Improved Margin of Stability of Interval Matrices*

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This paper is concerned with the stability analysis of interval matrices. Improved sufficient conditions for the stability of interval matrices are given. These conditions are based on a theorem that determines containment regions for the eigenvalues of a certain matrix which are smaller than the regions determined by the Gerschgorin Theorem and thus giving rise to larger margins of stability. We obtain sufficient conditions for the stability of a perturbed matrix using similarity transformations. Numerical examples are given to illustrate the results of the paper.

Key Words: Stability, Linear Systems, Interval Matrices, Gerschgorin Theorem

1. Introduction

Modelling time-invariant systems described in state-space form where the parameters of the systems are known only within certain closed intervals gives rise to interval dynamical systems of the form

\[ \dot{X}(t) = A_t X(t) \tag{1} \]

where \( A_t \) is an \( n \times n \) interval matrix, i.e.,

\[ A_t = [B, C] = [(a_{ij}) \mid b_{ij} \leq a_{ij} \leq c_{ij}, 1 \leq i, j \leq n], \tag{2} \]

\( B = [b_{ij}], C = [c_{ij}] \) are \( n \times n \) real matrices.

Interval dynamical systems appear in many engineering applications. For example, the coefficients of journal bearing oil film in rotating machinery change considerably with operating conditions. For instance, lubricating oil temperature usually changes over a wide range when the load changes. The dynamic system behavior is thus best described by state space equations incorporating interval matrices. Models with interval coefficients are also commonly used to represent stochastic systems in the presence of noise or when the parameters of the systems vary slowly with time. Designing robust stabilization controllers under these circumstances is a problem of great practical importance. The stability analysis of such dynamical systems reduces to the analysis of the stability of an interval matrix \( A_t \).

The interval matrix \( A_t \) is said to be stable with stability margin \( h \) (with \( h \geq 0 \)) if the set

\[ \{ \lambda \in C \mid \lambda \in \sigma(A), A \in A_t \} \]

is contained in the half plane

\[ \{ z \in C \mid \text{Re}(z) < -h \}. \]

In this case we say that \( A \) is \( h \)-stable. Based on Gershgorin Theorem, Hein"en\(^{10}\) gave the following sufficient conditions for the \( h \)-stability of \( A_t \).

**Theorem 1** If for the interval matrix \( A_t \) and \( h \geq 0 \), the following condition holds

\[ c_u + \sum_{j=1}^{n} \max_{i} \{ |b_{ij}|, |c_{ij}| \} < -h, \quad i = 1, \ldots, n, \]

then \( A_t \) is \( h \)-stable.

Because of its simplicity, several papers\(^{10-13}\) developed sufficient conditions for the margin of stability of \( A_t \) based on Theorem 1.

In this paper we develop sufficient conditions for the \( h \)-stability of an interval matrix \( A_t \). The condi-
tions to be developed are based on the following theorem due to Ref. (2) which may be regarded as a generalized version of Gerschgorin Theorem. In Ref. (2) it is shown that this Theorem, which is reproduced below for convenience, provides a smaller regions of containment of the eigenvalues of a general matrix than the regions provided by Gerschgorin Theorem.

**Theorem 2** Let \( A = [a_{ij}] \) be an \( n \times n \) real matrix and \( r \) an integer such that \( 1 \leq r \leq n \). Then each eigenvalue of \( A \) is contained in either one of the disks \( U(\theta)^r = \{ z \in \mathbb{C} \mid |z - a_{ij}| \leq S^{(r-1)} \}, \) \( j = 1, \ldots, n \), (3) or in one of the regions \( V(\theta)^r = \{ z \in \mathbb{C} \mid \sum_{j=1}^{n} |z - a_{ij}| \leq \sum_{j=1}^{n} R_j \} \), \( P \subseteq \{1, \ldots, n\}, |P| = r, \) (4)

where \( S^{(r-1)} \) denotes the sum of magnitudes of the \( n-r \) largest off diagonal elements in column \( j \), and \( R_j = \sum_{i \in P} |a_{ij}| \).

**Remark 1** 1. Since the regions of containment of the eigenvalues of a matrix \( A \) provided by Theorem 2 are in general smaller than those provided by Gerschgorin Theorem, it is expected that the former theorem will give rise to a larger margin of stability for an interval matrix \( A \).

2. The region \( \Gamma(\theta)^r = \bigcup_{j=1}^{n} U(\theta)^r \) is contained in the column-Gerschgorin region of \( A \), while the region \( \Gamma(\theta)^r \) \( = \bigcup_{j=1}^{n} V(\theta)^r \) consisting of \( \binom{n}{r} \) regions is contained in the row-Gerschgorin region of \( A \). The union \( \Gamma(\theta)^r \cup \Gamma(\theta)^r \) contains all the eigenvalues of \( A \).

3. For \( r > 2 \), the regions \( V(\theta)^r \) are not easily visualized. However, for \( r = 2 \) it can easily be seen that the regions \( V(\theta)^2 = \{ z \in \mathbb{C} \mid |z - a_{ij}| + |z - a_{ji}| \leq R_1 + R_2 \} \), \( i < j \), contain all the eigenvalues of \( A \).

4. If we set \( \Gamma(\theta)^r = \Gamma(\theta)^r \cup \Gamma(\theta)^r \) then \( \sigma(A) \subseteq \Gamma(\theta)^r \), \( r = 1, \ldots, n \). It follows that \( \sigma(A) \subseteq \bigcap_{r=1}^{n} \Gamma(\theta)^r \).

In Section 2 of this paper we derive a sufficient condition for the \( h \)-stability of an interval matrix \( A \), based on Theorem 2. In Section 3 sufficient conditions for the \( h \)-stability of a perturbed matrix are obtained using similarity transformations. Some illustrative examples are given in Section 4. Conclusions and remarks are the subject of Section 5.

2. **Sufficient Conditions ; The Extreme Matrix**

It follows from Theorem 2 above that if \( A = [a_{ij}] \) is a given matrix and \( \lambda \in \sigma(A) \) then \( \lambda \) must belong to either one of the disks \( U(\theta)^n \) or one of the regions \( V(\theta)^n \). Therefore \( \text{Re}(\lambda) \) must satisfy at least one of the

\[
\frac{n(n-1)}{2} \text{ conditions}
\]

\[
\text{Re}(\lambda) \leq a_{ii} + S^{(n)}, \quad j = 1, \ldots, n,
\]

and

\[
\text{Re}(\lambda) \leq \frac{a_{ii} + a_{jj}}{2} + \frac{R_i + R_j}{2}, \quad 1 \leq i < j \leq n.
\]

The following theorem follows readily from the foregoing remarks.

**Theorem 3** Let \( A_t \) be the interval matrix defined by (2) and suppose that for some \( h \geq 0 \) we have \( c_{ij} + h \leq 0 \), \( i = 1, \ldots, n \). Suppose also that

\[
\begin{align*}
(1) & \quad c_{ii} + \max_{i \neq j} \{ |b_{ij}|, |c_{ij}| \} < -h, \quad j = 1, \ldots, n, \\
(2) & \quad c_{ii} + c_{jj} + \frac{1}{2} \left[ \sum_{k \neq i} \left( |b_{ik}|, |c_{ik}| \right) + \sum_{k \neq j} \left( |b_{jk}|, |c_{jk}| \right) \right] < -h,
\end{align*}
\]

\( 1 \leq i < j \leq n \)

are satisfied. Then \( A_t \) is \( h \)-stable.

Corresponding to the interval matrix \( A_t \) we define a matrix \( A = [a_{ij}] \) by

\[
a_{ij} = \begin{cases}
\max \{ |b_{ij}|, |c_{ij}| \} & \text{if } i \neq j, \\
|c_{ii}| & \text{if } i = j
\end{cases}
\]

Then conditions (1), (2) may be written alternatively in the form

\[
\begin{align*}
(1)' & \quad a_{ii} + \max_{i \neq j} a_{ij} < -h, \quad j = 1, \ldots, n, \\
(2)' & \quad a_{ii} + a_{jj} + \left( \sum_{k \neq i} a_{ik} + \sum_{k \neq j} a_{jk} \right) < -2h,
\end{align*}
\]

\( 1 \leq i < j \leq n \).

**Remark 2** 1. Since \( A \) and \( A^T \) have the same eigenvalues, we may obtain a larger value for \( h \) by replacing \( A \) by \( A^T \) in Theorem 3.

2. In developing the sufficient conditions of Theorem 3 we used Theorem 2 with \( r = 2 \). Other sufficient conditions may be obtained by taking \( r = 3, 4, \ldots, n-1 \). For each \( r \) one gets two sets of sufficient conditions corresponding to \( A \) and \( A^T \). This way a largest possible margin of stability \( h \) may be arrived at.

3. **Sufficient Conditions ; The Perturbed Matrix**

The conditions stated in Theorem 3 are rather restrictive because of the requirement \( c_{ii} < 0 \). In this section we extend Theorem 2 and use this extension to obtain sufficient conditions that relax this restriction. We use here the same notations as Ref. (5). In what follows \( A \leq B \) for matrices \( A \) and \( B \) will denote elementwise inequalities and \( |A| \) will denote the matrix of absolute values of elements of \( A \).

A matrix \( A = A_0 + \delta A \) may be written in perturbed form as

\[
A = A_0 + \delta A,
\]

where \( |\delta A| \leq \Delta A \), and \( A_0, \delta A \) are \( n \times n \) matrices,

\[
\delta A = \left[ \frac{1}{2} (b_{ij} + c_{ij}) \right] \quad \text{and} \quad \Delta A = \left[ \frac{1}{2} (c_{ij} - b_{ij}) \right].
\]

Let \( T \) be a similarity transformation which
transforms \( A_0 \) into its Jordan form \( A_0, \lambda_1, \ldots, \lambda_n \) denote the eigenvalues of \( A_0 \), \( A=\text{diag}(\lambda_i) \) and let

\[
F = A_1 - A + |T^{-1}|A|T| = [f_{ij}].
\]  

(6)

The following theorem gives the conditions under which the eigenvalues of an interval matrix are included in the union of certain disks or regions.

**Theorem 4** Given any \( A \in A_1 \), each eigenvalue of \( A \) is included in the union of the disks

\[
U_j^{\lambda_j} = \{ z \in C \mid |z - \lambda_j| \leq \sum_{\alpha \neq j} |f_{\alpha j}| \}, \quad j = 1, \ldots, n
\]

(7)

or in the union of the regions

\[
V_j^{\lambda_j} = \{ z \in C \mid \sum_{\alpha \neq j} |z - \lambda_j| \leq \sum_{\alpha \neq j} |f_{\alpha j}| \},
\]

\[
P \subseteq \{ 1, \ldots, n \}, \quad |P| = r,
\]

(8)

where \( \bar{S}_{j-1}^{\lambda_j} \) denotes the sum of magnitudes of the \( r-1 \) largest off diagonal elements in the \( j \)th column of \( F \) plus \( f_{ij} \) and \( \bar{R}_j = \sum_{\alpha \neq j} |f_{\alpha j}| \). The same statement is true if we replace \( F \) by \( F^r \).

**Proof.** Given \( A \in A_1 \), write \( A = A_0 + \delta A \) as above and note first that

\[
\begin{align*}
|A_1 - A| + |T^{-1}\delta A T| &\leq |A_1 - A| + |T^{-1}\delta A T| \\
&\leq A_1 - A + |T^{-1}A_1 T| = F.
\end{align*}
\]

(9)

Now

\[
T^{-1}A_1 T = T^{-1}(A_0 + \delta A) T
\]

\[
= T^{-1}A_0 T + T^{-1}\delta A T
\]

\[
= A + (A_1 - A) + T^{-1}\delta A T,
\]

(10)

thus

\[
T^{-1}A_1 T - A = (A_1 - A) + T^{-1}\delta A T.
\]

(11)

Since \( \sigma(A) = \sigma(T^{-1}AT) \) we may apply Theorem 2 to the matrix \( T^{-1}AT \). Using (11) and the estimate (9) we obtain (7) and (8).

It should be noted that Theorem 4 above extends the main result of Ref. (5) to a more general situation by introducing the integer \( r \) of the theorem. In fact, the results of Ref. (5) can be derived from Theorem 4 as a special case by taking \( r = n \). The containment regions given by Theorem 4 are, in general, very hard to describe for large values of \( r \). It is relatively simple, however, to visualize such regions when \( r = 2 \). Therefore, we will apply Theorem 4 to obtain the following theorem which gives the conditions of stability for the case of \( r = 2 \).

**Theorem 5** If, for some \( h \geq 0 \) and all \( \lambda_i \in \sigma(A_0) \),

\[
\text{Re} (\lambda_i) + \bar{S}_{i-1}^{\lambda_i} < -h
\]

(12)

and

\[
\text{Re} (\lambda_i) + \text{Re} (\lambda_i) + \sum_{\alpha \neq j} f_{\alpha j} + \sum_{\alpha \neq j} f_{\alpha j} < 2h,
\]

\[
1 \leq i < j \leq n
\]

then the interval matrix \( A_1 \) is \( h \)-stable. The same statement holds if the matrix \( F \) is replaced by \( F^r \).

4. Examples

In this section we give two examples to illustrate the improvement of the margin of stability as a result of applying the sufficient conditions of Theorem 3-5 over that obtained by applying the conditions of Ref. (1), (4), (5).

**Example 1** In this example we illustrate the results of Section 2. Consider the interval matrix

\[
A_1 = \begin{bmatrix}
[-4,-3] & [0,1] \\
[0,1] & [-3,-2.1] & [0,1] \\
[0,1] & [0,1] & [-3,-2] 
\end{bmatrix}
\]

It cannot be concluded from Gerschgorin Theorem that \( A_1 \) is \( h \)-stable for any \( h \geq 0 \). However, applying Theorem 3 with

\[
A = \begin{bmatrix}
-3 & 1 & 1 \\
1 & -2.1 & 1 \\
1 & 1 & -2
\end{bmatrix}
\]

we see that the stability margin \( h = 0.5 \).

**Example 2** This example illustrates the results of Section 3. We will show that Gerschgorin theorem cannot conclude the stability of this example. However, stability is proved by applying Theorem 5. Consider the interval matrix

\[
A_1 = \begin{bmatrix}
[-.5,1] & [2.2] & [0,1] \\
[-5,-5] & [-6.5,-6] & [0,0] \\
[0,1] & [0,0] & [-1.5,-1]
\end{bmatrix}
\]

Computing the ingredients of Theorem 4 we get

\[
A_0 = \begin{bmatrix}
.75 & 2.0 & .05 \\
-5.0 & -6.25 & 0 \\
.05 & 0 & -1.25
\end{bmatrix},
\]

\[
\Delta A = \begin{bmatrix}
.25 & 0 & .05 \\
.05 & 0 & .25 \\
.05 & 0 & .25
\end{bmatrix}
\]

\[
\Lambda = \begin{bmatrix}
-4.2494 & 0 & 0 \\
0 & -1.1857 & 0 \\
0 & 0 & -1.3148
\end{bmatrix},
\]

\[
T = \begin{bmatrix}
-.3715 & -.6226 & -.6177 \\
-.9284 & .6147 & .6258 \\
-.0662 & -.4843 & .4763
\end{bmatrix}
\]

\[
F = \begin{bmatrix}
.5849 & .6037 & .6057 \\
.2650 & .4744 & .4714 \\
.2772 & .4915 & .4885
\end{bmatrix}
\]

Applying the conditions in Theorem 5 we get

\[
\lambda_1 + \lambda_2 + \bar{R}_1 + \bar{R}_2 = -2.4299,
\]

\[
\lambda_1 + \lambda_3 + \bar{R}_1 + \bar{R}_3 = -2.5127,
\]

\[
\lambda_2 + \lambda_3 + \bar{R}_2 + \bar{R}_3 = -.9324,
\]

so that \( \lambda_1 + \lambda_2 + \bar{R}_1 + \bar{R}_2 < 0, 1 \leq i < j \leq 3 \). Also condition (12) is easily checked. Notice that Gerschgorin Theorem cannot be used in this example also to conclude stability of the interval matrix \( A_1 \) as in Ref. (5) since

\[
\lambda_1 + \bar{R}_1 = -2.4551, \quad \lambda_1 + \bar{C}_1 = -3.1223,
\]

\[
\lambda_2 + \bar{R}_2 = -2.5127, \quad \lambda_2 + \bar{C}_2 = .0252,
\]

\[
\lambda_3 + \bar{R}_3 = -.0576, \quad \lambda_3 + \bar{C}_3 = .2508,
\]

where \( \bar{C}_i \) denotes the Gerschgorin column sum.

5. Conclusions

We developed sufficient conditions for the \( h \)-stability of a given interval matrix and have illustrated by means of examples that these sufficient conditions
provide a larger margin of stability than that obtained from applying Gerschgorin Theorem. The examples also show that our conditions can be used in cases where Gerschgorin Theorem fails to provide any information about the stability of the system. Such an improvement is important in such application areas where some of the system physical parameters change considerably with changing operating conditions. It is crucial for such cases to have available a reliable and, in the same time, easy to implement check for stability by only considering the upper and lower limits of the system parameters.

Because of the simplicity of our sufficient conditions and the advantage they have in insuring a larger margin of stability than that provided by the sufficient conditions of Ref. (1), (4), (5), we believe that our conditions are useful for an initial testing of marginal stability of interval matrices, and provide an improvement over those of Ref. (1), (4), (5).

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