Load Distribution Analysis Method for Cylindrical Worm Gear Teeth*

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Worm gears provide large reduction ratios and low noise for their load capacities. However, owing to manufacturing errors and bending deflections under load, a worm and a wheel do not operate at their theoretical load during meshing. It is important to predict the load distribution considering the tooth profile and assembling error. A method for determining the load distribution, Hertzian contact pressure and oil film thickness is presented in this paper. Using our empirical formula for bending deflections, we developed the transformation matrix. The deflection along the load direction at any point on the contact line at the same meshing position is expressed by the matrix of transformation. Using this matrix, the load distribution of a worm and wheel is evaluated using numerical analysis. Calculated results clearly agree with experimental ones obtained under actual operating conditions.

Key Words : Gear, Numerical Analysis, Load Distribution, Machine Element, Contact Problem, Worm, Wheel, Laser Holographic Interferometry

1. Introduction

Worm gears have been widely used in industry for many years because they achieve high reduction ratios and low speed and low noise operation. In the case of conventional design, gear specifications are determined using a standardized method. However, worm gears used for opening the cover of a high pressure vessel or those used in machine tools are expected to wear at an accelerated rate raising the problem of failure of the teeth due to fatigue. Large power capacity and small size for worm gears will be required more often in the future and development of

a method of rational design for such teeth is desired. If load distribution of gears along the contact line is determined, contact pressure, tooth root stress and oil film thickness at an arbitrary position on the tooth surface can be calculated. Many studies of load distribution along the contact lines for parallel-type gears have been conducted. The calculation for parallel-type gears has already been demonstrated⁽¹⁾. With regard to worm gears, Gunbara⁽²⁾ studied tooth contact pattern analysis but did not calculate load distribution on the tooth surface. Winter and Wilkesmann⁽³⁾ proposed a calculation method for load distribution on the tooth surface, but their method could not be used to calculate the tooth root stress. Therefore, we plan to develop a calculation method for contact pressure, tooth root stress and oil film thickness considering load distribution on the tooth surface along the contact lines. In order to calculate load distribution on the tooth surface, it is necessary to make compliance of tooth. Therefore, approximate equations for tooth deflection are formulated based on measurements of the tooth deflection of JIS 3-type and Niemann-type worms and wheels. Load distribution is calculated continuously, considering the

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influence of both machining and assembling error on tooth contact. Finally, we design a calculation method for contact pressure, tooth root stress and oil film thickness considering load distribution on the tooth surface. The calculated results are compared with experimental results for the contact pattern and tooth root stress.

2. Calculation Method of Load Distribution on the Tooth Surface

The force is transmitted along contact lines determined by the theoretical relative position of the worm and the wheel. We consider one mesh as shown in Fig. 1, set several load points along the contact lines and consider the elastic displacement of contact. As shown in Fig. 1, because the sum of elastic displacement $Wk (= Wk(1) + Wk(2))$ and initial gap k (gap due to tooth profile error and assembly error) at the contact point is equal to the meshing transmission error, gears can operate smoothly maintaining the deformed tooth profile. The condition of compatibility of deformation along the contact line is mathematically formulated as follows. Usually contact lines are divided into between ten and twenty sections.

$$\begin{bmatrix} [C_1]_1 & & & \\ & [C_1]_2 & & \\ & & \ddots & \\ & & & [C_1]_m \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} + \begin{bmatrix} [C_2]_1 & & & \\ & [C_2]_2 & & \\ & & \ddots & \\ & & & [C_2]_m \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \alpha e \quad (1)$$

$[C_1]$: compliance matrix of worm gears

$[C_2]$: compliance matrix of worm wheel gears

P : transmitted load vector

n : number of meshing teeth

α : meshing transmission error

e : unit vector

The condition of compatibility of deformation can be rewritten as follows by considering the initial

gap.

$$[C]P + \epsilon \geq \alpha e \quad (2)$$

where, $[C] = [C_1] + [C_2]$

ϵ : initial gap vector

An equality sign means that load is transmitted at the contact point and an inequality sign means that the load is not transmitted. Substitution of a variable vector Y into Eq. (2) yields the following equality.

$$[C]P + \epsilon - Y = \alpha e \quad (3)$$

The relation between P and Y is as follows.

$$\begin{cases} Y_i = 0 & P_i \geq 0 \\ Y_i > 0 & P_i = 0 \end{cases} \quad (4)$$

We obtain the following equation since the sum of transmitting load P_i is equal to total transmitting load F .

$$e^t \cdot P = F \quad (5)$$

where, e^t is the transposed unit vector.

The load distribution P_i can be obtained by simultaneously solving Eqs. (3), (4) and (5) using linear programming. In order to use linear programming, an object function is necessary, and therefore Eqs. (3) and (5) are rewritten as follows.

$$-[C]P + \alpha e + Y + Z = \epsilon \quad (6)$$

$$e^t \cdot P + Z_{N+1} = F \quad (7)$$

Next, we introduce an object function defined by the following equation.

$$Z_0 = e^t \cdot Z + Z_{N+1} \quad (8)$$

We set the value of Z_0 at zero and solve Eqs. (6) and (7) simultaneously. These methods are based on the linear programming developed by Conry and Seireg⁽⁴⁾. The compliance matrix of Eq. (1) contains (i) bearing and shearing deformation of the tooth, (ii) meshing deformation of the worm and worm wheel, (iii) bending deformation of the worm shaft.

3. Measurement of Tooth Deflections Using Laser Holographic Interferometry

3.1 Test conditions

3.1.1 Test gears The dimensions of the test gears are listed in Table 1. We select JIS 3-type and

Fig. 1 Meshing condition of worm gearing

Table 1 Dimensions of experimental gears

	Cylindrical worm gearing [JIS3-type]		Cylindrical worm gearing [Niemann-type]	
	Worm	Wheel	Worm	Wheel
Module (m)	20, 9		23.0	
Pressure angle (α)	22° 30'		24° 00'	
Number of teeth (Z)	5	81	3	50
Pitch circle diameter (do)	234.19 mm	1595.31 mm	250.00 mm	1150.00 mm
Lead angle (γ)	22° 48' 25"		17° 6' 18"	

Fig. 2 Tooth profile and appearance of plastic model

Fig. 3 Schematic of optical system of interferometer

Niemann-type worms and worm wheels. Figure 2 shows the tooth profile and the appearance of the model made of epoxy resin.

3.1.2 Optical system and coordinate system of interferometer

The optical system of the interferometer for measuring the deflections of the tooth is shown schematically in Fig. 3. A beam from a laser source is separated into a reference beam and an object beam reflected by a reference surface. The

Fig. 4 Coordinates of loading position

Fig. 5 Interference fringe patterns of gear tooth

reference beams are irradiated simultaneously on a thermoplastic the surface of which is coated with photosensitive material, which makes a hologram.

The tooth flank to be measured is placed at the same position as a reference flank. Interference between the real image of the tooth flank produced by the object beam and the image reconstructed by the reference beam produces a fringe pattern. To determine the deflection of the tooth at an arbitrary position, we select a loading position as shown in Fig. 4.

3.2 Test results

Figure 5 shows measured examples of interference fringe patterns for the worm and worm wheel.

The first interference fringe appears when the path difference between the object and reference beam is a half wave length of the laser beam. The first interference fringe is equivalent to a deflection of 0.257 μm , since the wavelength of the Argon laser is 5145 $\text{\AA}$. Under the test condition, loading force is chosen, so as to produce the 6th to 10th interference fringes, and the range of tooth deformation is estimated to be from 1.5 μm to 2 μm .

Large deflection occurs at the loading point as indicated by the decrease in the interval between the interference fringes.

4. Approximate Equation of Tooth Deflections

4.1 Modeling of approximate equation of tooth deflections

An empirical equation of tooth deflection is constructed from the test results. It is formulated based on the deflection equation developed by Umezawa⁽⁵⁾. This equation has already been applied to parallel gears.

$$W = \frac{V(\bar{r})}{\sqrt{F(\bar{\theta})} \sqrt{G(\bar{y})}} \times \sqrt{F(\theta)} \sqrt{F(\theta_p)} \sqrt{G(\bar{y})} \sqrt{G(\bar{y}_p)} U \frac{P}{m} \quad (9)$$

where, $\bar{y} = 1 - \frac{y}{h}$, $\bar{y}_p = 1 - \frac{y_p}{h}$, $\bar{y} = |\bar{y} - \bar{y}_p|$, $\bar{\theta} = |\theta - \theta_p|$

w : tooth deflection

h : whole depth

m : module

P : concentrated force

U : absolute tooth deflection

$v(\bar{r})$: characteristic function of tooth deflection

$G(y)$: characteristic function of tooth deflection in the direction of tooth height at the loading point

$F(\theta)$: characteristic function of tooth deflection in the direction of tooth width at the loading point

y, y_p : Y coordinates of measuring and loading points (tooth bottom is the origin)

Table 2 The empirical equations of absolute deflection

	Unit (μm)	
	Worm	Wheel
JIS3 type	$0.0111e^{3.15\alpha}$	$0.0074e^{3.2\alpha}$
Niemann type	$0.0105e^{3.2\alpha}$	$0.0040e^{3.13\alpha}$

Table 3 The empirical equations of corrective coefficient λ

	Worm	Wheel
JIS3 type	$0.203e^{2.34\alpha}$	$0.235e^{2.3\alpha}$
Niemann type	$0.227e^{2.32\alpha}$	$0.152e^{2.31\alpha}$

θ, θ_p : θ coordinates of measuring and loading points

4.2 Absolute tooth deflection [U]

We introduce absolute tooth deflection [U] which is, defined as the deflection when a unit load is applied to the center of face width and tooth height.

The values of deflection for steel and epoxy resin can be converted to those for steel and steel by multiplying by the elastic coefficient ratio. The influence of the pressure angle is considered by referring to the equation developed by Umezawa⁽⁵⁾. The empirical equation of deflection when unit load is applied is formulated as shown in Table 2.

4.3 Characteristic function of tooth deflection

Tooth deflection when load is applied to the tooth tip can be expressed by coaxial circles surrounding the loading point using circumferential coordinates. Introducing a corrective coefficient λ , tooth deflection takes the same value at the radial position expressed as

$$r = \sqrt{\left\{ \left(R \pm \frac{y + y_p}{2} \right) \lambda (\theta - \theta_p) \right\}^2 + (y - y_p)^2 / h} \quad (10)$$

R : tooth root radius of the worm or worm wheel.

In this equation, $+$ means worm and $-$ means worm wheel. The influence of the pressure angle is considered by referring to the equation developed by Umezawa⁽⁵⁾. The empirical equations for the corrective coefficient are formulated as shown in Table 3. The relationship between the characteristic function of tooth deflection $v(\bar{r})$ and the radial position $\bar{r}$ is shown in Fig. 6.

There is little difference between JIS 3-type and Niemann-type. We arrange the results shown in Fig. 6 and obtain the following equation by the least squares method.

$$v(\bar{r}) = 1 - 2.148\bar{r} + 1.785\bar{r}^2 - 0.584\bar{r}^3 \quad (11)$$

4.4 Characteristic function of tooth deflection in the direction of tooth height at the loading point

This function is the correction function necessary when the loading point is not put at the tooth tip. The tooth deflection in the direction of tooth height at the loading point is normalized by the tooth deflection

Table 4 The empirical equations of $G(y)$

Worm (JIS3 type)	$1-4.122y+7.771y^2-7.108y^3+2.46y^4$
Wheel (JIS3 type)	
Worm (Niemann type)	$1-2.457y+2.166y^2-0.617y^3-0.091y^4$
Wheel (Niemann type)	

Fig. 6 The relation between $v(\bar{r})$ and $\bar{r}$ Fig. 7 The relation between $G(y)$ and y

when load is applied to the tooth tip. The results are shown in Fig. 7. In this case the result for the Niemann-type worm wheel type is different from that for JIS 3-type and therefore we construct two empirical equations.

The reason for the difference between JIS 3-type and Niemann-type is the surface expansion of the Niemann-type worm wheel. The empirical equations are shown in Table 4. To satisfy Maxwell's theorem, we use $G(y)$ for $\sqrt{G(\bar{y})} \cdot \sqrt{G(\bar{y}_p)} / \sqrt{G(\bar{y})}$.

4.5 Characteristic function of tooth deflection in the direction of tooth width at the loading point

This function is the correction function which accounts for increase in tooth deflection at the edge of the tooth width. This function is only applied to worm

Table 5 The empirical equations of $F(\theta)$

Wheel (JIS3 type)	$\exp (2.187 \theta^{1.615})$
Wheel (Niemann type)	$\exp (4.375 \theta^{3.284})$

Fig. 8 The relation between $F(\theta)$ and θ

Fig. 9 The coordinates of tooth surface

wheels. The tooth deflection in the direction of tooth width is normalized by the tooth deflection when load is applied to the center of the tooth width. The results are shown in Fig. 8. The empirical equations are shown in Table 5. To satisfy Maxwell's theorem, we use $F(\theta)$ for $\sqrt{F(\theta)} \cdot \sqrt{F(\theta_p)} / \sqrt{F(\theta)}$.

5. Approximate Equation of Influence Coefficient of Root Stress

If the load distribution on the tooth surface is determined, root stress can be calculated using the following equation.

$$\sigma = [S]P \quad (12)$$

σ : stress vector

$[S]$: matrix of influence coefficient of root stress

P : load vector

It is necessary to set up a matrix of the influence coefficient of root stress in a generalized form. The measurement of root stress is carried out simultaneously with the measurement of tooth deflection. We set up a matrix of the influence coefficient of root stress, when the load is applied to arbitrary position on the tooth surface.

Then we construct an empirical equation as follows.

$$\sigma = \sigma_0 \cdot K_v \cdot K_s \cdot K_a \cdot K_\theta \quad (13)$$

Table 6 The empirical equations of σ_0

	Worm (root of tooth)	Worm (pitch diameter)	Wheel (root of tooth)
JIS3 type	$3.295 \left(\frac{a}{b} \right)^{0.627} \frac{w}{t}$	—	$3.474 \frac{w}{t}$
Niemann type	$3.295 \left(\frac{a}{b} \right)^{0.627} \frac{w}{t}$	$2.583 \frac{w}{t}$	$3.474 \frac{w}{t}$

a ; Worm outside diameter b ; Worm root diameter
w ; Unit load t ; Tooth thickness

Table 7 The empirical equations of K_v

	Worm (root of tooth)	Worm (pitch diameter)	Wheel (root of tooth)
JIS3 type	$\left(\frac{Y_F}{h} \right)^{0.752}$	—	$\left(\frac{Y_F}{h} \right)^{0.832}$
Niemann type	$\left(\frac{Y_F}{h} \right)^{0.752}$	$\left(\frac{Y_F}{h} \right)^{0.752}$	$\left(\frac{Y_F}{h} \right)^{0.832}$

Table 8 The empirical equations of K_s

	Worm (root of tooth)	Worm (pitch diameter)	Wheel (root of tooth)
JIS3 type	$\exp(-5.575 \theta - \theta_r ^{1.678})$	—	$\exp(-3.878 \theta - \theta_r ^{1.27})$
Niemann type	$\exp(-4.322 \theta - \theta_r ^{1.593})$	$\exp(-9.589 \theta - \theta_r ^{1.597})$	$\exp(-2.268 \theta - \theta_r ^{1.591})$

where σ_0 is fundamental nominal stress. The empirical equations are listed in Table 6.

K_v is the influence coefficient of tooth height which accounts for the difference in root stress when the load position moves in the direction of tooth height. The empirical equations obtained by arranging measured results are listed in Table 7. K_s is influence coefficient of tooth width that shows the distribution of root stress in the direction of tooth width. The empirical equations are listed in Table 8.

K_a is the stress concentration coefficient and is expressed in the following equation.

$$K_a = 1 + 0.193 \left(\frac{\rho}{t} \right)^{-0.373} \quad (14)$$

ρ : fillet radius

K_θ is the influence coefficient which accounts for the increase in root stress at the edge of tooth width of worm wheel and is expressed in the following equation.

$$K_\theta = 1 + 3.92 |\theta_p|^{2.5} \quad (15)$$

6. Calculation Example

In this section, we demonstrate an example calculated by the program developed.

6.1 Comparison with operating data

To examine the accuracy of the program devel-

Table 9 Specifications of experimental gears

	Worm	Wheel
Module (mm)	6.2	
Pressure angle (deg)	20	
Lead angle (deg)	17.896	
Center distance (mm)	125	
Number of teeth	3	31
Pitch circle diameter (mm)	57.6	192.4
Averaged diameter (mm)	57.8	192.2
Outside diameter (mm)	68.0	204.8
Root diameter (mm)	41.2	178.0
Face width (mm)	—	46
Worm rotational speed (rpm)	1800	—

Table 10 Test conditions

Test No.	Contents	Assembling error		Worm rotational speed (rpm)	Worm torque (N·m)	Lubricating oil
		Error of axial direction of wheel (mm)	Error of center distance (mm)			
1	Case of standard	-0.13*	—	1800	9.8 ~ 98	ISO VG 220
2	Case of error of axial direction	0.28	—	1800	9.8 ~ 98	ISO VG 220
3	Case of error of center distance	—	0.17	1800	9.8 ~ 98	ISO VG 220

*Necessary adjustment for standard tooth contact

oped in this study, an experiment on the tooth contact and root stress is carried out. Table 9 shows the dimensions of the worm gearing and Table 10 shows the test conditions. The tooth profile is JIS 3-type and profile modification is not carried out. In the case of a standard test, the error in the axial direction of the wheel is -0.13 mm as shown in Table 10.

This means that the worm wheel shifts -0.13 mm from the assembled position.

The computations account for the following three causes of error.

- 1) the machining error of the worm and worm wheel
- 2) the assembling error
- 3) the wear of the worm wheel

The machining error means the tooth profile error which is substituted in the place of the initial gap vector in Eq.(2) as the gap from the normal tooth profile. Comparison between the measured and calculated root stress and tooth contact of the worm wheel is shown in Fig. 10. The tooth surface is painted red and the contact pattern is obtained after meshing from the pattern of the removal of paint. In the

Fig. 10 Comparison between measured and calculated results

calculation, the contact area is defined by setting the gap between the worm and wheel at zero. Calculations are performed at intervals of 30 degrees of worm rotation.

When the actual tooth contact pattern is compared to the calculated tooth contact pattern, the agreement is quite good although the calculated tooth contact pattern is not as smooth as the measured one. The primary cause of the difference in tooth contact is wear of the convex surface of the worm and wheel which is not considered in the calculation. Comparison between the measured and calculated root stresses reveals good agreement. However, in the case of test No. 3, the calculated distribution of root stress during disengaging is larger than the measured result.

The disagreement between the calculated and measured results occurs under the condition that a strong contact pattern exists during disengaging and the tip of the worm wheel. In this area, the wear rate is considered to be high since the PV value (the

product of contact pressure and sliding velocity) is large. However, the continuous wear and fitting are not considered in the calculation. The continuous wear and fitting affect the accuracy of the calculation. Careful consideration should be given to prediction of fitting in the future.

6.2 Influence of assembling error

Next the calculation of the effect of assembling error on tooth contact of the worm wheel without machining error is carried out. The dimensions of the tested worm gearing are listed in Table 9. The orthogonal error of the shaft varies in the region between -0.1 and 0.1 degree and the axial gap of the worm wheel varies in the region between -0.3 and 0.3 mm. Figure 11 shows the change of root stress and PV value of the worm and worm wheel. The calculated torque is 96 Nm. However, the PV value increases by a certain kind of the assembling error. The position of the maximum PV value shifts in the direction of disengaging.

Fig. 11 The influence of assembling error

7. Conclusion

In the present paper, we design a program for calculating load distribution along the contact lines, in order to develop a method of tooth contact analysis of worm gearing. The calculated results are compared with experimental results of tooth contact and tooth root stress.

The results obtained are summarized as follows ;

(1) An empirical equation of deflection of the worm and worm wheel is formulated.

(2) The matrix of the influence coefficient is formulated in order to calculate root stress of the worm and worm wheel.

(3) The calculation program of contact pressure, tooth root stress and oil film thickness considering the effect of load distribution along the contact lines on

machining and assembling error is developed.

(4) Calculated results of tooth contact and tooth root stress agree reasonably well with the experimental results. It is confirmed that the calculation program is practical and useful.

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