Bifurcations of Higher Subharmonics and Chaos in a Forced Vibratory System with an Asymmetrical Restoring Force*

Masaharu KURODA**, Mikio NAKAI***,
Takeshi HIKAWA**** and Yutaka MATSUKI*****

In this paper, we describe the behavioral characteristics of a nonlinear oscillator derived from gear-meshing vibration, which exhibits various successive bifurcations ending in chaos. There are two types of sudden change from a chaotic to a periodic attractor, one of which is called "hysteresis". In a period-doubling bifurcation, a heteroclinic connection between the outset of an inversely unstable fixed point of period $2n$ and the inset of an inversely unstable fixed point of period n is necessary to generate an n -band chaotic attractor. If a directly unstable fixed point exists in the vicinity of any band attractor and the attractor collides with the inset of the directly unstable fixed point, the attractor expands due to an "interior catastrophe", and transforms into a one-band attractor. In the bifurcation diagram, periodic subharmonic oscillations appear discontinuously as "windows" belonging to the same "family". These "families" are divided into two types: type-I, in which fold bifurcation occurs repeatedly, and type-II, in which fold bifurcation occurs only at both ends of the solution curve. Depending upon whether or not both of the chaotic attractors exist in the same area of the phase plane, either before or after the bifurcation, the discontinuous bifurcation of the chaotic attractor is referred to as an "interior catastrophe" or "hysteresis", respectively.

Key Words: Nonlinear Vibration, Chaos, Gear, Piecewise Linear System, Window, Hysteresis, Interior Catastrophe, Heteroclinic Point, Band Attractor

1. Introduction

There are many piecewise linear equations of motion which describe physical phenomena such as the vibration caused by collision or that occurring in machinery containing gears⁽¹⁾⁻⁽⁴⁾. Since most of these equations describe chaotic vibration, it is necessary to investigate the generation mechanism of chaotic

vibration thoroughly. This study is concerned with gear-meshing vibration, which is well known as a vibratory system in which chaos is produced as a result of collision^{(5),(6)}. First, the equation of motion describing a vibratory system with a piecewise-linear asymmetric restoring force is derived by considering the vibration in the circumferential direction of the gear as a function of the mesh error. In the first half of this study, the bifurcation structure below the first resonance of the gear-meshing vibration was investigated extensively, focusing on various bifurcation processes, in particular, on band attractor and "interior catastrophe" processes which lead from period doubling to chaos. As a result, almost all bifurcations and chaos-producing processes have been clarified by comparing analytical solutions with numerical solutions. The latter half of the study deals with bifurcation phenomena occurring in the range of gear rotation number around and above the first resonance. Around the first resonance, it is shown that numerous "windows" exist between the chaotic regions, which arise from the appearance of various

* Received 4th September, 1995. Japanese originals: Trans. Jpn. Soc. Mech. Eng., Vol. 61, No. 583, C (1995), pp. 808-814 (Received 24th August, 1994); Vol. 61, No. 583, C (1995), pp. 815-822 (Received 26th August, 1994).

** Sound & Vibration Div., Mechanical Engineering Laboratory, MITI, 1-2 Namiki, Tsukuba Science City, Ibaraki 305, Japan

*** Department of Precision Engineering, Kyoto University, Yoshidahon-machi, Sakyo-ku, Kyoto City, Kyoto 606-01, Japan

**** DAIKIN Co., Ltd., 1304 Kanaoka-cho, Sakai City, Osaka 591, Japan

***** YAMAHA Co., Ltd., 10-4 Nakazawa-cho, Hamamatsu City, Sizuoka 430, Japan

super/subharmonic vibrations. However, these super/subharmonic vibrations never occur at random. We also report that super/subharmonic vibrations of the same period belong to the same "family", and these families can be divided into two types. Finally, above the first resonance, it is proved that if another directly unstable fixed point exists in the vicinity of the 2^{n-1} -band attractor into which a 2^n -band attractor transforms without the existence of a fold-bifurcated directly unstable fixed point, the attractor abruptly transforms into a one-band attractor.

2. Equation of Motion for Gear-Meshing Vibration

To take account of the gear-tooth separation occurring during the rotation of gears with nonzero mesh, it is assumed that a gear tooth does not make contact with the tooth flank opposite to its meshing tooth during the separation time and that the gear stiffness, $\bar{K}$, remains constant over one period of gear rotation. First, the dimensionless equation of motion describing the gear-meshing vibration in the circumferential direction is expressed as⁽⁷⁾

$$\frac{d^2\xi}{d\tau^2} + c\frac{d\xi}{d\tau} + \bar{K}\eta H(\eta) = 1, \quad (1)$$

where $\eta = \xi - B \cos(\omega\tau)$; $\xi = \bar{K}X/W$; $B = e_0\bar{K}/W$; $c = D/\sqrt{M\bar{K}}$; $\tau = \omega_0 t$; $\omega_0 = \sqrt{\bar{K}/M}$; $\omega = \omega_z/\omega_0$; $\omega_z = 2\pi/T_z$; X is the reduced displacement of the relative rotational angle between the driving gear and the driven gear on the line of action; W is the static mass in the direction normal to the tooth face; e_0 is the amplitude in the gear-mesh error which is approximated as a sinusoidal wave; D is the damping coefficient; t is the time; ω_0 is the natural angular frequency; M is the equivalent mass of the gear on the line of action; and T_z is the gear-meshing period. $H(\eta)$ is a step function describing the gear-face separation for $\eta < 0$. $\bar{K}$ is set equal to 1.0. Using the variable conversion $x = \xi - B \cos(\omega\tau) - 1$, Eq. (1) gives

$$\ddot{x} + c\dot{x} + g(x) = Bc\omega \sin(\omega\tau) + B\omega^2 \cos(\omega\tau), \quad (2)$$

where the dot denotes differentiation with respect to τ . $g(x)$ is defined as

$$g(x) = (x+1)H(x+1) - 1, \quad (3)$$

which implies that

$$g(x) = \begin{cases} x & (x \geq -1) \\ -1 & (x < -1) \end{cases} \quad (4)$$

Therefore $g(x)$ is the asymmetrical restoring force caused by the gear-tooth separation.

Selecting region A in which $x \geq -1.0$ and region B in which $x < -1.0$, Eq. (2) can be solved, because a strictly analytical solution exists in both regions A and B . The general solutions of Eq. (2) in regions A

and B are respectively given by

$$x(\tau) = e^{-\frac{c}{2}\tau} (A_1 \sin(\mu\tau) + A_2 \cos(\mu\tau)) + E \sin(\omega\tau) + F \cos(\omega\tau), \quad (5)$$

$$x(\tau) = B_1 + B_2 e^{-c\tau} + \frac{1}{c}\tau - B \cos(\omega\tau), \quad (6)$$

where $\mu = \sqrt{4 - c^2}/2$, $E = Bc\omega/(\omega^2 c^2 + (1 - \omega^2)^2)$, and $F = B\omega^2(1 - \omega^2 - c^2)/(\omega^2 c^2 + (1 - \omega^2)^2)$. The constants A_1 , A_2 , B_1 , and B_2 are determined from the initial conditions and the conditions necessary to obtain a periodic solution.

The periodic solutions are divided into two categories according to the number of times the waveform crosses the boundary line ($x = -1$), the period number and one of the initial conditions, as follows⁽⁴⁾.

nAm : an m -periodic solution starting in region A at $\tau = \tau_0$, which crosses the boundary ($x = -1$) $2n$ times over m periods.

nBm : an m -periodic solution starting in region B at $\tau = \tau_0$, which crosses the boundary ($x = -1$) $2n$ times over m periods.

The nAm and nBm periodic solutions can be obtained by following the procedure given below. First, the number of times, $2n$, that the solution waveform crosses the boundary ($x = -1$) in m periods can be denoted by $\tau_1, \dots, \tau_{2n}$. Assuming that the solution at τ which satisfies $\tau_{k-1} \leq \tau < \tau_k$ (where $1 \leq k \leq 2n+1$, $\tau_0 = 0$, $\tau_{2n+1} = mT$, $T = 2\pi/\omega$, and T is the period) is represented as $x_k(\tau)$, then the necessary conditions that must be satisfied for the solution to be m -periodic are

$$\begin{aligned} x_1(0) &= x_{2n+1}(mT) = x_0, \\ \dot{x}_1(0) &= \dot{x}_{2n+1}(mT) = \dot{x}_0 \\ x_k(\tau_k) &= x_{k+1}(\tau_k) = -1, \\ \dot{x}_k(\tau_k) &= \dot{x}_{k+1}(\tau_k) \quad (1 \leq k \leq 2n), \end{aligned} \quad (7)$$

where k is an integer. As mentioned above, depending on whether x_0 exists in region A or B , the m -periodic solution is classified as being nAm or nBm . Therefore, there exist $2n+2$ unknown parameters, x_0 , $\dot{x}_0$, $\tau_1, \dots, \tau_{2n}$, and $4n+2$ unknown coefficients for each x_k . A_1 , A_2 , B_1 and B_2 can be determined from the $6n+4$ equations given by Eq. (7).

Finally, using the variational function $y(\tau)$ to determine the stability of the periodic solutions, the first-order variational equation derived from Eq. (2) is

$$\ddot{y} + c\dot{y} + a(\tau)y = 0, \quad (8)$$

where

$$a(\tau) = \begin{cases} 1 & (x \geq -1) \\ 0 & (x < -1) \end{cases} \quad (9)$$

According to the eigenvalues derived from the characteristic equation (Eq. (8)), it can be determined whether the periodic solution is stable or unstable. If we denote the eigenvalues by λ_1 and λ_2 ($|\lambda_1| > |\lambda_2|$), then $|\lambda_1| < 1$ and $|\lambda_2| < 1$ give a completely stable solution, $|\lambda_1|$

>1 and $|\lambda_2|>1$ give a completely unstable solution, $\lambda_1>1>\lambda_2>0$ give a directly unstable solution, and $0>\lambda_2>-1>\lambda_1$ give an inversely unstable solution. Moreover, at the boundary between the inversely unstable solution and the completely stable solution, $\lambda_1=-1$ and $\lambda_2=-e^{-cmT}$, and $\lambda_1=1$ and $\lambda_2=e^{-cmT}$ at the boundary between the directly unstable solution and the completely stable solution.

3. Bifurcations below the First Resonance

3.1 Bifurcation diagram over the whole range

First, the bifurcation phenomena produced by Eq. (2) are investigated under the conditions $B=3.0$ and $c=0.08$. Figure 1(a) shows a bifurcation diagram in which ω increases from 0.35 to 1.85 with the initial condition $(x_0, \dot{x}_0)=(0, 0)$ at $\omega=0.35$. In this figure, the values of x at $\tau \bmod 2\pi=0$ are displayed as dots. Starting from a 1-periodic solution and progressing through various periodic solutions and chaos, x shows a complicated bifurcation structure ending with another 1-periodic solution at $\omega=1.78$. The maximum Lyapunov exponent $\lambda^{(8)}$ and the winding number (rotation number⁽⁸⁾) are shown in Figs. 1(b) and 1(c), respectively. When a periodic solution is nAm or nBm , the winding number takes the value n/m . Figure 1 shows that a periodic solution can exist when the winding number is rational and the Lyapunov exponent is negative, and that chaos can appear when the Lyapunov exponent is positive.

Fig. 1 (a) Bifurcation diagram, (b) maximum Lyapunov exponent and (c) winding number in the range $0.35 \leq \omega \leq 1.85$

Although it has been reported that a quasi-periodic solution can exist when the winding number is irrational and the Lyapunov exponent is zero, there is no frequency for which the Lyapunov exponent is zero in this nonlinear vibratory system. The above facts imply that the existence of a periodic solution can be detected using the maximum Lyapunov exponent, and the type of periodic solution can be distinguished by means of the winding number.

3.2 Bifurcation diagram below the first resonance

In this section, the details of bifurcation phenomena, in particular those appearing in the range $\omega=0.35 \sim 0.87$, are investigated. Figure 2 shows the bifurcation diagram, and analytical solution curves indicating the values of fixed points at $\tau \bmod 2\pi=0$ in the (ω, x) plane obtained from Eq. (2). In the figure, ${}^iD^j$, ${}^iI^j$ and ${}^iS^j$ denote, respectively, a directly unstable fixed point of the i -th period- j solution, an inversely unstable fixed point of the i -th period- j solution and a completely stable fixed point of the i -th period- j solution. The index i is omitted when its value is obvious from the notation. Moreover, in order to make the figure easier to understand, for the period-6 solutions (3B6, 4B6) and the period-3 solution (2B3), only one of the

Fig. 2 Bifurcation diagram (upper part) and solution curves (lower part, solid line: completely stable fixed point, broken line: directly unstable fixed point, dotted line: inversely unstable fixed point, all at $\tau \bmod 2\pi=0$) in the range $0.35 \leq \omega \leq 0.87$

fixed points is plotted in each case. From Fig. 2, the fact that the stable solution curves (fixed points) obtained by the analytical solutions correspond to the presence of periodic solutions in the bifurcation diagram provides further evidence for the validity of this analysis. Furthermore, the transformation processes from periodic solutions to chaotic solutions occur via several patterns rather than via only one pattern, as expected. For further insight into the transformation from a periodic solution to a chaotic solution, the bifurcation phenomena ranging from ① to ④ in Fig. 2 are now investigated.

3.2.1 Range ①: $0A1 \rightarrow 1A1 \rightarrow 2A2 \rightarrow 1A2 \rightarrow 2$ -band attractor $\rightarrow$ 1-band attractor $\rightarrow 1B1$ Figure 3 shows a magnification of both the solution curves and the bifurcation diagram corresponding to the chaotic regime in the vicinity of ① in Fig. 2. In this study, the outset and the inset of invariant curves from a fixed point are called the “ α -branch” and the “ ω -branch”, respectively.

(1) Fold-bifurcated $1A2 \rightarrow 2$ -band attractor $\rightarrow$ 1-band attractor

Figure 4(a) shows that the S^1 of a $1B1$ is the only solution which can exist outside the area surrounded by the ω -branches emanating from D^1 at $\omega=0.5438$ (corresponding to ① in Fig. 3). From Fig. 4(b), which is a magnification of Fig. 4(a), the $^1S^2$ s are solutions of and the $^1D^2$ s are fixed points of a $1A2$ which undergoes a fold bifurcation at $\omega=0.544$. As ω increases to the value at which these fixed points coincide, the $1A2$ solution must disappear. After the disappearance of these fixed points, the ensuing bifurcation behavior is determined by which basin of the attractor the Poincaré point lies in. In the case of the bifurcation diagram in Fig. 3, as shown in Fig. 4(c) (corresponding to ② in Fig. 3), the α -branches and ω -branches from the $^1I^2$ s of a $2A2$ located inside the ω -branches emanating from the D^1 make homoclinic intersections with each other to produce a 2-band attractor. However, α -branches from the $^1I^2$ s and ω -branches from the $^1I^1$ of the $1A1$ still do not touch each other, as is seen in the magnification in Fig. 4(c); no heteroclinic intersection exists. Therefore, a 1-band attractor cannot be generated, as explained in the next section. However, as ω increases to a value at which an α -branch from the $^1I^2$ s and an ω -branch from the $^1I^1$ have a heteroclinic intersection, a 1-band attractor can appear.

(2) 1-Band attractor $\rightarrow 1B1$ periodic solution

When $\omega=0.5507$, where the 1-band attractor intersects the D^1 in Fig. 4(d) (corresponding to ③ in Fig. 3), the ω -branches from the D^1 can no longer comprise the boundary between the basin of the period-1 solution and the basin of the 1-band

Fig. 3 Magnification of the solution curves and bifurcation diagram of the chaotic part in range ① of Fig. 2

Fig. 4 Invariant curves for $1A2 \rightarrow 2$ -band attractor $\rightarrow$ 1-band attractor $\rightarrow 1B1$ in range ① of Fig. 2 (solid line: ω -branch, bold solid line: α -branch, broken line: ω -branch from $^1I^2$, dot: Poincaré point, $\circ$: unstable fixed point, $\bullet$: stable fixed point)

attractor. Therefore, no solution except the S^1 of the $1B1$ can survive for all initial conditions after this intersection has occurred. A bifurcation following the process outlined above, in which an attractor transforms into an S^1 located far from the attractor in the phase plane, is called “hysteresis”⁽⁹⁾⁻⁽¹²⁾.

3.2.2 Range ②: $1B1 \rightarrow$ period-doubling bifurcation $\rightarrow$ chaos $\rightarrow 1B2$ Figure 5(a) shows a bifurcation diagram in the vicinity of ② in Fig. 2 which shows the evolution from a periodic to a chaotic solution. As demonstrated in the Poincaré maps in Figs. 5(b)~(d), the occurrence of a series of band attractors (a 4-band attractor in Fig. 5(b), a 2-band attractor in Fig. 5(c), and a 1-band attractor in Fig. 5(d)) is one of the characteristics of period-doubling

Fig. 5 (a) Bifurcation diagram, (b) Poincaré map: 4-band attractor, (c) Poincaré map: 2-band attractor and (d) Poincaré map: 1-band attractor, in range ⑥ of Fig. 2

Fig. 6 Invariant curves and fixed points for the 4-band $\rightarrow$ 2-band $\rightarrow$ 1-band attractor in range ⑥ of Fig. 2((a)~(d) bold solid lines: α -branches from the I^4 s, ${}^2I^2$ s and ${}^2I^1$, broken line: ω -branch from the ${}^2I^1$, dotted line: ω -branch from the ${}^2I^2$ s and solid line: ω -branch from the I^4 s, (e) solid line: ω -branch from the ${}^2I^1$ and bold solid line: α -branch from the ${}^2I^2$ s)

bifurcation. This phenomenon is closely related to the invariant curves from an inversely unstable fixed point with each period.

(1) 4-Band attractor $\rightarrow$ 2-band attractor $\rightarrow$ 1-band attractor

Figures 6(a)~(c) show invariant curves around the two fixed points on the left in Fig. 5(b) in which a 4-band attractor appears at $\omega=0.7195$ (corresponding to ④ in Fig. 5(a)). Figure 6(a) demonstrates that the α -branches from the I^4 s intersect some of the ω -branches from the I^4 s while the α -branches do not intersect the ω -branches from the ${}^2I^1$ and ${}^2I^2$ s. Moreover, comparing Figs. 5(b)~(d) with Figs. 6(a)~(c), it is apparent that the 4-band attractor existing in a stable form is contained within the area bounded by the α -branches from the ${}^2I^2$ s and the ${}^2I^1$. Some invariant curves from the ${}^2I^2$ s and the ${}^2I^1$ make homoclinic intersections at $\omega=0.7195$, as is obvious from Figs. 6(b) and (c). Nevertheless, the parts consisting of the α -branches from the I^4 s can only exist in a stable form. Figure 6(d) shows the invariant curves which produce the 2-band attractor at $\omega=0.722$ (corresponding to ⑤ in Fig. 5(a)). In this figure, it is shown that a heteroclinic intersection of the α -branches from the I^4 s with the ω -branches from the ${}^2I^2$ s changes the 4-band attractor (consisting of two parts separated by the ω -branches from the ${}^2I^2$ s) into a 2-band attractor. Because the α -branches from the ${}^2I^2$ s never touch the ω -branches from the ${}^2I^1$ while the invariant curves from the ${}^2I^1$ make homoclinic intersections (i.e., no heteroclinic intersection exists in this case as well as in Fig. 6(c)), there cannot be a 1-band attractor. In Fig. 6(e), since ω -branches from the ${}^2I^1$ and α -branches from the ${}^2I^2$ s have heteroclinic intersections at $\omega=0.7224$ (corresponding to ⑥ in Fig. 5(a)), a 1-band attractor is able to form a stable structure. In the simulations described above, it has been proved that even when an α -branch and an ω -branch from an inversely unstable fixed point of period n have homoclinic intersections, an n -band attractor still might not appear. Furthermore, it is clear that the existence of a heteroclinic intersection between an α -branch from an inversely unstable fixed point I^{2n} of period $2n$ and an ω -branch from an inversely unstable fixed point I^n of period n is needed to produce an n -band attractor.

In order to understand the observation that the 1-band attractor transforms into a 1B2 periodic solution after it expands rapidly around $\omega=0.725$, the relevant invariant curves are shown in Figs. 7(a)~(d). In these figures, the ω -branches from the ${}^2D^2$ s (see Fig. 2) which fold-bifurcate at $\omega=0.652$ are illustrated, and it is demonstrated that the domain surrounded by the ω -branches from the ${}^2D^2$ s becomes the basin of the

$^2S^2$ s of the 1B2 periodic solution.

(2) Enlargement of the 1-band attractor

Figure 7(a) is of a larger area than Fig. 6(e), and shows the α -branches from the $^2I^1$ and the ω -branches from one of the D^3 s of the 2B3 periodic solution obtained using the solution curves in Fig. 2, although these do not appear in the bifurcation diagram in Fig. 2. At $\omega=0.7224$ (Fig. 7(a)), the ω -branches from the D^3 never intersect the α -branches from the $^2I^1$. As ω increases to $\omega=0.725$ (Fig. 7(b), corresponding to ⑦ in Fig. 5(a)), a rapid expansion of the α -branches from the $^2I^1$ occurs due to their passing through the D^3 s and the I^3 s, i.e., a so-called "interior catastrophe" takes place. When the ω -branches from the D^3 s and α -branches from the $^2I^1$ make heteroclinic intersections, the attractor becomes a part of the α -branches from the D^3 s leading to rapid enlargement of the attractor.

(3) Chaos $\rightarrow$ 1B2 periodic solution

As ω increases further, the attractor approaches the ω -branches from the $^2D^2$ s at $\omega=0.7387$ (Fig. 7(c)), in which the dots indicate Poincaré points, corresponding to ⑧ in Fig. 5(a)). Figure 7(d) shows the attractor and invariant curves at $\omega=0.739$ (corresponding to ⑨ in Fig. 5(a)), where $\times$ denotes the Poincaré map after 12 000 periods. The Poincaré map indicates the manner in which the attractor transforms into the periodic solution of the $^2S^2$ s via the $^2D^2$ points as soon as one of the Poincaré points of the attractor lies on an ω -branch from the $^2D^2$ s. In this way, even though the attractor is in a chaotic state

Fig. 7 Invariant curves leading to the 1B2 periodic solution after the 1-band attractor expands in range ⑥ (solid line: ω -branch from $^2D^2$, bold solid line: α -branch from $^2I^1$, dotted line: ω -branch from D^3 , dot: Poincaré point)

initially, after some time it moves into the basin of the period-2 solution, and it finally converges to the period-2 solution denoted by $\bullet$. In this case, the transformation to the periodic solution turns out to be a hysteresis phenomenon, which is similar to range ①.

3.2.3 Range ③: 1B2 $\rightarrow$ period-doubling bifurcation $\rightarrow$ chaos $\rightarrow$ period-6 solution $\rightarrow$ period-doubling bifurcation $\rightarrow$ chaos The bifurcation diagrams around $\omega=0.80$ and $\omega=0.83$ (where the window of a period-6 solution appears) are shown in Fig. 8(a).

Fig. 8 Bifurcation diagrams and invariant curves for the 2-band attractor $\rightarrow$ 4B6 periodic solution, and for the 6-band attractor $\rightarrow$ 1-band attractor in range ③ of Fig. 2 (solid line: ω -branch, bold solid line: α -branch, dot: Poincaré point)

(1) 2-Band attractor $\rightarrow$ expanded 2-band attractor

Figure 8(b) shows only two ω -branches from the ${}^1D^6$ s of a 3B6 at $\omega=0.815$ (corresponding to ⑩ in Fig. 8(a)), which transforms into a 2-band attractor after period-doubling bifurcation. Although the attractor is located close to the ${}^1D^6$ points and the ω -branches from the ${}^1D^6$ s, as shown in the magnification, the attractor does not intersect the ω -branches from the ${}^1D^6$ s, so it cannot reach any ${}^2D^2$ points. However, as ω increases to $\omega=0.8175$ (corresponding to ⑪ in Fig. 8(a)), the attractor is enlarged discontinuously upon intersecting the ${}^1D^6$ points, as shown in Fig. 8(c) (ω -branches from the ${}^2D^2$ s). Furthermore, the attractor expands again upon intersecting ω -branches from the ${}^2D^2$ s. This implies that the α -branches from the ${}^1D^6$ s act as a catalyst for expansion of the attractor⁽¹³⁾.

(2) 2-Band attractor $\rightarrow$ 4B6 periodic solution

When $\omega=0.8312$ (Fig. 8(d), corresponding to ⑫ in Fig. 8(a)), the attractor has expanded completely, but it still remains in the form of a 2-band attractor; it never becomes a 1-band attractor. Figure 8(e) shows the α -branches from the ${}^2I^2$ s and the ${}^3I^2$ s, in addition to the ω -branches from the ${}^2I^1$. As shown in the magnifications, no heteroclinic intersection exists between the α -branches from the ${}^2I^2$ s and the ${}^3I^2$ s and the ω -branches from the ${}^2I^1$. Therefore, the chaotic attractor never forms a 1-band attractor. Moreover, the ${}^2S^6$ fixed points of the 4B6 solution which fold-bifurcate at $\omega=0.83122$ are also shown in Fig. 8(d). The ${}^2S^6$ points lie within the area of the attractor, and the attractor is transformed into the set of ${}^2S^6$ points of period 6 when ω exceeds 0.83122. In this situation, the attractor cannot undergo a “hysteresis” transformation. Since the range of ω in which the ${}^2S^6$ points exist is narrow, it is manifested as a “window”.

(3) 6-Band attractor $\rightarrow$ 1-band attractor

After period-doubling bifurcation, the 4B6 periodic solution becomes a 12-band attractor and then a 6-band attractor. As ω increases further, the 6-band attractor (the α -branches from the ${}^2I^6$ s) touches the ${}^2D^6$ points, or ω -branches from the ${}^2D^6$ s, at $\omega=0.8333$ (corresponding to ⑬ in Fig. 8(a); the fixed points ① and ② in Fig. 8(a) are evaluated only in Fig. 8(f)). This leads to a rapid expansion of the attractor, which forms a 1-band attractor at $\omega=0.83337$ (Fig. 8(g), corresponding to ⑭ in Fig. 8(a)). This assessment is confirmed by the fact that the attractor is enlarged at the ω value at which the directly unstable fixed points ${}^2D^6$ s come in contact with the attractor. The ${}^2D^6$ s are indicated by the dashed lines in Fig. 8(a). The “interior catastrophe” occurs in the manner mentioned above. Range ① in

the sequence chaos $\rightarrow$ 1-band attractor $\rightarrow$ period-2 solution (1B2) displays the reverse behavior to that of range ③.

3.3 Bifurcation structure from a period- n solution to a one-band attractor

As a consequence of the results obtained in the previous sections, but usually requiring the existence of an inversely unstable fixed point I^1 , a period- n solution transforms into a 1-band attractor via the following process: period- n solution $\rightarrow$ period doubling $\rightarrow \dots \rightarrow 2^3n$ -band attractor $\rightarrow 2^2n$ -band attractor $\rightarrow 2n$ -band attractor $\rightarrow n$ -band attractor $\rightarrow$ 1-band attractor. This process is designated “BA-1” here. However, if a directly unstable fixed point D^m exists in the vicinity of a band attractor which participates in this “BA-1” process, the band attractor expands as a result of the “interior catastrophe” effect. This occurs when the band attractor touches the D^m point or an ω -branch from the D^m , i.e., when a heteroclinic intersecting structure is formed; the attractor transforms into a 1-band attractor. Therefore this is the condition in which a directly unstable fixed point D^m of period m exists in the vicinity of one of the band attractors as a precursor to the 1-band attractor in the “BA-1” process. The attractor is simultaneously contained within the area bounded by the ω -branches from the D^m . This process is called “BA-2” here. An example of the above is when a ${}^2D^6$ of the 4B6 in the ③ range intersects the 6-band attractor. In this example, as a result of the heteroclinic intersection the attractor changes to a 1-band attractor, even though no heteroclinic point exists between the α -branches from the inversely unstable fixed points, I^2 s and I^4 s, and the ω -branches from the inversely unstable fixed point, I^1 . The 1B2 solution in the ① range is another example. However, in the case where no fold-bifurcated directly unstable fixed points exist in the range of ω for which the band attractors appear (for example, the fold-bifurcated 1B1 in the ⑥ range), the usual “BA-1” process is observed.

Furthermore, there is a case in which the 1-band attractor expands further after the “BA-1” process. This is when, after the 1-band attractor is formed, a directly unstable fixed point D^m of period m exists in the vicinity of the 1-band attractor, and simultaneously the attractor lies in the domain bounded by the ω -branches from the D^m s. In this instance, the attractor is enlarged due to the “interior catastrophe” caused by the intersection of the 1-band attractor with D^m points or their ω -branches. An example of this process is the attractor expansion resulting from the intersection of the 1-band attractor and the ω -branches from the D^3 fixed points of the 2B3 in the ⑥ range.

4. Bifurcations around the First Resonance

In this and the next sections we investigate the “windows” appearing around $\omega=0.86$, and the “interior catastrophes” occurring at the $\mathbb{M}$ and $\mathbb{N}$ points in Fig. 1.

4.1 Windows and families

4.1.1 Period-7 solutions and their families ($\omega=0.85\sim 0.95$) Figure 9(a) shows a bifurcation diagram with periodic solution trajectories corresponding to “windows” appearing in the range $0.85\leq\omega\leq 0.95$. The period-7 solutions and chaotic solutions appear alternately starting at $\omega=0.86$. The winding numbers are indicated in Figure 9(b) in order to enable the periodic solutions to be classified. As mentioned in section 3.1, any region where the wind-

Fig. 9 (a) Bifurcation diagram showing the trajectories of the periodic solutions, (b) the winding number and (c) the solution curves (bold solid line: completely stable fixed point, dotted line: inversely unstable fixed point, broken line: directly unstable fixed point) in the range $0.85\leq\omega\leq 0.95$

ing number is constant and rational is a region where a periodic solution can exist. This condition is called the “phase-locking condition”⁽¹⁴⁾. While somewhat noisy periodicity is observed in the solution trajectories around $\omega=0.89$, the winding numbers have rational values. Figures 9(a) and 9(b) show that period-7 solutions which include many super/subharmonic vibrations exist. The winding numbers of these solutions indicate that the number of times the trajectory goes around the equilibrium point during the seven periods decreases in the sequence $5/7\rightarrow 4/7\rightarrow 3/7$ with increasing ω in the range $0.85\leq\omega\leq 0.95$. In order to determine the mechanisms which allow many period-7 solutions to appear intermittently in this manner, strictly analytical solutions of Eq.(1) are obtained. Figure 9(c) shows the numerically calculated results for the fixed points of period 7 as a function of ω . These periodic solutions all belong to the nBm group. From a comparison of Fig. 9(a) with Fig. 9(c), it is apparent that the “windows” correspond to the bold solid curves, in other words, the fold-bifurcated completely stable fixed points in Fig. 9(c). Moreover, the “windows” can be generated without the hysteresis process, because the fold-bifurcated completely stable fixed points are located within the chaotic attractor before the appearance of the “windows”.

Fig. 10 The $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{C}$ families in the period-7 solution curves (bold solid line: completely stable fixed point, dotted line: inversely unstable fixed point, broken line: directly unstable fixed point). Subsequent figures have the same definition for each type of line.

Fig. 11 Solution curves for period 7 (enlargement)

In order to determine the bifurcation route of each periodic solution, we have plotted Figs. 10 and 11. Figure 10 shows only one of the fixed points among the period-7 solutions in the range $0.8 \leq \omega \leq 1.8$, and Fig. 11 shows an enlargement of the part of Fig. 10 in the range $0.85 \leq \omega \leq 0.95$. In these figures, some bold solid lines have been omitted if the region in which the completely stable fixed points exist is too narrow to display them, although completely stable fixed points must exist at both sides of the inversely unstable fixed points indicated by the dotted lines. From Figs. 10 and 11, we conclude the following.

(1) The various period-7 solutions appearing intermittently as “windows” in the bifurcation diagram comprise just a few “families”⁽¹⁵⁾.

(2) There are three families, (E), (F) and (G) in Figs. 10 and 11. These families can be categorized into two types: one in which fold-bifurcation is repeated several times, and one in which fold-bifurcation occurs only at both ends of the solution curve. In this study, the former is called type-I, and the latter is called type-II.

A **type-I family** consists of multifoldings⁽¹⁶⁾ in which fold bifurcations (saddle-node bifurcations) are iterated, as in the (E) and (F) parts. For instance, for the (E) family in Fig. 11 the solution curves form one family by the repetition of several multifoldings in the process: directly unstable fixed point of $5B7 \rightarrow$ fold bifurcation $\rightarrow$ completely stable fixed point of $5B7 \rightarrow$ inversely unstable fixed point of $4B7$ from that of $5B7 \rightarrow$ completely stable fixed point of $4B7 \rightarrow$ fold bifurcation $\rightarrow$ directly unstable fixed point of $3B7$ from that of $4B7 \rightarrow \dots$. Even though there are cases in which the solution curves apparently do not have this kind of bifurcation upon first inspection, in reality they do, as shown in enlargements (b) and (c) in Fig. 10. Because one of the eigenvalue is -1 at the boundary between a completely stable fixed point and an inversely unstable fixed point, the periodicity of the solution is doubled through flip-bifurcation at the boundary.

A **type-II family** consists of the solution curves which fold-bifurcate only at both ends, as in (G) at around $\omega = 0.905$ in Fig. 11. Here, the family consists of only the completely stable fixed points and the directly unstable fixed points of $4B7$.

4.1.2 Period-5 and -9 solutions and their families ($\omega = 0.95 \sim 1.20$) Figures 12(a) and (b) show calculated results for the bifurcation diagram and winding numbers for the ω range immediately above that in Fig. 9. In this figure, “windows” of period-5 and period-9 solutions appear. Figures 13 and 14 show calculated results for one of the fixed points in each of the period-5 and period-9 solutions, respectively. The

period-5 solutions in Fig. 13 still comprise (H) and (I) families. The (H) family, which consists of $1B5$ and $2B5$ with relatively small winding numbers, is a type-II family. On the other hand, the (I) family is a type-I family. In addition, the completely stable fixed

Fig. 12 (a) Bifurcation diagram with the trajectories of periodic solutions and (b) the winding number in the range $0.95 \leq \omega \leq 1.20$

Fig. 13 The (H) and (I) families in the period-5 solution curves

Fig. 14 The (J) family in the period-9 solution curves

points at ③, ④, ⑤ and ⑥ in the enlargement in Fig. 13 correspond to “windows” ③ in Fig. 9(b), and ④, ⑤ and ⑥ in Fig. 12(b), respectively. For the period-9 solutions in Fig. 14, the fixed points comprise the ① family which is type-I. However, in the ⑧ “window” in Fig. 12(b), it is apparent that the inversely unstable fixed points are missing, and that the completely stable fixed points and the directly unstable fixed points are related; that is, there is a region in which the connection of the solutions is slightly different from that usually seen in a type-I family.

4.1.3 Period-3, -4 and -6 solutions and their families

In the case of the period-4 solutions in

solution transforms into a 1-band attractor, and by which the 1-band attractor converges to a $2B6$ solution into which a $1B3$ solution bifurcated, can be explained by use of the invariant curves in Fig. 16(a) and (b). Only the invariant curves needed in the explanation below have been included in the figure. Besides the 4-band attractor indicated by the arrows, Fig. 16(a) shows the invariant curves at $\omega=0.970$ (corresponding to ⑮ in Fig. 12(a)). In addition, one of the 4 bands of the 4-band attractor and its associated ω -branches are shown in the enlargement in Fig. 16(a). The hatched area in this figure containing the $^3S^6$ s is the basin of the $^3S^6$ solution ($2B6$). As seen in the enlargement, the 4-band attractor never intersects the ω -branch from the $^2I^1$ at $\omega=0.970$. The 4-band attractor intersects the ω -branch from the $^2I^1$ at $\omega=$

0.9703 (corresponding to ⑮ in Fig. 12(a)). The intersection point is heteroclinic, thereby immediately changing the attractor into a 1-band attractor. However, the 1-band attractor enters the basin of the $^3S^6$ s (the ω -branches from the $^1D^3$ s are its boundary) in Fig. 16(b), requiring about 200 periods to do so, and then it must converge to the $^3S^6$ solution. As ω increases further, the $2B6$ (the period-doubled $1B3$) transforms, via a 6-band attractor resulting from a period-doubling bifurcation, to form a 3-band attractor.

4.2.3 3-Band attractor $\rightarrow$ 1-band attractor

When $\omega=0.99746$ (corresponding to ⑰ in Fig. 12(a)), the invariant curves in Fig. 16(c) show that this 3-band attractor is enlarged so that it is about to touch ω -branches from the $^1D^3$ s, after which it forms

Fig. 17 Bifurcation diagrams and invariant curves around ⑮ (solid line: ω -branch from $^2D^4$ s, dot: Poincaré point, ●: stable fixed point, ○: unstable fixed point)

heteroclinic intersections. Consequently, as ω increases further to $\omega=0.9975$ (corresponding to ⑩ in Fig. 12(a)), the attractor changes completely into a chaotic attractor characterized by a single band, as shown in Fig. 16(d). Thus the “interior catastrophe” promotes formation of a 1-band attractor in accordance with the “BA-2” process.

5. Bifurcations above the First Resonance

5.1 Expansion of the chaotic attractor in the ⑩ region

As is clear from Fig. 1, the chaotic attractor expands in the ⑩ region. Figure 17(a) shows an enlargement of the bifurcation diagram in the ⑩ region in Fig. 1. Only a chaotic attractor can be observed in this bifurcation diagram. However, a period-4 solution of $1B4$ (corresponding to the $^2S^4$ solution in Fig. 15) can coexist with the chaotic attractor in the vicinity of $\omega=1.18$ as shown by the invariant curves in Fig. 17(c). The area surrounded by the ω -branches emanating from the $^2D^4$ s is the basin of the period-4 solution, $^2S^4$ s. Therefore, the remaining area must be the basin of the chaotic attractor. A bifurcation diagram has been obtained using a calculation in which the initial condition is in the basin of the $1B4$ solution; the result is shown in Fig. 17(b). From this figure, the solution is obtained in the order as follows: period-4 solution $\rightarrow$ period doubling $\rightarrow \dots \rightarrow$ 8-band attractor $\rightarrow$ 4-band attractor. It then transforms rapidly into the 1-band attractor which coexists with the 4-band attractor at around $\omega=1.2196$. This transformation can be clarified as follows using the invariant curves at $\omega=1.218$ (corresponding to ②⑩ in Fig. 17(b)) and those at $\omega=1.2196$ (corresponding to ②⑪ in Fig. 17(b)). Figure 17(d) and an enlargement of it (Fig. 17(e)) show that the 4-band attractor and the 1-band attractor coexist at $\omega=1.218$, where the hatched area is the basin of the 4-band attractor. Neither attractor ever intersects the ω -branches from the $^2D^4$ s. However, when $\omega=1.2196$ (Fig. 17(f)), the 4-band attractor transforms into the 1-band attractor when it intersects the ω -branches from the $^2D^4$ s, as indicated by the crosses. When ω increases beyond 1.2196, the 1-band attractor is the only attractor which is generated, regardless of the initial conditions in the phase plane. Consequently, the bifurcation diagram in Fig. 17(b) becomes identical to that in Fig. 17(a) for $\omega>1.2196$. According to the procedure described above, a 4-band chaotic attractor can transform into a 1-band chaotic attractor, which exists in a different location in the phase plane; this is a “hysteresis”-like transformation^{(9),(12)}.

When ω increases further to $\omega=1.23$ (Fig. 17(g)),

corresponding to ②⑫ in Fig. 17(b)), the 1-band attractor intersects ω -branches from the $^2D^4$ s once again to form heteroclinic intersections; the size of the attractor increases at $\omega=1.235$ (Fig. 17(h)), corresponding to ②⑬ in Fig. 17(b)) and thus an “interior catastrophe” occurs. The bifurcations leading to the enlargement of the chaotic attractor in the ⑩ part of Fig. 1 behave identically to the global structure of this ⑩ part.

5.2 The chaotic attractor in the ⑪ region $\rightarrow 2B2 \rightarrow 1B1$

Finally, the ⑪ part of Fig. 1 is examined, where the solution transforms into a period-1 solution via a period-2 solution from a chaotic solution. Figure 18(a) shows a bifurcation diagram in which the ⑪ part is enlarged. This figure shows that the chaotic attractor changes abruptly into a 4-band attractor. The solution curves for this region are shown in enlargement ⑪ in Fig. 15. No directly unstable fixed points into which a $1B1$ solution fold-bifurcated exist in the ⑪ region. Consistent with the definition, the solution proceeds as a “BA-1” sequence with increasing ω : chaos $\rightarrow$ 1-band attractor $\rightarrow$ 2-band attractor $\rightarrow$ 4-band attractor $\rightarrow \dots$. However, the bifurcation diagram in Fig. 18(a) does not imply that the solution follows this sequence. In other words, the chaotic behavior in Fig. 18(b) transforms rapidly into the 4-band attractor in Fig. 18(c), as shown by the Poincaré maps at $\omega=1.67785$ in Fig. 18(b) (corresponding to ②⑭ in Fig. 18(a)) and at $\omega=1.678$ in Fig. 18(c) (corresponding to ②⑮ in Fig. 18(a)). This observation can also be explained using invariant curves. Figures 18(d) and (e) show the invariant curves and Poincaré maps located in the lower left part of Fig. 18(c). Additionally, the crosses denote the Poincaré points after 1400 periods. At $\omega=1.678$, the 4-band attractor is about to intersect ω -branches from the $^3I^2$. Even though a new 2-band attractor appears at $\omega=1.67785$, because the 4-band attractor has intersected ω -branches from the $^3I^2$ before 1400 periods have elapsed, a 1-band attractor can never exist since the 2-band attractor does not intersect the ω -branches from the $^2I^1$; that is, no heteroclinic intersection point exists. However, after 1400 periods, the Poincaré points indicated by the crosses intersect an ω -branch from one of the $^4D^6$ s situated close to the ω -branches from the $^3I^2$, and then the attractor transforms abruptly into a new large attractor, as seen in Fig. 18(b). Consequently, the 1-band attractor transforms into the 4-band attractor owing to the lack of a stable 2-band attractor. The above solution bifurcates in accordance with the “BA-2” sequence. This is different from the usual sequence for cases in which no fold-bifurcated directly unstable fixed

Fig. 18 Bifurcation diagram and invariant curves around ② (solid line: ω -branch from ${}^4D^6$ s, broken line: ω -branch from ${}^2I^1$, dotted line: ω -branch from ${}^3I^2$, dot: Poincaré point)

points exist. Now suppose that a 2^n -band attractor transforms into a 2^{n-1} -band attractor due to contact between the 2^n -band attractor and an ω -branch from the inversely unstable fixed points $I^{2^{n-1}}$'s of period 2^{n-1} . Provided that a directly unstable fixed point D^m of period m exists close to some of the ω -branches from the $I^{2^{n-1}}$'s, the 2^{n-1} -band attractor will come into contact with the ω -branches from the directly unstable fixed point D^m , requiring a few thousand periods, and will then abruptly transform into a 1-band attractor.

6. Conclusions

Bifurcation procedures occurring in a forced vibratory system with an asymmetric piecewise-linear restoring force were investigated, the system being derived from the circumferential vibration of a gear with a meshing error. Comparing analytical results with numerically integrated results obtained in the gear-revolution number range $\omega=0.35\sim 1.80$, we reach the following conclusions.

(1) When an attractor changes from being chaotic to being periodic, there can be two types of transformation. One is the "hysteresis" type in which the chaotic attractor transforms into the periodic solution situated far from the chaotic attractor in the phase plane; the other is that in which the chaotic solution transforms into a periodic solution located within the chaotic attractor boundaries before the transition.

(2) When an n -periodic solution changes into a

chaotic solution via period doubling, this is normally followed by a process in which the number of bands in the attractor decreases discretely and exponentially to 1. This sequence of band attractors is closely related to the invariant curves which emanate from the inversely unstable fixed points with each period. In order to obtain an n -band attractor, the existence of a heteroclinic intersection between an α -branch from an I^{2n} of period $2n$ and an ω -branch from an I^n of period n is required.

(3) In the process in which an n -period attractor transforms into a 1-band attractor, if another directly unstable fixed point exists in the vicinity of a band attractor and the attractor has a heteroclinic intersection with an ω -branch starting from the directly unstable fixed point as ω increases, an "interior catastrophe" causes the attractor to expand, and it transforms into a 1-band attractor.

(4) A series of periodic solutions belonging to various super/subharmonic vibrations which appear irregularly as "windows" in the bifurcation diagram, comprise one "family", which can be categorized as either type-I or type-II, as follows.

Type-I is a family characterized by multifolding, that is, iterating fold bifurcations (saddle-node bifurcations). The solution curve repeats the following pattern: directly unstable $\rightarrow$ fold bifurcation $\rightarrow$ completely stable $\rightarrow$ inversely unstable $\rightarrow$ completely stable $\rightarrow$ fold bifurcation $\rightarrow$ directly unstable.

Type-II is a family which is fold-bifurcated only at both ends of the solution curve.

(5) The intermittent bifurcations of the chaotic attractor can be referred to as either "interior catastrophe" or "hysteresis" according to whether or not the bifurcated chaotic attractor remains within the area in which the attractor was located in the phase plane before the bifurcation. Furthermore, it is shown that both of these bifurcation phenomena can occur in a vibratory system with a piecewise-linear asymmetric restoring force.

(6) When a 2^n -band attractor transforms into a 2^{n-1} -band attractor by intersecting an ω -branch from the inversely unstable fixed point I^{2^n-1} of period 2^{n-1} without a fold-bifurcated directly unstable fixed point, and simultaneously, another directly unstable fixed point of a certain period is located close to an ω -branch from the I^{2^n-1} , no stable 2^{n-1} -band attractor can be produced. The result is that the 2^{n-1} -band attractor changes abruptly to a 1-band attractor after intersecting an ω -branch from the directly unstable fixed point.

References

- (1) Shaw, S.W. and Holmes, P.J., A Periodically Forced Piecewise Linear Oscillator, *J. Sound Vib.*, Vol. 90, No. 1 (1983), p. 129.
- (2) Kodera, T., Chaotic Behaviour in Impact Vibration System, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), C Vol. 52, No. 477 (1986), p. 1508.
- (3) Imamura, H. and Suzuki, K., Dynamic Behaviour in a Vibro-Impact Mechanical System (Bifurcation Phenomena of $1/n$ Subharmonic Solutions and Chaotic Behaviour), *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), C Vol. 55, No. 510, (1989), p. 267.
- (4) Shirao, Y., Inagaki, Y., Kawabata, H., Takehara, N., Nagahara, T. and Kido, M., On the Periodic Solutions of Nonautonomous Piecewise Linear Systems with Unsymmetrical Restoring Force (IX), *IEICE Technical Report*, (in Japanese), NLP, Vol. 85, No. 16 (1985), p. 25.
- (5) Sato, K., Yamamoto, S. and Fujishiro, S., Dynamical Distinctive Phenomena in a Gear System (Bifurcation Sets of Periodic Solutions and Chaotically Transitional Phenomena), *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 54, No. 507, C (1988), p. 2735.
- (6) Kahraman, A. and Singh, R., Non-linear Dynamics of a Geared Rotor-bearing System with Multiple Clearances, *J. Sound Vib.*, Vol. 144, No. 3 (1991), p. 469.
- (7) e.g., Terauchi, Y., Haguruma no Shindou Souon to Sono Boushi (Vibration and Noise Control in Gear-Systems) (2), *Science of Machine*, (in Japanese), Vol. 24, No. 8 (1972), p. 1070.
- (8) Takayasu, H. (ed.), *Fractal Kagaku (Science of Fractals)*, (in Japanese), (1987), p. 58, Asakura-Syoten, Tokyo.
- (9) Thompson, J.M.T. and Stewart, H.B., (translated by Musya, T. and Hashiguchi, S.) *Nonlinear Dynamics and Chaos*, (in Japanese), (1988), p. 275, Ohm-sya, Tokyo.
- (10) Stewart, H.B., Ueda, Y. and Yoshida, S., Window and Destroyer, *IEICE Technical Report*, (in Japanese), NLP, Vol. 88, No. 70 (1988), p. 25.
- (11) Tsukamoto, A., Tsuji, K. and Matsumoto, T., Global Structures of Stable and Unstable Manifolds and Behavior of Chaos of Nonlinear Systems Described by Duffing's Equation, *IEICE Technical Report*, (in Japanese), NLP, Vol. 90, No. 37 (1990), p. 19.
- (12) Tsukamoto, A., Tsuji, K. and Matsumoto, T., Chaotic Behavior and Global Structures of Nonlinear Systems Described by Duffing's Equation with Soft Nonlinearity, *IEICE Technical Report*, (in Japanese), NLP, Vol. 91, No. 12 (1991), p. 15.
- (13) Stewart, H.B., Ueda, Y. and Yoshida, S., On a Chaotic Saddle Catastrophe (II), *IEICE Technical Report*, (in Japanese), NLP, Vol. 88, No. 14 (1988), p. 3.
- (14) Takayasu, H. (ed.), *Fractal Kagaku (Science of Fractals)*, (in Japanese), (1987), p. 72, Asakura-Syoten, Tokyo.
- (15) Kawakami, H. (ed.), *Some Problems on the Theory of Dynamical Systems in Applied Science*, (1992), p. 119, World Scientific, Singapore.
- (16) Noguchi, J., Tanaka, S., Matsumoto, T. and Chua, L.O., Multi Folding: A New Attractor Formation Mechanism in a R-L-Diode Circuit, *IEICE Technical Report*, (in Japanese), NLP, Vol. 88, No. 24 (1988), p. 37.