Timoshenko's Shear Coefficient for Transverse Vibrations of Trigonal Crystals of Rectangular Cross Section*

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In this paper, we derive the shear coefficient K for trigonal crystals of rectangular cross section in Timoshenko's beam theory of three dimensional (3-D) elasticity. The theoretical result gives K as a function of the transformed elastic stiffness constants c'_{22} , c'_{44} , c'_{66} . For an isotropic material it agrees completely with that derived by G. R. Cowper. In addition, values of K versus cut angle for quartz crystal, lithium tantalate, and lithium niobate which belong to the trigonal system are calculated, so that K has a value of 0.796-0.867 for quartz crystal, 0.834-0.857 for lithium tantalate, and 0.831-0.863 for lithium niobate, when Z-plates of the crystals are rotated with a cut angle θ of 0-180° about the x -axis. The values of K are then compared with those for isotropic materials obtained by other authors.

Key Words: Piezoelement, Timoshenko's Shear Coefficient, Trigonal Crystal, Rectangular Cross Section, Transverse Vibration

1. Introduction

The one-dimensional (1-D) theory of beams can be extended in the range of applicability by taking account of rotary inertia and transverse shear deformations. The equations with these effects are generally referred to as Timoshenko's beam equations⁽¹⁾. In these equations, the effective transverse shear strain is taken as equal to the average shear stress on a cross section divided by the product of the shear modulus (elastic stiffness constant) and the shear coefficient, K . In addition, the shear coefficient K is commonly defined as the ratio of the average shear strain on a section to the shear strain at the centroid.

However, there occurs a big difference between the results analyzed using the commonly defined K and the experimental results⁽²⁾. Leibowitz and Kennard have criticized the definition, pointing out an unwarranted assumption in the underlying analysis,

and Cowper has agreed with their criticism and has pointed out that further study of the shear coefficient is needed⁽²⁾. In addition, Mindlin, et al.⁽³⁾, and Goodman⁽⁴⁾ have pointed out that Timoshenko's beam equations with the customary values of K lead to unsatisfactory results when the high resonance frequencies of vibrating beams are calculated, and have insisted that K should be adjusted arbitrarily in order to obtain better results. However, they do not discuss the basic definition of K . Instead, they point out that the distribution of shear strain over a cross section varies with the mode of vibration, i.e., frequency.

According to the thinking of Leibowitz and Kennard, Cowper has examined the shear coefficient K for an isotropic material in detail and better results are obtained⁽²⁾. It must be noted, however, that K for an anisotropic material such as quartz crystal, lithium tantalate, and lithium niobate is not examined at all.

In this paper, the shear coefficient K for trigonal crystals which are anisotropic is described. In analysis procedure, first, the shear coefficient K in Timoshenko's beam theory is derived using Cowper's definition and is found to be given as a function of stress-component T_6 and residual displacement v_y .

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Next, T_6 and v_y are derived from the equations of equilibrium and the boundary conditions. Substitution of T_6 and v_y into K gives a function of harmonic function χ' . The shear coefficient K is, therefore, calculated using χ' derived. Finally, the calculated values of K for quartz crystal, lithium tantalate, and lithium niobate are shown to a cut angle rotated about the x -axis.

2. Derivative of shear coefficient K

Figure 1 shows a flexural crystal resonator of a beam which belongs to a trigonal system, and its coordinate system (a), and front view and sectional view when lateral force Q is applied at the end (b). Now, as length $2y_0$ is much larger than width $2x_0$ and thickness $2z_0$, stress-components T_1 , T_3 , T_5 become

$$T_1 = T_3 = T_5 = 0 \quad (1)$$

For the rotation about the x -axis (rotation angle θ), a relationship between stress T_p and strain S_q for trigonal crystals is given as

$$\begin{aligned} T_1 &= c_{11}S_1 + c_{12}S_2 + c_{13}S_3 + c_{14}S_4 \\ T_2 &= c_{21}S_1 + c_{22}S_2 + c_{23}S_3 + c_{24}S_4 \\ T_3 &= c_{31}S_1 + c_{32}S_2 + c_{33}S_3 + c_{34}S_4 \\ T_4 &= c_{41}S_1 + c_{42}S_2 + c_{43}S_3 + c_{44}S_4 \\ T_5 &= c_{55}S_5 + c_{56}S_6 \\ T_6 &= c_{65}S_5 + c_{66}S_6 \end{aligned} \quad (2)$$

where c_{pq} is elastic stiffness constant. Accordingly, from the relation of Eq. (1), Eq. (2) is simplified into

$$\begin{aligned} T_2 &= c'_{22}S_2 + c'_{24}S_4 \\ T_4 &= c'_{42}S_2 + c'_{44}S_4 \\ T_6 &= c'_{66}S_6 \end{aligned} \quad (3)$$

Fig. 1 Flexural quartz crystal resonator of beam and its coordinate system (a), and front view and sectional view when lateral force Q is applied at the end (b)

where,

$$\begin{aligned} c'_{pq} &= \bar{c}_{pq} - \bar{c}_{1p}\bar{c}_{q1}/\bar{c}_{11} \quad (p, q=2, 4) \\ \bar{c}_{pq} &= c_{pq} - c_{3p}c_{q3}/c_{33} \quad (p, q=1, 2, 4) \\ c'_{66} &= c_{66} - c_{56}c_{65}/c_{55} \end{aligned}$$

The definition of displacement and rotation angle accords to Ref. (2). The transverse mean deflection $U(y)$ of the cross section and the mean displacement $V(y)$ of the cross section in the y -axis direction are defined as

$$U(y) = \frac{1}{A} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} u_x dx dz \quad (4)$$

and

$$V(y) = \frac{1}{A} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} u_y dx dz \quad (5)$$

where A is the area of the cross section, u_x and u_y are, respectively, the x -component and the y -component of displacement of a point of the beam. In addition, the mean angle $\Phi(y)$ of rotation of a cross section about the neutral axis is also defined as

$$\Phi(y) = \frac{1}{I_z} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} x u_y dx dz \quad (6)$$

where,

$$I_z = \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} x^2 dx dz$$

Further, residual displacements v_x , v_y are defined by the following relation, namely

$$\begin{aligned} u_x &= U(y) + v_x \\ u_y &= V(y) + x\Phi(y) + v_y \end{aligned} \quad (7)$$

It follows from Eq. (7) that

$$\begin{aligned} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} v_x dx dz &= \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} v_y dx dz \\ &= \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} x v_y dx dz = 0 \end{aligned} \quad (8)$$

The residual displacement v_y essentially represents the warping of the cross section. From Eq. (3) the stress-strain relation is

$$T_6 = c'_{66} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \quad (9)$$

Substitution of Eq. (7) into Eq. (9) gives

$$\frac{\partial U(y)}{\partial y} + \Phi(y) = \frac{T_6}{c'_{66}} - \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \quad (10)$$

Integration of Eq. (10) over the cross section then yields, in view of Eq. (8),

$$\frac{\partial U(y)}{\partial y} + \Phi(y) = \frac{1}{Ac'_{66}} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} \left(T_6 - c'_{66} \frac{\partial v_y}{\partial x} \right) dx dz \quad (11)$$

The right term of Eq. (11) can be replaced by

$$\frac{Q}{KA'c'_{66}} \quad (12)$$

The K of Eq. (12) is called Timoshenko's shear coefficient. From the relation of Eqs. (11) and (12) the shear coefficient K is rewritten as

$$K = \frac{Q}{\int_{-z_0}^{z_0} \int_{-x_0}^{x_0} \left(T_6 - c'_{66} \frac{\partial v_y}{\partial x} \right) dx dz} \quad (13)$$

That is, the shear coefficient per unit force K/Q is

given as a function of stress T_6 , elastic stiffness constant c'_{66} and residual displacement v_y . The stress T_6 and the residual displacement v_y are derived next.

3. Derivative of stress T_6 and residual displacement v_y

3.1 Equation of equilibrium

The bending moment M at the cross section distant y from the fixed end is

$$M = Q(2y_0 - y) \quad (14)$$

Here, we assume that stress-component T_2 on any element of this section is given by the equation

$$T_2 = -Q(2y_0 - y)x/I_z \quad (15)$$

Since two of the equations of equilibrium from the relation of Eq.(1) must be satisfied with

$$\frac{\partial T_6}{\partial y} = 0 \text{ and } \frac{\partial T_4}{\partial y} = 0 \quad (16)$$

it follows that T_6 and T_4 must be independent of y . By use of the relation of Eq.(15), the equation of equilibrium reduces to

$$\frac{\partial T_6}{\partial x} + \frac{\partial T_4}{\partial z} + \frac{Q}{I_z}x = 0 \quad (17)$$

In addition, there is a relationship with respect to strain S_q ($q=3$ to 6) as follows:

$$\frac{\partial}{\partial z} \left(\frac{\partial S_4}{\partial x} + \frac{\partial S_5}{\partial y} - \frac{\partial S_6}{\partial z} \right) = 2 \frac{\partial^2 S_3}{\partial x \partial y} \quad (18)$$

From the relation of Eqs.(2) and (16), it follows that $\partial S_5 / \partial y = 0$. Equation (18) is, therefore transformed into

$$\frac{\partial S_4}{\partial x} - \frac{\partial S_6}{\partial z} = 2\tau - \frac{2\sigma_z}{c'_{22}} \frac{Q}{I_z} z \quad (19)$$

where 2τ is a constant of integration and σ_z is a dimensionless constant which is given by elastic stiffness constants; and from this equation it follows that S_4 and S_6 which satisfy Eq.(19) can be expressed in the forms

$$\begin{aligned} S_4 &= \tau x + \frac{\partial \phi_0}{\partial z} \\ S_6 &= -\tau z + \frac{\partial \phi_0}{\partial x} + \frac{\sigma_z}{c'_{22}} \frac{Q}{I_z} z^2 \end{aligned} \quad (20)$$

where ϕ_0 is a function of x and z .

From Eqs.(20) and (3) the equation of equilibrium of Eq.(17) reduces to

$$c'_{66} \frac{\partial^2 \phi_0}{\partial x^2} + \bar{c}_{44} \frac{\partial^2 \phi_0}{\partial z^2} + \frac{Q}{I_z} x = 0 \quad (21)$$

where, $\bar{c}_{44} = c_{44} - c_{24}^2 / c'_{22}$ ($\because c_{24} = c'_{42}$).

In addition, we change the variables by putting

$$\begin{aligned} x' &= x \sqrt{\frac{\bar{c}_{44} + c'_{66}}{2c'_{66}}}, \quad z' = z \sqrt{\frac{\bar{c}_{44} + c'_{66}}{2\bar{c}_{44}}}, \\ \phi'_0 &= \phi_0 \frac{\bar{c}_{44} + c'_{66}}{2\sqrt{\bar{c}_{44}c'_{66}}} \end{aligned} \quad (22)$$

The equation of equilibrium of Eq.(21) is then rewritten as

$$\frac{\partial^2 \phi'_0}{\partial x'^2} + \frac{\partial^2 \phi'_0}{\partial z'^2} + \frac{Q}{c'_{22}I_z} \Xi x' = 0 \quad (23)$$

where,

$$\Xi = \frac{c'_{22}}{\bar{c}_{44}} \sqrt{\frac{2\bar{c}_{44}}{\bar{c}_{44} + c'_{66}}}$$

3.2 Boundary condition

On the other hand, the condition that the bounding surface is free from traction must be

$$T_6 \cos(x, \nu) + T_4 \cos(z, \nu) = 0 \quad (24)$$

Here ν denotes the direction of the normal to a plane, and the general form of $\cos(a, \nu)$ has the following relation

$$\cos(a, \nu) = \begin{cases} 1 & |\nu| = a \\ 0 & |\nu| \neq a \end{cases} \quad (25)$$

From the relations of Eqs.(3), (15) and (20), the equation of the boundary condition becomes

$$\begin{aligned} \frac{\partial \phi_0}{\partial \nu} &= \tau \{ z \cos(x, \nu) - x \cos(z, \nu) \} \\ &\quad - \frac{\sigma_z}{c'_{22}} \frac{Q}{I_z} z^2 \cos(x, \nu) \\ &\quad + \frac{Q}{c'_{22}I_z} \frac{c'_{42}}{\bar{c}_{44}} (2y_0 - y)x \cos(z, \nu) \end{aligned} \quad (26)$$

which holds at all points of the bounding curve of any cross section. Here $d\nu$ denotes the element of the outward-drawn normal to this curve. In addition, if $f(x, z) = 0$ is the equation of the bounding curve, $\cos(x, \nu)$ and $\cos(z, \nu)$ become equal to $\partial f / \partial x$ and $\partial f / \partial z$, respectively. Now, when a new function $F(x', z')$ is satisfied with

$$F(x', z') \equiv f \left(x' \sqrt{\frac{2c'_{66}}{\bar{c}_{44} + c'_{66}}}, z' \sqrt{\frac{2\bar{c}_{44}}{\bar{c}_{44} + c'_{66}}} \right) \quad (27)$$

the equation $f(x, z) = 0$ becomes $F(x', z') = 0$. Therefore, by taking account of Eq.(27) and the relation of $\partial F / \partial x' = \cos(x', \nu')$ and $\partial F / \partial z' = \cos(z', \nu')$, the equation of the boundary condition of Eq.(26) can be rewritten as

$$\begin{aligned} \frac{\partial \phi'_0}{\partial \nu'} &= \tau \{ z' \cos(x', \nu') - x' \cos(z', \nu') \} \\ &\quad - \frac{Q}{c'_{22}I_z} \Xi' \sigma_z z'^2 \cos(x', \nu') \\ &\quad + \frac{Q}{c'_{22}I_z} \frac{c'_{42}}{\bar{c}_{44}} (2y_0 - y)x' \cos(z', \nu') \end{aligned} \quad (28)$$

where,

$$\Xi' = \sqrt{\frac{2\bar{c}_{44}}{\bar{c}_{44} + c'_{66}}}$$

Here $d\nu'$ is the element of the normal to the transformed boundary. Equations (23) and (28) are further simplified by putting

$$\begin{aligned} \phi'_0 &= \tau \phi' - \frac{Q}{c'_{22}I_z} \left\{ \chi' - \frac{1}{6} \left(1 - \frac{1}{2} \Xi \right) x'^3 \right. \\ &\quad \left. + \frac{1}{2} \left(1 + \frac{1}{2} \Xi \right) x' z'^2 \right\} \end{aligned} \quad (29)$$

then ϕ' is the torsion function for the cross section. Substitution of Eq.(29) into Eq.(23) gives χ' and ϕ' which are, respectively, satisfied with

$$\frac{\partial^2 \chi'}{\partial x'^2} + \frac{\partial^2 \chi'}{\partial z'^2} = 0 \quad (30)$$

and

$$\frac{\partial^2 \phi'}{\partial x'^2} + \frac{\partial^2 \phi'}{\partial z'^2} = 0 \quad (31)$$

From Eq.(29) $\partial \phi'_0 / \partial \nu'$ becomes

$$\begin{aligned} \frac{\partial \phi'_0}{\partial \nu'} = & \tau \left\{ \frac{\partial \phi'}{\partial x'} \cos(x', \nu') + \frac{\partial \phi'}{\partial z'} \cos(z', \nu') \right\} \\ & - \frac{Q}{c'_{22} I_z} \left[\left\{ \frac{\partial \chi'}{\partial x'} - \frac{1}{2} \left(1 - \frac{1}{2} \varepsilon \right) x'^2 \right. \right. \\ & + \frac{1}{2} \left(1 + \frac{1}{2} \varepsilon \right) z'^2 \left. \right\} \cos(x', \nu') \\ & + \left. \left\{ \frac{\partial \chi'}{\partial z'} + \left(1 + \frac{1}{2} \varepsilon \right) x' z' \right\} \cos(z', \nu') \right] \quad (32) \end{aligned}$$

Substituting this relation into Eq.(28), $\partial \chi' / \partial \nu'$ yields

$$\begin{aligned} \frac{\partial \chi'}{\partial \nu'} = & \left\{ \frac{1}{2} \left(1 - \frac{1}{2} \varepsilon \right) x'^2 \right. \\ & - \frac{1}{2} \left(1 + \frac{1}{2} \varepsilon - 2 \varepsilon' \sigma_z \right) z'^2 \left. \right\} \cos(x', \nu') \\ & - \left\{ \left(1 + \frac{1}{2} \varepsilon \right) z' + \frac{c'_{42}}{c'_{44}} (2y_0 - y) \right\} x' \cos(z', \nu') \quad (33) \end{aligned}$$

3.3 Stress T_6

Since strains S_4 and S_6 are, respectively, given as a function of $\partial \phi_0 / \partial x$ and $\partial \phi_0 / \partial z$, they are calculated from Eq.(29), so that Eq.(20) yields strains S_4 and S_6 as follows:

$$\begin{aligned} S_4 = & \tau \left(x + \varepsilon'' \frac{\partial \phi'}{\partial z'} \right) \\ & - \frac{Q}{c'_{22} I_z} \left\{ \varepsilon'' \frac{\partial \chi'}{\partial z'} + \frac{1}{2} \frac{c'_{22}}{c'_{44}} \left(1 + \frac{2}{\varepsilon} \right) x z \right\} \quad (34) \end{aligned}$$

and

$$\begin{aligned} S_6 = & \tau \left(-z + \varepsilon' \frac{\partial \phi'}{\partial x'} \right) - \frac{Q}{c'_{22} I_z} \\ & \times \left[\varepsilon' \frac{\partial \chi'}{\partial x'} + \frac{1}{4} \frac{c'_{22}}{c'_{44}} \left\{ \frac{c'_{44}}{c'_{66}} \left(1 - \frac{2}{\varepsilon} \right) x^2 \right. \right. \\ & + \left. \left. \left(1 + \frac{2}{\varepsilon} - 4 \frac{c'_{44}}{c'_{22}} \sigma_z \right) z^2 \right\} \right] \quad (35) \end{aligned}$$

where,

$$\varepsilon'' = \sqrt{\frac{2c'_{66}}{c'_{44} + c'_{66}}}$$

Consequently, stress T_6 is given as

$$\begin{aligned} T_6 = & c'_{66} \tau \left(-z + \varepsilon'' \frac{\partial \phi'}{\partial x'} \right) - \frac{c'_{66} Q}{c'_{22} I_z} \\ & \times \left[\varepsilon'' \frac{\partial \chi'}{\partial x'} + \frac{1}{4} \left\{ \frac{c'_{22}}{c'_{66}} \left(1 - \frac{2}{\varepsilon} \right) x^2 \right. \right. \\ & + \left. \left. \frac{c'_{22}}{c'_{44}} \left(1 + \frac{2}{\varepsilon} - 4 \frac{c'_{44}}{c'_{22}} \sigma_z \right) z^2 \right\} \right] \quad (36) \end{aligned}$$

where,

$$\varepsilon''' = \varepsilon' \varepsilon'' = \frac{2\sqrt{c'_{44} + c'_{66}}}{c'_{44} + c'_{66}}$$

3.4 Residual displacement v_y

As $\partial S_5 / \partial y = 0$ is taken into account, strains S_2 , S_4 and S_6 have the relation

$$2 \frac{\partial^2 S_2}{\partial z \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial S_4}{\partial x} + \frac{\partial S_6}{\partial z} \right) \quad (37)$$

Deriving $\partial S_4 / \partial x$ and $\partial S_6 / \partial z$ from Eqs.(34) and (35), substitution of the derivatives into Eq.(37) gives

$$\begin{aligned} 2 \frac{\partial^2 S_2}{\partial z \partial x} = & \frac{\partial}{\partial y} \left[2 \tau \varepsilon''' \frac{\partial^2 \phi'}{\partial x \partial z} - \frac{Q}{c'_{22} I_z} \right. \\ & \times \left. \left\{ 2 \varepsilon''' \frac{\partial^2 \chi'}{\partial x \partial z} + \frac{c'_{22}}{c'_{44}} \left(1 + \frac{2}{\varepsilon} - 2 \frac{c'_{44}}{c'_{22}} \sigma_z \right) z \right\} \right] \quad (38) \end{aligned}$$

Integration of Eq.(38) over the cross section and the relation of $S_2 = \partial u_y / \partial y$ yield the displacement u_y as

$$\begin{aligned} u_y = & x f_z(y) - \frac{Q}{c'_{22} I_z} \left\{ \varepsilon''' \chi' + \frac{1}{4} \frac{c'_{22}}{c'_{44}} \right. \\ & \times \left. \left(1 + \frac{2}{\varepsilon} - 2 \frac{c'_{44}}{c'_{22}} \sigma_z \right) x z^2 \right\} \quad (39) \end{aligned}$$

where the function $f_z(y)$ is a polynomial whose exact form is dependent upon the end conditions of the beam and which has a form of $\alpha y + \beta$ (α, β : constant) to satisfy $\partial S_4 / \partial y = 0$. The derivative of displacement u_y makes it possible to calculate $V(y)$ and $\Phi(y)$ from the relation of Eqs.(5) and (6). From the second relation of Eq.(7), the residual displacement v_y is transformed into

$$\begin{aligned} v_y = & u_y - V(y) - x \Phi(y) \\ = & x f_z(y) - \frac{Q}{c'_{22} I_z} \left\{ \varepsilon''' \chi' + \frac{1}{4} \frac{c'_{22}}{c'_{44}} \right. \\ & \times \left. \left(1 + \frac{2}{\varepsilon} - 2 \frac{c'_{44}}{c'_{22}} \sigma_z \right) x z^2 \right\} - \frac{1}{A} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} u_y dx dz \\ & - \frac{x}{I_z} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} x u_y dx dz \quad (40) \end{aligned}$$

In Eq.(36) we neglect the torsional term, namely

$$\tau \left(-z + \varepsilon'' \frac{\partial \phi'}{\partial x'} \right) = 0 \quad (41)$$

Substitution of the T_6 and $\partial v_y / \partial x$ into the denominator of the right side in Eq.(13) gives

$$\begin{aligned} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} \left(T_6 - c'_{66} \frac{\partial v_y}{\partial x} \right) dx dz = & \frac{Q}{K} \\ = & \frac{c'_{66}}{c'_{22}} \frac{Q}{I_z} \left[\sigma_z I_x - \frac{1}{4} \left\{ \frac{c'_{22}}{c'_{44}} \left(1 + \frac{2}{\varepsilon} \right) I_x \right. \right. \\ & + \left. \left. \frac{c'_{22}}{c'_{66}} \left(1 - \frac{2}{\varepsilon} \right) I_z \right\} - \frac{A}{I_z} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} \varepsilon''' x \chi' dx dz \right] \quad (42) \end{aligned}$$

where,

$$I_x = \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} z^2 dx dz$$

Thus, we find that the denominator of the right side in Eq.(13) is given as a function of the harmonic function χ' .

4. Derivative of harmonic function χ'

The equations of the boundaries are two of $x' = \pm x'_0$ ($x'_0 = x_0 / \varepsilon''$) and $z' = \pm z'_0$ ($z'_0 = z_0 / \varepsilon'$). From Eq.(33) the boundary condition at $x' = \pm x'_0$ is

$$\begin{aligned} \frac{\partial \chi'}{\partial x'} = & \frac{1}{2} \left(1 - \frac{1}{2} \varepsilon \right) x_0'^2 - \frac{1}{2} \left(1 + \frac{1}{2} \varepsilon - 2 \varepsilon' \sigma_z \right) z_0'^2 \\ & (-z'_0 < z' < z'_0) \quad (43) \end{aligned}$$

while the boundary condition at $z' = \pm z'_0$ is

$$\begin{aligned} \frac{\partial \chi'}{\partial z'} = & \left\{ \mp \left(1 + \frac{1}{2} \varepsilon \right) z_0' - \frac{c'_{42}}{c'_{44}} (2y_0 - y) \right\} x' \\ & (-x'_0 < x' < x'_0) \quad (44) \end{aligned}$$

We introduce a new function $\bar{\chi}$ by the equation

$$\bar{\chi} = \chi' - \frac{1}{6} \left[\left(1 + \frac{1}{2} \bar{\epsilon} \right) x'^3 - 3 \left\{ \left(1 + \frac{1}{2} \bar{\epsilon} \right) x' z'^2 + \frac{2C'_{42}}{\bar{c}_{44}} (2y_0 - y) x' z' \right\} \right] \quad (45)$$

Then $\bar{\chi}$ is a plane harmonic function within the rectangle, $\partial \bar{\chi} / \partial z'$ vanishes at $z' = \pm z'_0$, and the condition at $x' = \pm x'_0$ becomes

$$\frac{\partial \bar{\chi}}{\partial x'} = -\frac{1}{2} \bar{\epsilon} x'^2 + \bar{\epsilon}' \sigma_z z'^2 + \frac{C'_{42}}{\bar{c}_{44}} (2y_0 - y) z' \quad (46)$$

Now when $-z'_0 < z' < z'_0$, the function z'^2 can be expanded in a Fourier's series⁽⁵⁾ as follows:

$$z'^2 = \frac{z_0'^2}{3} + \frac{4z_0'^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi z'}{z_0'} \quad (47)$$

Hence, $\bar{\chi}$ can be expressed in the form

$$\begin{aligned} \bar{\chi} = & \left(-\frac{1}{2} \bar{\epsilon} x'^2 + \frac{1}{3} \bar{\epsilon}' \sigma_z z'^2 \right) x' \\ & + \frac{C'_{42}}{\bar{c}_{44}} (2y_0 - y) x' z' + \bar{\epsilon}' \sigma_z \frac{4z_0'^3}{\pi^3} \\ & \times \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \frac{\sinh \frac{n\pi x'}{z_0'}}{\cosh \frac{n\pi x_0'}{z_0'}} \cos \frac{n\pi z'}{z_0'} \end{aligned} \quad (48)$$

and, by means of this relation and Eq.(45), χ' can be written down, so that the harmonic function χ' which includes the torsional term becomes

$$\begin{aligned} \chi' = & \frac{1}{\bar{\epsilon}'''} \left\{ \left(-\frac{1}{2} \frac{C'_{22}}{C'_{66}} x_0'^2 + \frac{1}{3} \sigma_z z_0'^2 \right) x \right. \\ & + \frac{1}{6} \left(\frac{1}{2} \frac{C'_{22}}{C'_{66}} + \frac{\bar{c}_{44}}{C'_{66}} \frac{1}{\bar{\epsilon}'} \right) x^3 \\ & - \frac{1}{2} \left(\frac{1}{2} \frac{C'_{22}}{\bar{c}_{44}} + \frac{1}{\bar{\epsilon}'} \right) x z^2 \left. \right\} + \frac{\sigma_z}{\bar{\epsilon}'''} \frac{4z_0'^3}{\pi^3} \\ & \times \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \frac{\sinh \left(n\pi \sqrt{\frac{\bar{c}_{44}}{C'_{66}}} \frac{x}{z_0'} \right)}{\cosh \left(n\pi \sqrt{\frac{\bar{c}_{44}}{C'_{66}}} \frac{x_0'}{z_0'} \right)} \cos \frac{n\pi z}{z_0'} \end{aligned} \quad (49)$$

The derivative of the harmonic function χ' makes it possible to calculate the shear coefficient K .

5. Calculation of shear coefficient K and its calculated values

From the relation of Eq.(42) the shear coefficient K is given as

$$K = \frac{c'_{22} I_z}{c'_{66} \left[\sigma_z I_x - \frac{1}{4} \left\{ \frac{C'_{22}}{\bar{c}_{44}} \left(1 + \frac{2}{\bar{\epsilon}} \right) I_x + \frac{C'_{22}}{C'_{66}} \left(1 - \frac{2}{\bar{\epsilon}} \right) I_z \right\} - \frac{A}{I_z} \int_{-z_0}^{z_0} \int_{-x_0}^{x_0} \bar{\epsilon}''' x \chi' dx dz \right]} \quad (50)$$

As a result of calculation, we find that the shear coefficient K in Timoshenko's beam theory for trigonal crystals of rectangular cross section, derived using Cowper's definition is

$$K = \frac{10c'_{22}}{11c'_{22} + \sqrt{2} \bar{c}_{44} (\bar{c}_{44} + c'_{66})} \quad (51)$$

As is apparent from Eq.(51), K of the rectangular cross section is independent of a thickness-to-width ratio. In case of an isotropic material, when Poisson's

ratio is taken as σ , the constants c'_{22} , $\bar{c}_{44}$ and c'_{66} have the relation

$$\bar{c}_{44} = c'_{66} = \frac{c'_{22}}{2(1+\sigma)} \quad (52)$$

Substituting this into Eq.(51), the shear coefficient K for the isotropic material becomes

$$K = \frac{10(1+\sigma)}{12+11\sigma} \quad (53)$$

This result agrees completely with that obtained by G. R. Cowper⁽²⁾. The following shows an example of the calculated results. In the calculation the constants of Refs.^{(6),(7)} are used.

Figure 2 shows the relationship of the shear coefficient K and a cut angle θ rotated about the x -axis. K varies with the cut angle θ and has a value of 0.796 - 0.867 when the cut angle θ varies from -90° to $+90^\circ$.

Figure 3 shows the relationship of K and θ for lithium tantalate rotated about the x -axis. K has a value of 0.834 - 0.857 when the cut angle θ varies from -90° to $+90^\circ$.

Figure 4 shows the relationship of K and θ for lithium niobate rotated about the x -axis. K has a

Fig. 2 Relationship of the shear coefficient K and a cut angle θ rotated about the x -axis

Fig. 3 Relationship of the shear coefficient K and a cut angle θ for lithium tantalate rotated about the x -axis

Table 1 Comparison of our values of K for quartz crystal, lithium tantalate, and lithium niobate, and those⁽²⁾ for isotropic material obtained by other authors

	K
Timoshenko	0.667
Mindlin	0.822
Goodman ($\sigma = 1/3$)	0.870
Roark	0.833
Cowper ($\sigma = 0$)	0.833
($\sigma = 0.3$)	0.850
($\sigma = 1/2$)	0.870
The author	0.796~0.867 for Q.C. 0.834~0.857 for L.T. 0.831~0.863 for L.N

Fig. 4 Relationship of the shear coefficient K and a cut angle θ for lithium niobate rotated about the x -axis

value of 0.831–0.863 versus a variation of the above-mentioned cut angle.

Table 1 compares our values of K for quartz crystal (Q. C.), lithium tantalate (L. T.), and lithium niobate (L. N.) with those for the isotropic material obtained by other authors. With the exception of Timoshenko's value, the foregoing results almost include those obtained by other authors and agree fairly well with each other.

6. Conclusions

In this paper, the shear coefficient K in Timoshenko's beam theory for trigonal crystals of the rectangular cross section that are anisotropic has

been derived using Cowper's definition. It was shown that the shear coefficient K is given as a function of the transformed elastic stiffness constants c'_{22} , $\bar{c}_{44}$, c'_{66} only and is independent of a thickness-to-width ratio of the cross section. In addition, K for quartz crystal, lithium tantalate, and lithium niobate versus a cut angle θ was calculated, so that it was found to be very dependent upon the cut angle θ .

For the coming subject, this K will be applied to a vibration analysis of flexural mode trigonal crystal resonators, and the influence by K will be clarified on resonance frequency and frequency temperature behavior for the flexural mode resonators.

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