller

$$A(\varepsilon) = G_M(\varepsilon)h^*(\varepsilon), B(\varepsilon) = \frac{\beta(\varepsilon)}{\lambda(\varepsilon)}, C(\varepsilon) = \frac{\lambda(\varepsilon)}{\alpha(\varepsilon)}. \quad (9)$$

By applying the PIM method to $\bar{M}_B(s)$ as in pole placement control where $\rho(\varepsilon)$ is set equal to the matched-pole-zero model of $\bar{\lambda}(s)$, the direct digital design of model reference control can be achieved. The design requires the solution of the Diophantine equation (3), which now has the form of

$$\alpha(\varepsilon)d(\varepsilon) + \beta(\varepsilon)n(\varepsilon) = h^*(\varepsilon)n(\varepsilon)\lambda(\varepsilon). \quad (10)$$

Since the left-hand side of Eq. (10) represents the actual characteristic polynomial of the feedback loop, the numerator of an appropriate invariant plant model should be used for $n(\varepsilon)$. On the other hand, the right-hand side of Eq. (10) represents the desired polynomial which includes the plant numerator to achieve pole-zero cancellation. While this presents no problem in the CT case, a problem can arise in the DT case, since the numerator on the right-hand side of Eq. (10) is a characteristic polynomial of a filter to be implemented later and must be stable. If the invariant model gives an unstable numerator, such as the numerator $n_s(\varepsilon)$ of the step invariant model, it must be approximated by one that is stable. A conventional approach^{(6),(7)} to this approximation is to simply truncate the numerator by ignoring the zeros which do not correspond to the zeros of the CT plant. This can be achieved easily using the Euler or delta operator, since the sampling zeros arise due to terms whose coefficients tend to zero as $T \rightarrow 0$. Ignoring these terms leads to the truncated numerator, $n_{ts}(\varepsilon)$, which is used for both sides of Eq. (10); that is,

$$\alpha(\varepsilon)d_s(\varepsilon) + \beta(\varepsilon)n_{ts}(\varepsilon) = h^*(\varepsilon)n_{ts}(\varepsilon)\lambda(\varepsilon). \quad (11)$$

Since the left-hand side of this equation does not represent the closed-loop characteristic equation which includes the actual D/A-plant-A/D combination, stability can be guaranteed only when $T \rightarrow 0$, in general. A better approach, which is suggested by the mapping of the PITF, is to use the numerator of the step invariant model in the left-hand side of Eq. (10), and that of the matched-pole-zero model $n_m(\varepsilon)$ in the right-hand side; that is,

$$\alpha(\varepsilon)d_s(\varepsilon) + \beta(\varepsilon)n_s(\varepsilon) = h^*(\varepsilon)n_m(\varepsilon)\lambda(\varepsilon). \quad (12)$$

The resulting overall transfer function is $G_M(\varepsilon) \frac{n_s(\varepsilon)}{n_m(\varepsilon)}$, which is stable. Since both the step invariant model and matched-pole-zero model are valid DT models of the same plant and, thus, the fractional part $\frac{n_s(s)}{n_m(s)}$ approaches unity as $T \rightarrow 0$, the closed-loop system approaches the reference model as $T \rightarrow 0$.

4. Simulation Studies

Consider the unstable, fourth-order, minimum-

phase plant given by

$$\bar{G}(s) = \frac{s+3}{s(s+2)(s-2)(s-3)}. \quad (13)$$

The reference model is selected to be

$$\bar{G}_M(s) = \frac{6}{(s^2+2s+2)(s+3)}. \quad (14)$$

Since the degrees of $\bar{h}^*(s)$ and $\bar{\lambda}(s)$ happen to be identical in this example, the design parameters are selected such that

$$\bar{h}^*(s) = \bar{\lambda}(s) = (s+2)^3. \quad (15)$$

The desired PITF is calculated to be $\bar{M}(s) = \bar{M}_F(s)\bar{M}_B(s)$, where

$$\begin{aligned} \bar{M}_F(s) &= \frac{6(s+2)^3}{(s^2+2s+2)(s+3)}, \\ \bar{M}_B(s) &= \frac{s(s+2)(s-2)(s-3)(s+2)^3}{(s+2)^6(s+3)}. \end{aligned} \quad (16)$$

It should be noted that all the design specifications are given in the continuous-time domain. The digital model reference control system is designed using the PIM method, as described in section 3. Figure 4(a) shows the simulation result for the unit step reference input with the sampling interval of $T=0.2$ seconds (about 10 samples per rise time). It depicts the control inputs to and the responses of the CT plant under analog control and digital control. For $T < 0.2$ seconds, the output difference between the two curves is very small. As the sampling interval increases, the response of the digitally controlled CT plant deteriorates gradually, but at least stability is maintained (Fig. 4(b) shows the result for $T=2$ seconds).

The model reference control system is designed also using the conventional direct method where the numerator of the step invariant model of the plant is truncated so that there is only one finite zero which corresponds to $s=-3$. Although the step invariant model is usually used for the reference model, the matched-pole-zero model is used so that the reference model need not be truncated. By this means, the reference outputs for both the PIM and conventional designs are made almost identical, and the digital controller performances can be compared easily. Figure 5(a) shows the simulation result for the unit step reference input with the sampling interval of $T=0.02$ seconds. Compared with Fig. 4(a), it can be seen that the response is not good, even though the sampling rate is ten times faster. When $T=0.05$ seconds, the response is stable but highly oscillatory (Fig. 5(b)).

All the designs were carried out using the Euler (delta) form, while response calculations were performed using the shift (z) form. For the calculation of the CT plant response, the 4th-order Runge-Kutta method was used and the zero-order hold was assumed for the D/A converter. It should be noted

that due to a relatively high order of the PITF, the controller designed in the z operator form resulted in numerically unstable systems when T was smaller than about 0.01 second, even with double precision, while that designed in the Euler form gave stable systems. For sampling intervals of the same value, the proposed method provides results that are much

superior to those obtained using the previous method.

5. Conclusion

The PIM method proposed in earlier studies as a digital redesign method has been extended to the direct digital design method. As in the digital redesign PIM method, the direct digital PIM method guarantees

Fig. 4 Model reference control with the PIM method

Fig. 5 Model reference control with the conventional method

stability of the combined CT and DT systems for almost any sampling interval. This method is applicable to cases where the desired transfer function from the reference input to the plant input (plant-input transfer function) can be determined from the available information such as the plant model and

design specifications. This approach was explained with particular reference to the design of pole placement controllers and model reference control systems. Simulation studies have been carried out to show that the proposed model reference control method is preferable to the conventional method which uses the

truncated step invariant model, because its performance is much closer to that of the analog controller. For the case of pole placement control, no significant improvement was noted (results not shown in this paper) in terms of performance over the conventional direct digital design. However, for the case of model reference control, the conclusion drawn from extensive computer simulations for second-, third-, and fourth-order plants is that the performance with the proposed method is better than that with the conventional method even when the sampling period is about ten times longer. Also, it is recommended that the design be carried out using the Euler operator form, rather than the z operator form, for better parameter representation, especially at high sampling rates. Using the all-stabilizing-controller formulation, the performance of the digital controller can be improved without redesigning the analog controller⁽⁴⁾. Unlike the indirect PIM method presented in previous papers, the direct PIM method proposed in this paper for pole placement and model reference control results in the same order as the analog controller.

The Diophantine equation appearing in the design may also be solved using other numerator polynomials of the plant, such as

$$\alpha(\varepsilon)d_s(\varepsilon) + \beta(\varepsilon)n_m(\varepsilon) = h^*(\varepsilon)n_m(\varepsilon)\lambda(\varepsilon) \quad (17)$$

$$\alpha(\varepsilon)d_s(\varepsilon) + \beta(\varepsilon)n_s(\varepsilon) = h^*(\varepsilon)n_{Ts}(\varepsilon)\lambda(\varepsilon). \quad (18)$$

Similarly to Eq. (11), Eq. (17) does not ensure stability unless $T \rightarrow 0$ and, therefore, is not recommended for the design. The simulation results obtained using Eq. (17), although not shown in the present paper, are almost identical to those obtained using Eq. (11) for almost all values of T . On the other hand, Eq. (18) gives the overall transfer function of $G_M(\varepsilon) \frac{n_s(\varepsilon)}{n_{Ts}(\varepsilon)}$ which is stable. The simulation results with this design (not shown here) are very similar to those using Eq. (12) for almost all values of T . In all cases, the desired performance was obtained as $T \rightarrow 0$. However, for a finite value of T , it is best to use the model which represents the actual plant model (i.e., the step invariant model if the zero-order hold is used for implementation) in the left-hand side of the Diophantine equation and a stable approximation of the numerator (such as $n_m(\varepsilon)$ and $n_{Ts}(\varepsilon)$) in the right-hand side.

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