Decentralized Controller Design of Interconnected Systems with $H^\infty$ and Variance Constraints*

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The problem of disturbance attenuation and covariance assignment for an interconnected system using decentralized controls is addressed. We solve the $H^\infty$ and variance constraints using the Riccati equation approach. To achieve the specified upper boundaries of disturbance attenuation and state covariance, a condition which ensures the existence of local controllers is derived. The entire set of state covariances which may be assigned to each subsystem by local state feedback is characterized and all corresponding local state feedback controllers are derived. Complete solutions to the decentralized controller design of interconnected systems with $H^\infty$ and variance constraints are achieved merely by iteratively solving a set of local algebraic Riccati equations.

Key Words: Interconnected System, Decentralized Control, $H^\infty$-Norm Constraint, Variance Constraint

1. Introduction

When designing a control system, one often begins with a plant which is subject to external disturbances. A common design objective is to reduce the effect of these disturbances to an acceptable level\(^{(1)-(9)}\). For example, in the disturbance decoupling problem of decentralized systems researched by Hu and Zang\(^{(4)}\) and Hu and Zheng\(^{(5)}\), the local state feedback control was identified such that the disturbances were completely decoupled from the output. However, it may be impossible to reduce the effect of the disturbances below a certain threshold value for some systems and hence such a problem would have no solution. In addition, in many regulation control problems, it is desirable to design the controller so that various system states have acceptable root mean squared values. The covariance control theory, which guarantees that such individual performance objectives will be met for linear stochastic systems, has been developed recently by Hotz and Skelton\(^{(6)}\), Collins and Skelton\(^{(7)}\) and Skelton and Ikeda\(^{(8)}\). Thus, reducing the effect of disturbances on the whole system to an acceptable level and specifying the individual variance constraints simultaneously by decentralized control is an alternative interesting topic.

In this paper, we utilize the $H^\infty$ theory and covariance control theory to explore the problem of $H^\infty$ norm constraint and individual variance constraints for interconnected systems. The decentralized $H^\infty$ local controllers are constructed from the solutions of a set of local algebraic Riccati equations only. A strategy for choosing decentralized controllers is found such that the state variance of each subsystem is bounded by a specified value and the $H^\infty$ norm of a specified closed-loop transfer function is less than a given scalar simultaneously.

This paper is organized as follows. In Section 2, we briefly describe the system concerned and then the problem formulation is given. Then, in Section 3, an approach which deals with the $H^\infty$ norm and state variance constraints using decentralized control is introduced. An illustrated example of the motion of a
satellite in a circular and equatorial orbit is given in Section 4. Finally, in Section 5 we summarize our results and draw some conclusions.

2. System Description and Problem Formulation

Let \( S \) be an interconnected system composed of \( N \) subsystems \( S_i, i=1, \ldots, N \). Each \( S_i \) is described by using the Eq.

\[
\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + \sum_{j \neq i} M_{ij} L_j(\sigma_i) H_j x_j(t) + M_{ii} w_i(t) \quad (1. a)
\]

\[
y_i(t) = C_i x_i(t), \quad \text{and} \quad (1. b) \]

\[
z_i(t) = E_i x_i(t), \quad (1. c)
\]

where \( A_i \in \mathbb{R}^{n_i \times n_i} \) and \( B_i \in \mathbb{R}^{n_i \times n_k} \) are the nominal system matrix and input matrix respectively, \( x_i \in \mathbb{R}^{n_i} \) is the state, \( u_i \in \mathbb{R}^{n_i} \) is the control, \( w_i \in \mathbb{R}^{n_k} \) is the white noise signal of unity intensity, \( y_i \in \mathbb{R}^{n_i} \) is the measured output, and \( z_i \in \mathbb{R}^{n_z} \) is the controlled output; moreover, \( M_i \in \mathbb{R}^{n_i \times n_k} \) and \( H_i \in \mathbb{R}^{n_i \times n_i} \) characterize the structure of the interaction, \( L_i(\sigma) \in \mathbb{R}^{n_i \times n_k} \) is the interaction function between the \( i \)th and \( j \)th subsystems, and \( \sigma_i \) is a parameter vector belonging to a compact set, \( \Pi_i \subset \mathbb{R}^{n_i} \). It should be emphasized that \( L_i(\sigma) \) discussed throughout this paper is assumed to be linear time-invariant.

Assumption 1. Let

\[
G_i(\sigma) := [L_i(\sigma) L_i(\sigma) \cdots L_{i-1}(\sigma) L_{i+1}(\sigma) \cdots L_{N}(\sigma)]^T \in \Theta_i, \quad (2)
\]

where the set \( \Theta_i \) is defined as

\[
\Theta_i := \{ (\theta_i, \Theta_i) \subseteq \mathbb{I} ; \text{ the elements of } \theta_i \text{ are Lebesgue measurable} \}, \quad (3)
\]

that is, \( G_i(\sigma) G_i^T(\sigma) \leq I \).

Assumption 2. \( C_i = I \) and \( (A_i, B_i) \) is controllable for \( i=1, 2, \ldots, N \).

Remark 1. Assumption 2 allows us to focus attention on constant local state feedback controllers of the form

\[
u_i(t) = K_i x_i(t) \quad (4)
\]

Assumption 3. \( (A_i + B_i K_i, M_i) \) is a stabilizable pair for any choice of \( K_i, i=1, 2, \ldots, N \).

Remark 2. If \( M_i = [B_i M_i] \), i.e., the actuators are a disturbance source, then Assumption 3 can be guaranteed.

The problem considered in this paper is the characterization of an entire set of state covariances which may be assigned to each subsystem by local state feedback and determine the set of all local state feedback gains which will overbound an admissible state covariance to the system.

The closed-loop transfer function \( \mathcal{E}(s) \) from \( w(t) \) to \( z(t) \) is \( \mathcal{E}(s) = E(sI - A)^{-1} M \) where

\[
\bar{A} = A + BK, \quad A = \begin{bmatrix} A_1 & A_{12}(\sigma) & \cdots & A_{1N}(\sigma) \\ A_{21}(\sigma) & A_2 & \cdots & A_{2N}(\sigma) \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1}(\sigma) & \cdots & \cdots & A_N \end{bmatrix},
\]

\[
B = \text{diag}(B_1, \ldots, B_N), \quad K = \text{diag}(K_1, \ldots, K_N), \quad E = \text{diag}(E_1, \ldots, E_N), \quad M = \text{diag}(M_1, \ldots, M_N), \quad \text{and} \quad A_{ij}(\sigma) = M_i L_j(\sigma) H_j, \quad \text{for } i \neq j.
\]

Here, we define \( \chi \) as the steady state covariance matrix of the state vector of the whole closed-loop interconnected system, i.e.,

\[
\chi = \lim_{t \to \infty} \mathbb{E}[x(t)x^T(t)], \quad (5)
\]

where \( \mathbb{E} \{ \cdot \} \) denotes the expectation of \( \cdot \) and \( x(t) = [x_1(t) x_2(t) \cdots x_N(t)]^T \).

The steady state covariance \( \chi \) of the closed-loop system is the solution to the following Lyapunov equation

\[
\dot{\chi} + \chi A^T + MM^T = 0. \quad (6)
\]

In order to enforce the disturbance attenuation constraint \( \gamma > 0 \) and overbound the closed-loop steady state covariance simultaneously, we should replace the Lyapunov equation (6) by the algebraic Riccati equation stated in the following theorem.

Theorem 1. Let the constant scalar \( \gamma > 0 \) and the matrix \( K_i \) be given. If there exist a positive-definite matrix \( P_i \) and a constant \( \eta_i > 0 \) satisfying

\[
(A_i + B_i K_i) P_i + P_i (A_i + B_i K_i)^T + \gamma^2 P_i H_i F_i P_i + \frac{1}{\eta_i} P_i H_i F_i P_i + \sum_{i=1}^{N} \eta_i M_i M_i^T = 0, \quad (7)
\]

then we have

(i) \( \bar{A} \) is asymptotically stable,

(ii) \( \| \mathcal{E}(s) \|_{\infty} \leq \gamma \), \quad (8)

(iii) \( [\chi]_{ii} \leq P_i \eta_i \), \quad (9)

where \( [\cdot]_i \) denotes the \((i, i)\) block diagonal matrix of the matrix \([\cdot]\) with the dimensions \( n_i \times n_i \).

Proof: Using the positive-definite solution \( P_i \) of Riccati equation (7), this results in a closed-loop system matrix \( \bar{A} = \bar{A} + \bar{B} \) such that

\[
\bar{A} \bar{P} + \bar{P} \bar{A}^T = (\bar{A} + B K) P_i + P_i (A_i + B K)^T + \gamma^2 P_i H_i F_i P_i + \frac{1}{\eta_i} P_i H_i F_i P_i + \sum_{i=1}^{N} \eta_i M_i M_i^T, \quad (10)
\]

where \( \bar{A}_{i, i} = \begin{bmatrix} 0 & A_{i2}(\sigma) & \cdots & A_{iN}(\sigma) \\ A_{2i}(\sigma) & 0 & \cdots & A_{2N}(\sigma) \\ \vdots & \vdots & \ddots & \vdots \\ A_{Ni}(\sigma) & \cdots & \cdots & 0 \end{bmatrix} \) and \( H_i \) is a
block-diagonal matrix constructed by $H_i$, for $i=1, 2, \cdots, N$.

Since

$$
A_{eq}P + P A_{eq}^T = A_{eq}[\text{diag}[P_0, 0, \cdots, 0] + \text{diag}[0, P_0, 0, \cdots, 0] + \cdots + \text{diag}[0, \cdots, 0, P_N]]
$$

$$
+ \{\text{diag}[P_1, 0, \cdots, 0] + \text{diag}[0, P_1, 0, \cdots, 0] + \cdots + \text{diag}[0, \cdots, 0, P_N]\} A_{eq}
$$

$$
= \{\text{diag}[0, M_0, \cdots, M_N] [L_0 \cdots L_N]^T H_i\}

\times [P_0 0 \cdots 0] + [P_0 0 \cdots 0]^T
$$

$$
\times \{\text{diag}[0, M_0, \cdots, M_N] [L_0 \cdots L_N]^T H_i\}
$$

$$
+ \{\text{diag}[M_0, 0, \cdots, M_N] [L_0 \cdots L_N]^T H_i\}
$$

$$
= \{\text{diag}[0, M_0, \cdots, M_N] [L_0 \cdots L_N]^T H_i\}

\times [P_0 0 \cdots 0] + [P_0 0 \cdots 0]^T
$$

$$
\times \{\text{diag}[M_0, 0, \cdots, M_N] [L_0 \cdots L_N]^T H_i\}
$$

$$
+ \cdots
$$

$$
+ \{\text{diag}[M_0, \cdots, M_{N-1}, 0] [L_0 \cdots L_{N-1}, 0]^T H_i\}
$$

$$
\times [P_0 \cdots P_{N-1}] + [P_0 \cdots P_{N-1}]^T
$$

$$
\times \{\text{diag}[M_0, \cdots, M_{N-1}, 0] [L_0 \cdots L_{N-1}, 0]^T H_i\}
$$

$$
\leq \gamma h_1 \text{diag}[0, M_0 M_0^T, \cdots, M_N M_N^T]
$$

$$
+ \frac{1}{\gamma h_1} \text{diag}[P_0 H_i^T H_i P_0, 0, \cdots, 0]
$$

$$
+ \frac{1}{\gamma h_1} \text{diag}[P_0 H_i^T H_i P_0, 0, \cdots, 0]
$$

$$
+ \gamma n \text{diag}[0, M_0 M_0^T, \cdots, M_N M_N^T, 0]
$$

$$
+ \frac{1}{\gamma n} \text{diag}[0, \cdots, 0, P_n H_i^T H_i P_n]
$$

$$
= \text{diag}[\frac{1}{\gamma h_1} I, \cdots, \frac{1}{\gamma h_1} I] PH_i^T PH
$$

$$
+ \text{diag}[(\gamma h_1 + \gamma n) I, \cdots, (\gamma h_1 + \gamma n) I] MM^T
$$

then,

$$
\tilde{A} P + P \tilde{A}^T + \gamma^2 PE^T EP + MM^T \leq 0.
$$

(11)

Assume, in contrast, that $\tilde{A}$ is not asymptotically stable, i.e., there exists a $\lambda$ with $\text{Re}(\lambda) > 0$ and a nonzero vector $x \in \mathbb{R}^n$ such that $\tilde{A} x = \lambda x$, where $n = \sum_{i=1}^{N} n_i$, and $x = [x_1 \cdots x_N]^T$. From Eq. (11), it follows that

$$
M^T x = 0, \text{i.e., } M^T x_i = 0, \text{ for } i = 1, 2, \cdots, N.
$$

(12)

Thus,

$$
\tilde{A} x = \text{diag}[(A_1 + B_1 K_1, \cdots, (A_N + B_N K_N)]
$$

$$
+ MGH_i^T x
$$

$$
= \text{diag}[(A_1 + B_1 K_1)^T, \cdots, (A_N + B_N K_N)^T] x = \lambda x
$$

(13)

i.e., $(A_i + B_i K_i)^T x_i = \lambda x_i$, for $i = 1, 2, \cdots, N$, where

$$
G = \begin{bmatrix}
0 & L_{10}(\sigma) & \cdots & L_{1N}(\sigma) \\
L_{20}(\sigma) & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
L_{N0}(\sigma) & \cdots & L_{N1}(\sigma) & 0
\end{bmatrix}
$$

Thus, there exists at least a pair $(A_i + B_i K_i, M_i)$ that is locally nonstabilizable. This contradicts the assumption that $(A_i + B_i K_i, M_i)$ is stabilizable for $i = 1, 2, \cdots, N$. Therefore, we can conclude that $\tilde{A}$ must be asymptotically stable. Using Lemma 1 of Willems\cite{willems}, we have

$$
\|\mathcal{J}(e^{jw})\| \leq \gamma^2 I,
$$

i.e., $\|\mathcal{J}(s)\| \leq \gamma$.

To prove Eq. (9), subtract Eq. (6) from Eq. (11) to obtain

$$
\tilde{A} (P - \tilde{A}) + (P - \tilde{A}) \tilde{A}^T + \gamma^2 PE^T EP \leq 0.
$$

(14)

Since $\tilde{A}$ is asymptotically stable, we have

$$
(P - \tilde{A}) \geq \int_0^\infty e^{s\tilde{A}} [\gamma^2 PE^T EP] e^{s\tilde{A}} dt \geq 0.
$$

(15)

Thus $P \geq \chi$, and it follows that Eq. (9) holds.

In the following, we describe the appropriate condition under which a local state covariance upper boundary, $P_i > 0$, is assignable for the consistency of Eq. (7) and determine all local controllers $K_i$ which achieve the specified local state covariance upper boundary, $P_i$, such that variance constraint Eq. (9) and $H^*$ norm constraint Eq. (8) are satisfied simultaneously. Before stating the theorem, we should introduce some notations. Let $T_i$ be a factor of the specified covariance $P_i$ for the $i$th subsystem, i.e., $P_i = T_i^T T_i$,

$$
A_i = T_i^{-1} A_i T_i, \quad B_i = T_i^{-1} B_i, \quad E_i = T_i^{-1} E_i T_i,
$$

$$
\tilde{H}_i = \left(\frac{1}{\eta_i}\right)^{1/2} H_i T_i,
$$

$$
\tilde{M}_i = \left(1 + \sum_{i=1}^{N} \eta_i\right)^{1/2} T_i^{-1} M_i, \quad \tilde{K}_i = K_i T_i
$$

and $[\cdot]^*$ denotes the Moore–Penrose inverse of $[\cdot]$. Then Eq. (7) becomes

$$
(A_i + B_i K_i)^T + E_i E_i^T + \tilde{H}_i^T \tilde{H}_i + \tilde{M}_i \tilde{M}_i^T = 0.
$$

(16)

Since $\{E_i E_i^T + \tilde{H}_i^T \tilde{H}_i + \tilde{M}_i \tilde{M}_i^T\} \geq 0$, the matrix Eq. (16) has the solution\cite{19}

$$
B_i K_i = -\frac{1}{2} \left[ E_i E_i^T + \tilde{H}_i^T \tilde{H}_i + \tilde{M}_i \tilde{M}_i^T + \tilde{A}_i - \mathcal{L}_i\right],
$$

(17)

where $\mathcal{L}_i$ denotes the skew-symmetric part of $B_i K_i$. The consistency of Eq. (17) depends, in part, upon the choice of $\mathcal{L}_i$. Thus Eq. (17) is consistent if and only if

$$
(I - B_i B_i^T) \mathcal{L}_i = (I - B_i B_i^T) (E_i E_i^T + \tilde{H}_i^T \tilde{H}_i + \tilde{M}_i \tilde{M}_i^T)
$$

(18)

for some skew-symmetric matrix $\mathcal{L}_i$. It should be noted that the consistency of Eq. (17) cannot always be forced by an appropriate choice of $\mathcal{L}_i$.

**Theorem 2.** The matrix Eq. (17) has a solution, i.e., there exists a local feedback controller $K_i$ that achieves a specified state covariance upper boundary matrix $P_i = P_i^T > 0$ for Eq. (7), if and only if

$$
\begin{bmatrix}
\mathcal{F} [\tilde{A}_i + \tilde{A}_i^T + E_i E_i^T + \tilde{H}_i^T \tilde{H}_i + \tilde{M}_i \tilde{M}_i^T] \mathcal{F} & 0 \\
0 & \mathcal{F} \end{bmatrix} = 0
$$

(19)

Moreover, if Eq. (19) holds, then all controllers that
assign $P_i$ to Eq. (7) will be given by
\[
x = \left[ K_i \right] \left[ R_i \right] = - \frac{1}{2} B_i^T \left[ \tilde{A}_i + \tilde{A}_i^T + E_i \tilde{E}_i \right]
+ \tilde{H}_i \tilde{H}_i + \tilde{M}_i \tilde{M}_i - \tilde{L}_i \tilde{T}_i^{-1}
= - \frac{1}{2} B_i^T \left[ \tilde{P}_i \tilde{A}_i \tilde{P}_i + \tilde{A}_i \tilde{P}_i + \gamma^2 P_i \tilde{E}_i \tilde{E}_i P_i \right]
+ \frac{1}{\eta_i} P_i \tilde{H}_i \tilde{H}_i P_i
+ \left( 1 + \sum_{j \neq i} \eta_j \right) \tilde{M}_i \tilde{M}_i - \tilde{T}_i \tilde{L}_i \tilde{T}_i^{-1} \tilde{P}_i^{-1},
\]
where $[-]_{n \times m}$ denotes the unitary model matrix of $B_i B_i^T$, and
\[
\left[ \begin{array}{c}
R_{i1} \\
R_{i2}
\end{array} \right] = \tilde{F}_i \left[ \begin{array}{c}
\tilde{L}_i \\
\tilde{R}_{i2}
\end{array} \right] - \tilde{R}_{i2}
\tilde{F}_i^{-1}
\]
where $\tilde{R}_{i2} = 0$ and $\tilde{L}_i$ an arbitrary skew-symmetric matrix.

Proof: Assume that there exists a gain $K_i$ that achieves the state covariance upper boundary $P_i$ for Eq. (7). Then there exists a skew-symmetric matrix $\tilde{L}_i$ to satisfy Eq. (18). Pre- and post-multiplying both sides of Eq. (18) by $\tilde{F}_i$ and $\tilde{F}_i^{-1}$ respectively, and using the fact that $\tilde{F}_i \tilde{F}_i^{-1} = \tilde{I}$ yields,
\[
\tilde{F}_i (I - B_i \tilde{B}_i) \tilde{F}_i^{-1} \tilde{L}_i = \tilde{F}_i (I - \tilde{B}_i \tilde{B}_i) \tilde{F}_i^{-1} \tilde{L}_i \tilde{F}_i^{-1}
= \tilde{F}_i (I - \tilde{B}_i \tilde{B}_i) \tilde{F}_i^{-1} \tilde{L}_i \tilde{F}_i^{-1}
+ \tilde{H}_i \tilde{H}_i + \tilde{M}_i \tilde{M}_i \tilde{F}_i.
\]
As $\tilde{F}_i$ is the unitary modal matrix of $B_i B_i^T$, we have $\tilde{F}_i \tilde{B}_i \tilde{F}_i^{-1} \tilde{L}_i = \begin{bmatrix}
I_{p \times p} & 0 \\
0 & 0
\end{bmatrix}$. Denoting $\tilde{F}_i \tilde{L}_i \tilde{F}_i^{-1} = \begin{bmatrix}
\tilde{L}_{i1} \\
\tilde{L}_{i2}
\end{bmatrix}$ and using Eq. (22), then Eq. (23) can be rewritten as
\[
\begin{bmatrix}
0 & I_{(n-p) \times (n-p)} \\
0 & 0
\end{bmatrix} \left[ \begin{array}{c}
\tilde{L}_{i1} \\
\tilde{L}_{i2}
\end{array} \right] = \begin{bmatrix}
0 & I_{(n-p) \times (n-p)} \\
0 & 0
\end{bmatrix} \left[ \begin{array}{c}
\tilde{R}_{i1} \\
\tilde{R}_{i2}
\end{array} \right].
\]
Expanding Eq. (24), we can determine that $\tilde{L}_{i1}$ is an arbitrary skew-symmetric matrix, $\tilde{L}_{i1} = - \tilde{R}_{i2}$ and $\tilde{L}_{i2} \tilde{R}_{i2}$ is symmetric, thus $\tilde{L}_{i2} \tilde{R}_{i2} = \tilde{R}_{i2} \tilde{L}_{i2}$ and only if $\tilde{R}_{i2} = 0$, i.e., Eq. (19) holds.

Conversely, assume Eq. (19) holds, then according to Eq. (7), any choice of skew-symmetric $\tilde{L}_{i1} = \tilde{L}_i$ satisfies Eq. (22). Thus Eq. (17) is consistent and there exists a $K_i$ that achieves $P_i$.

Furthermore, it is easy to see that Eq. (20) is the general solution to Eq. (17) as long as $B_i B_i^T \neq 0$.

**Remark 3.** Equation (19) is equivalent to
\[
(I - B_i B_i^T) \left[ \gamma^2 P_i \tilde{E}_i \tilde{E}_i P_i + \frac{1}{\eta_i} P_i \tilde{H}_i \tilde{H}_i P_i \right] + \left( 1 + \sum_{j \neq i} \eta_j \right) \tilde{M}_i \tilde{M}_i - \tilde{T}_i \tilde{L}_i \tilde{T}_i^{-1} \tilde{P}_i^{-1} = 0
\]
When checking condition Eq. (19), we may adjust the parameter $\eta_i$ to meet it. Once condition Eq. (19) is satisfied and the local controller Eq. (20) is adopted, the requirements Eqs. (8) and (9) are met from Theorem 1 immediately.

**Remark 4.** The transformation matrix $T_i$ may be evaluated by using the Cholesky decomposition from the specified local covariance $P_i$, but it is not unique. We found that it is not very important to actually evaluate the transformation matrix $T_i$, because the state covariance upper boundary assignment theory is not concerned with the particular choice of the factor $T_i$.

4. Illustrative Example

Consider the equations of motion for a satellite in a circular, equatorial orbit. Small perturbations about this orbit are governed by the following equations which are linearized about an orbit of radius and angular velocity. For the sake of simplicity, units are normalized so that the mass of the satellite and the radius are unity. Let angular velocity be equal to one, then the system with the assumed noise is represented as
\[
\begin{align*}
\dot{x}_{1} &= \begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 2 \end{bmatrix} w_1 \\
&\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 2 \end{bmatrix} w_2, \\
\dot{x}_{2} &= \begin{bmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} w_2 \\
&\quad + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ -2 \end{bmatrix} w_2. 
\end{align*}
\]
where $(x_{11}, x_{21}) = (r, \theta)$ describe the satellite's position in the equatorial plane and $(x_{12}, x_{22})$ are the corresponding velocity and angular velocity. The controls are thrusters using gas jets and the controlled outputs are
\[
\begin{align*}
x_1(t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1(t), \\
x_2(t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_2(t).
\end{align*}
\]
Here, let the interactions be decomposed as
\[
M_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad L_{11} = 1, \quad H_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \\
M_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad L_{21} = 1, \quad H_2 = \begin{bmatrix} 0 \end{bmatrix}.
\]
It can be easily shown that Assumption 3 is satisfied. Our objective is to design decentralized feedback controls such that the overall system is asymptotically stable, $\gamma = 0.5$, and
\[
\lim_{t \to \infty} \epsilon [x_{11}(t)] \leq 0.30, \quad \lim_{t \to \infty} \epsilon [x_{21}(t)] \leq 0.40,
\]
\[
\lim_{t \to \infty} \epsilon [x_{12}(t)] \leq 0.25, \quad \lim_{t \to \infty} \epsilon [x_{22}(t)] \leq 0.30.
\]
Let $\eta_1 = \eta_2 = 1$, and
\[ P_1 = \begin{bmatrix} 0.3 & -0.2 \\ -0.2 & 0.4 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.25 & -0.135 \\ -0.135 & 0.3 \end{bmatrix} \] and
\[ (28) \]
condition Eq. (19) is satisfied. Using the Cholesky decomposition, we have
\[ T_1 = \begin{bmatrix} 0.5477 & 0 \\ -0.3651 & 0.5164 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0.5 & 0 \\ -0.27 & 0.4766 \end{bmatrix} \]
and then $F_1 = F_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Furthermore using Eqs. (21) and (22), it follows that
\[ \mathcal{L}_1 = \begin{bmatrix} 0 & 3.4649 \\ -3.4649 & 0 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 0 & 0.5185 \\ -0.5185 & 0 \end{bmatrix} \]
Finally, using Eq. (20), we obtain the desired local feedback gains $K_i, i = 1, 2$, as follows
\[ K_1 = \begin{bmatrix} 13.8 \\ 15.8 \end{bmatrix}, \quad (30a) \]
\[ K_2 = \begin{bmatrix} 10.359 \\ 18.269 \end{bmatrix}, \quad (30b) \]

5. Conclusions

The main objective of this work is to demonstrate the close relationship among the $H^\infty$ control theory, covariance control theory, and the decentralized controller design for interconnected systems. We have developed a methodology to achieve the $H^\infty$ norm and variance constraint performance requirements simultaneously for interconnected systems. Appropriate conditions for the existence of constant linear local feedback controllers are proposed and the set of all local state feedback gains which will overbound the admissible state variance of the system is also derived.

References