Leakage-Flow-Induced Vibrations of an Axisymmetric Body*

(Part 1: Analysis of the Moment Acting on an Axisymmetric Body for Rotational Motion)

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Using a narrow annular passage between a tapered outer cylinder and a tapered axisymmetric body, we analyzed the moment of the fluid-dynamic forces acting on the body, which vibrates harmonically in the rotational mode. The moments were divided into those of inertia, damping, and stiffness. The following conclusions were obtained by study of these coefficients. When the pivot is situated downstream of the passage inlet, intense dynamic instability due to negative damping and static instability due to negative stiffness occur when the passage is divergent. If the pivot is situated at the passage inlet, the instability does not occur. When the angles of the wall surfaces of the axisymmetric body and outer cylinder to the body axis are varied respectively, the boundary of negative damping is approximately given by the straight line. It was found that negative damping is caused by the effect of steady flow.

Key Words: Vibration Coupled with Fluid Motion, Fluid Force, Stability, Damping, Moment, Leakage Flow, Axisymmetric Body

1. Introduction

When fluid energy is effectively used as power, the fluid flow is throttled by valve, rod and seal because it is necessary to control the flow rate and pressure. The fluid system then becomes locally leakage flow and fluid-drive energy is often supplied to the vibrations of the structural system by the work of the fluid-dynamic force. In that case, unstable vibrations due to dynamic instability occur in the system. The unstable vibrations result in widely varying difficulties which are damage, wear of the structural system, and generation of noise.

In Ref. (1), the authors have already reported on the analysis of the fluid dynamic forces acting on the

wall surface of an axisymmetric body (hereafter referred as the center body) for translational vibrations. Results in Ref. (1) revealed that the instabilities due to negative damping and stiffness occur mainly from the value of the area ratio of the exit to inlet passage. In Ref. (2), using a narrow annular tapered passage we examined the stability of the center body which vibrates with one degree of freedom of translational motion. The analysis was found to be valid as the experimental data were in good agreement with theoretical results.

There are a number of unstable vibrations caused by rotational motion in the leakage flow such as the control rods of reactors, diffuser, and so forth. Inada and Hayama⁽³⁾ have analyzed the moment acting on the walls of a one-dimensional, narrow, tapered passage when one wall is vibrating in rotational mode and have examined the mechanism of instability. However, in most cases of the examples above, the working fluid for the actual unstable vibrations flows into the narrow annular passage. For the annular passage, using potential flow theory, Mateescu and Paidoussis⁽⁴⁾ have analyzed the unsteady moment of fluid force acting on the body. The analytical and

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experimental studies on the effect of negative damping on an axisymmetric cylindrical rod, which vibrates in rotational mode, have been carried out by Fujita and Ito⁽⁵⁾. Nevertheless, in the case of an axial annular flow, many points remain to be clarified as to the mechanism of the instability.

In this paper, from a practical point of view, the moments of the fluid dynamic-forces acting on the rigid center body in a narrow annular passage are analyzed by considering the turbulent skin friction. The moments are divided into those of inertia, damping, and stiffness, which are proportional to the angular acceleration, velocity, and displacement, respectively. For the rotational vibrations, the dynamic and static stabilities of the system are studied by evaluating the characteristics of those coefficients.

Nomenclature

- A_r : area ratio of exit to inlet
 C_{aa} : fluid-dynamic damping for rotational vibration
 C_f : coefficient of friction
 I_a : added moment of inertia
 g : circumferential volume flow rate $[(r_2 - r_1)v]$ m²/s
 K_{aa} : fluid-dynamic stiffness for rotational vibration
 l_e : pivot location in z direction m
 $2l_s$: length of outer cylinder m
 l_1, m_1, n_1 : outward direction cosine of boundary surface to r, θ, z axes, respectively
 m_x, m_y, m_z : moment about rectangular coordinates x, y, z axes, respectively N·m
 p : pressure Pa
 p_{in}, p_{ex} : pressure just before inlet and just after exit Pa
 q : volume flow rate in z direction $[(r_2^2 - r_1^2)w/2]$ m³/(rad·s)
 r_{1c}, r_{2c} : radiuses of center body and outer cylinder at inlet m
 r_{1l}, r_{2l} : radiuses of center body and outer cylinder at exit m
 t : time s
 u, v, w : component of flow velocity in r, θ, z directions m/s
 u_{re}, v_{re}, w_{re} : velocity of u, v, w relative to the moving velocity of center body m/s
 u_w, v_w, w_w : moving velocity of boundary surface on center body in r, θ, z direction m/s
 u_{wn} : moving velocity of boundary surface on center body in normal direction m/s
 α : angle of rotation of center body to z -axis rad
 β : pressure loss coefficient of passage =

$$2C_f l_s / (r_{2c} - r_{1c})$$

ν_1, ν_2 : angles of wall surface of center body and outer cylinder to z -axis deg.

ρ : fluid density kg/m³

σ : normal stress Pa

τ : frictional shear stress Pa

ω : circular frequency rad/s

ξ_{in}, ξ_{ex} : pressure loss coefficients at inlet and exit subscript

1 : wall surface of center body

2 : wall surface of outer cylinder

a : fluid-dynamic coefficient for rotational vibration

i : 1, 2

o : steady state

2. Analysis

The analytical model of an axisymmetric body, which vibrates with rotational mode is shown in Fig. 1. The leakage passage corresponds to the annular section between the center body and outer cylinder. The annular width ($r_2 - r_1$) is assumed to be small enough compared to the total length of the passage. ν_1 and ν_2 represent the angles of the wall surfaces of the center body and the outer cylinder to the z -axis, respectively, where the counterclockwise angles are defined positive. That is, the leakage passage is the tapered configuration. The pressures just before the inlet and just after the exit are expressed as p_{in}, p_{ex} , respectively. In addition, the fluid velocity at each location is assumed to be negligible.

As shown by the broken line, the center body is rotated in the x - z plane about the pivot o , which is located at $z = l_e$ and the counterclockwise angular displacement, $\Delta\alpha$ is defined as positive. The amplitude of the angular displacement, $\Delta\bar{\alpha}$ is assumed to be very small compared with the annular width, where the over bar represents the amplitude.

2.1 Basic equations

When integrating both sides of the equations of continuity and motion in cylindrical coordinates with respect to r from r_1 to r_2 , we assume the following conditions. (1) Since the annular width is small

Fig. 1 Analytical model

enough compared to the total length of the passage, the pressure is assumed to be constant in the r direction. (2) Considering the turbulent flow, the fluid velocities v, w in the θ, z directions, respectively are assumed to be constant in the r direction. (3) The normal stresses are assumed to be equal to the pressure ($-p$). Similar to Eqs.(4)-(6) in Ref.(1), the integrated equations are as follows:

$$\frac{\partial g}{\partial \theta} + \frac{\partial q}{\partial z} + \frac{\partial r_q}{\partial t} = 0 \quad (1)$$

$$\frac{\partial}{\partial t} \left(\frac{r_s}{r_g} g \right) + \frac{\partial}{\partial \theta} \left(\frac{r_q}{r_g^2} g^2 \right) + \frac{\partial}{\partial z} \left(\frac{r_s}{r_g r_q} g q \right) = - \frac{r_q \partial p}{\rho \partial \theta} + \frac{1}{\rho} \{ r_2^2 (\tau_{r\theta})_2 - r_1^2 (\tau_{r\theta})_1 \} \quad (2)$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial \theta} \left(\frac{1}{r_q} g q \right) + \frac{\partial}{\partial z} \left(\frac{q^2}{r_q} \right) = - \frac{r_q \partial p}{\rho \partial z} + \frac{1}{\rho} \{ r_2 (\tau_{rz})_2 - r_1 (\tau_{rz})_1 \}, \quad (3)$$

where

$$r_g = r_2 - r_1, \quad r_q = (r_2^2 - r_1^2)/2 \\ r_s = (r_2^3 - r_1^3)/3, \quad g = r_g v, \quad q = r_q w,$$

provided that g represents the volume flow rate per unit length with respect to the circumferential velocity v and q represents the volume flow rate per unit θ with respect to the axial velocity w in the z direction.

Fluid velocity u_1 and u_2 in the r direction can be derived by considering the continuity of the normal velocity components at both wall surfaces of the center body and outer cylinder:

$$u_1 = \frac{\partial r_1}{r_1 \partial \theta} v + \frac{\partial r_1}{\partial z} w + \frac{\partial r_1}{\partial t} \quad (4)$$

$$u_2 = \frac{dr_2}{dz} w. \quad (5)$$

The shear stresses $(\tau_{r\theta})_i$ and $(\tau_{rz})_i$ on the wall surfaces may be derived by considering the turbulent skin friction which is proportional to the square of the resultant velocity with respect to u_i, v and w . In addition, it is necessary to consider the movement of the boundary surface on the center body in both cases of $(\tau_{r\theta})_1$ and $(\tau_{rz})_1$. The shear stresses may be written as

$$\left. \begin{aligned} (\tau_{r\theta})_1 &= \frac{v_{re}}{u_{r1}} \tau_1 = - \frac{C_f \rho}{2} S_1 v_{re} u_{r1} \\ (\tau_{r\theta})_2 &= \frac{v}{u_{r2}} \tau_2 = - \frac{C_f \rho}{2} S_2 v u_{r2} \end{aligned} \right\} \quad (6)$$

$$\left. \begin{aligned} (\tau_{rz})_1 &= \frac{w_{re}}{u_{r1}} \tau_1 = - \frac{C_f \rho}{2} S_1 w_{re} u_{r1} \\ (\tau_{rz})_2 &= \frac{w}{u_{r2}} \tau_2 = - \frac{C_f \rho}{2} S_2 w u_{r2} \end{aligned} \right\}, \quad (7)$$

where

$$\tau_1 = \frac{C_f \rho}{2} S_1 u_{r1}^2, \quad \tau_2 = - \frac{C_f \rho}{2} S_2 u_{r2}^2 \\ u_{r1} = \sqrt{u_{re}^2 + v_{re}^2 + w_{re}^2}, \quad u_{r2} = \sqrt{u_2^2 + v^2 + w^2} \\ u_{re} = u_1 - u_w, \quad v_{re} = v - v_w, \quad w_{re} = w - w_w \\ S_1 = \sqrt{1 + \left(\frac{\partial r_1}{r_1 \partial \theta} \right)^2 + \left(\frac{\partial r_1}{\partial z} \right)^2}, \quad S_2 = \sqrt{1 + \left(\frac{dr_2}{dz} \right)^2}$$

$$u_w = l_1 u_{wn} = - \frac{1}{S_1^2} \frac{\partial r_1}{\partial t} \\ v_w = m_1 u_{wn} = - \frac{1}{S_1^2} \frac{\partial r_1}{r_1 \partial \theta} \frac{\partial r_1}{\partial t} \\ w_w = n_1 u_{wn} = - \frac{1}{S_1^2} \frac{\partial r_1}{\partial z} \frac{\partial r_1}{\partial t}, \quad u_{wn} = \frac{1}{S_1} \frac{\partial r_1}{\partial t}.$$

The pressures just after the inlet $p(0)$ and just before the exit $p(2l_s)$ are given by using the pressure loss coefficients of the passage ξ_{in} and ξ_{ex} , respectively. In addition, the circumferential velocity just after the inlet is assumed to be negligible. The following boundary conditions are then obtained.

$$p(0) = p_{in} - \frac{\xi_{in} \rho w^2(0)}{2} \quad (8)$$

$$p(2l_s) = p_{ex} + \frac{\xi_{ex} \rho w^2(2l_s)}{2} \quad (9)$$

$$v(0) = 0. \quad (10)$$

2.2 Moment of fluid dynamic force

For the steady state, the center body and outer cylinder are assumed to be concentric and the fluid velocity v in the θ direction is assumed to be equal to zero. That is, the fluid in the clearance flows only in the r, z directions. Similar to Eq.(15) in Ref.(1), the nondimensional radii of the center body and outer cylinder are given as follows:

$$\left. \begin{aligned} R_{1o} &= H_1 Z + \Gamma_1 \\ R_{2o} &= H_2 Z + \Gamma_2 \end{aligned} \right\}, \quad (11)$$

where

$$R_{io} = r_{io}/r_{1c}, \quad Z = (z - l_s)/l_s \\ H_1 = (r_{1l} - r_{1c})/(2r_{1c}), \quad H_2 = (r_{2l} - r_{2c})/(2r_{1c}) \\ \Gamma_1 = (r_{1l} + r_{1c})/(2r_{1c}), \quad \Gamma_2 = (r_{2l} + r_{2c})/(2r_{1c}).$$

From the above nondimensional equations, the values of Z at the inlet and exit of the leakage passage are found to be -1 and $+1$, respectively.

As shown in Fig. 1, the center body in an unsteady state is assumed to be vibrating harmonically with the amplitude of the angular displacement $\Delta \bar{\alpha}$ in the x direction. From Fig. 1, the nondimensional radius of the center body is as follows:

$$R_1 = \sqrt{R_{1o}^2 - \{ (L_{es} + L_o Z) \Delta \alpha \} \sin \theta \}^2} \\ + \{ (L_{es} + L_o Z) \Delta \alpha \} \cos \theta \\ \cong R_{1o} + \{ (L_{es} + L_o Z) \Delta \bar{\alpha} \} \cos \theta \exp(j\Omega T), \quad (12)$$

where

$$\Delta \alpha = \Delta \bar{\alpha} \exp(j\Omega T) \\ R_1 = r_1/r_{1c}, \quad L_o = l_s/r_{1c}, \quad \Omega = \omega t_o \\ L_{es} = \frac{(l_s - l_e)}{r_{1c}}, \quad T = \frac{t}{t_o}, \quad t_o = \frac{r_{qo}(0) l_s}{q_o}.$$

The subscript o in the above equations expresses the quantities in the steady state. From Eq.(20) in Ref.(1), the steady flow rate q_o is given as follows:

$$q_o = r_{1c}^2 \sqrt{\frac{p_{in}}{\rho [Y_o(-1) + \xi_{in} / \{ 2R_{qo}^2(-1) \}]}}, \quad (13)$$

where

$$R_{qo}(-1) = r_{qo}(0)/r_{1c}^2 = (R_{2c}^2 - 1)/2$$

$$Y_o(Z) = \sum_{n=0}^N C_{on} P_n(Z)$$

$$P_n(Z) : \text{Legendre's polynomial, } = \frac{1}{2^n n!} \frac{d^n}{dZ^n} \{(Z^2 - 1)^n\}$$

C_{on} : a constant

N : maximum degree of polynomial.

The radius R_1 in Eq. (12) and the other quantities are expressed by the sum of the steady component and unsteady component :

$$\left. \begin{aligned} R_1 &= R_{1o} + \Delta R_1(Z, \theta, T) \\ P &= P_o(Z) + \Delta P(Z, \theta, T) \\ Q &= 1 + \Delta Q(Z, \theta, T) \\ G &= \Delta G(Z, \theta, T) \end{aligned} \right\} \quad (14)$$

where

$$\begin{aligned} \Delta R_1(Z, \theta, T) &= \{(L_{es} + L_o Z) \Delta \bar{a}\} \cos \theta \\ &\quad \times \exp(j\Omega T) \\ \Delta P(Z, \theta, T) &= \Delta \bar{P}(Z) \cos \theta \exp(j\Omega T) \\ \Delta Q(Z, \theta, T) &= \Delta \bar{Q}(Z) \cos \theta \exp(j\Omega T) \\ \Delta G(Z, \theta, T) &= \Delta \bar{G}(Z) \sin \theta \exp(j\Omega T) \\ P &= \frac{p_o(Z) + \Delta p}{p_{co}}, \quad Q = \frac{q_o + \Delta q}{q_o}, \quad G = \frac{l_s \Delta g}{q_o} \\ P_o(Z) &= Y_o(Z), \quad p_{co} = \rho q_o^2 / r_{1c}^4. \end{aligned}$$

The nondimensional steady pressure $P_o(Z)$ in the above equation is given by the equations of continuity and motion in the steady state and equal to $Y_o(Z)$ in Eq. (13).

After nondimensionalizing Eqs. (1)-(3) and linearizing these equations by Eq. (14), the equations of continuity and motion in an unsteady state are as follows :

$$\begin{aligned} \frac{\partial(\Delta G)}{\partial \theta} + \frac{\partial(\Delta Q)}{\partial Z} - \frac{R_{1o}}{R_{qo}(-1)} \frac{\partial(\Delta R_1)}{\partial T} &= 0 \quad (15) \\ \frac{R_{qo}}{R_{qo}(-1)} \frac{\partial(\Delta G)}{\partial T} + \frac{\partial(\Delta G)}{\partial Z} - \left(\frac{1}{R_{qo}} \frac{dR_{qo}}{dZ} \right. \\ &\quad \left. + \frac{1}{R_{qo}} \frac{dR_{qo}}{dZ} - \frac{1}{R_{so}} \frac{dR_{so}}{dZ} \right) \Delta G \\ &= L_o^2 \frac{R_{qo} R_{so}}{R_{so}} \left\{ -R_{qo} \frac{\partial(\Delta P)}{\partial \theta} + R_{2o}^2 \Delta(T_{rz})_2 \right. \\ &\quad \left. - R_{1o}^2 \Delta(T_{rz})_1 \right\} \quad (16) \\ \left(\frac{R_{qo}}{R_{qo}(-1)} \frac{\partial}{\partial T} + 2 \frac{\partial}{\partial Z} - \frac{2}{R_{qo}} \frac{dR_{qo}}{dZ} \right) \Delta Q + \frac{\partial(\Delta G)}{\partial \theta} \\ &\quad + \left(\frac{R_{1o}}{R_{qo}} \frac{\partial}{\partial Z} - \frac{2R_{1o}}{R_{qo}^2} \frac{dR_{qo}}{dZ} + \frac{H_1}{R_{qo}} \right) \Delta R_1 \\ &= R_{qo} \left[-R_{qo} \frac{\partial(\Delta P)}{\partial Z} + R_{1o} \frac{dP_o}{dZ} \Delta R_1 \right. \\ &\quad \left. + L_o \{ R_{2o} \Delta(T_{rz})_2 - R_{1o} \Delta(T_{rz})_1 - (T_{rz})_{1o} \Delta R_1 \} \right]. \quad (17) \end{aligned}$$

After nondimensionalizing the fluid velocity u_i, v and w in Eqs. (6) and (7) by the steady velocity in the z direction, $w_o (= q_o / r_{qo})$ and linearizing these equations by Eq. (14), the dimensionless shear stresses $\Delta(T_{rz})_i$ and $\Delta(T_{rz})_i$ in the right-hand sides in Eqs. (16) and (17) are obtained as follows.

$$\Delta(T_{rz})_i = \pm \frac{C_f}{L_o} \frac{S_{io}^2}{B_p B_m^2} \Delta G \quad (18)$$

$$\begin{aligned} \Delta(T_{rz})_1 &= -\frac{C_f}{R_{qo}^2} \left[S_{1o}^2 \left\{ \Delta Q + \frac{R_{1o}}{R_{qo}} \Delta R_1 \right\} \right. \\ &\quad \left. + \frac{H_1}{L_o} \left\{ \frac{R_{qo}}{R_{qo}(-1)} \frac{\partial(\Delta R_1)}{\partial T} + \frac{\partial(\Delta R_1)}{\partial Z} \right\} \right] \quad (19) \end{aligned}$$

$$\Delta(T_{rz})_2 = -\frac{C_f S_{2o}^2}{R_{qo}^2} \left\{ \Delta Q + \frac{R_{1o}}{R_{qo}} \Delta R_1 \right\}, \quad (20)$$

where

$$\begin{aligned} B_p &= H_p Z + \Gamma_p, \quad B_m = H_m Z + \Gamma_m, \quad H_{p,m} = H_2 \pm H_1 \\ \Gamma_{p,m} &= \Gamma_2 \pm \Gamma_1, \quad S_{io} = \sqrt{1 + (H_i / L_o)^2}. \end{aligned}$$

The governing equations for $\Delta \bar{Q}$, $\Delta \bar{G}$, and $\Delta \bar{P}$ may be written in matrix form by substituting Eqs. (18)-(20) into Eqs. (16) and (17), and taking Eq. (15) into consideration; i.e.,

$$\begin{aligned} \begin{bmatrix} \frac{d}{dZ} & 1 \\ 0 & \frac{d}{dZ} + j\Omega K_{2o} + K_{3o} \\ \frac{d}{dZ} + j\Omega \frac{K_{2o}}{2} + K_{5o} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \Delta \bar{Q} \\ \Delta \bar{G} \\ \Delta \bar{P} \end{bmatrix} \\ = \begin{bmatrix} j\Omega (L_{es} + L_o Z) K_{1o} \\ 0 \\ K_{7o} + (j\Omega K_{8o} + K_{9o})(L_{es} + L_o Z) \end{bmatrix} \Delta \bar{a}, \quad (21) \end{aligned}$$

where

$$\begin{aligned} K_{1o} &= \frac{(H_1 Z + \Gamma_1)}{R_{qo}(-1)}, \quad K_{2o} = \frac{B_p B_m}{2 R_{qo}(-1)} \\ K_{3o} &= \frac{E_{1o} Z^2 + E_{2o} Z + E_{3o}}{B_s} - \frac{3 H_p H_m Z + \Gamma_c + 3 \Gamma_k}{B_p B_m} \\ K_{4o} &= -3 L_o^2 \frac{B_p^2 B_m^3}{4 B_s} \\ K_{5o} &= \frac{(C_f L_o D_{1o} - 2 H_p H_m) Z + C_f L_o D_{2o} - 2 \Gamma_k}{B_p B_m} \\ K_{6o} &= \frac{(B_p B_m)^2}{8}, \quad K_{7o} = -(C_f H_1 + L_o) \frac{H_1 Z + \Gamma_1}{B_p B_m} \\ K_{8o} &= -\frac{C_f H_1}{L_o} \frac{H_1 Z + \Gamma_1}{2 R_{qo}(-1)} \\ K_{9o} &= 2 \frac{E_{4o} Z^2 + E_{5o} Z + E_{6o}}{(B_p B_m)^2} - \frac{C_f L_o S_{1o}^2 + 2 H_1}{2 B_p B_m} \\ &\quad + \frac{B_p B_m (H_1 Z + \Gamma_1)}{4} \frac{dP_o}{dZ} \\ B_s &= H_s Z^3 + H_r Z^2 + \Gamma_h Z + \Gamma_s, \quad H_s = H_2^3 - H_1^3 \\ H_r &= 3(H_2^2 \Gamma_2 - H_1^2 \Gamma_1), \quad \Gamma_h = 3(H_2 \Gamma_2^2 - H_1 \Gamma_1^2) \\ \Gamma_s &= \Gamma_2^3 - \Gamma_1^3 \\ E_{1o} &= 3(H_s + (C_f L_o D_{3o})/2) \\ E_{2o} &= 2(H_r + (3 C_f L_o D_{4o})/2) \\ E_{3o} &= \Gamma_h + (3 C_f L_o D_{5o})/2 \\ E_{4o} &= 2 H_p H_m H_1 - C_f L_o D_{3o}^* \\ E_{5o} &= 2(H_p H_m \Gamma_1 + H_1 \Gamma_h - C_f L_o D_{4o}^*) \\ E_{6o} &= 2 \Gamma_h \Gamma_1 - C_f L_o D_{5o}^* \\ \Gamma_c &= H_2 \Gamma_1 - H_1 \Gamma_2, \quad \Gamma_k = H_2 \Gamma_2 - H_1 \Gamma_1 \end{aligned}$$

$$\begin{aligned}
D_{1o} &= H_1 S_{1o}^2 + H_2 S_{2o}^2, \quad D_{2o} = \Gamma_1 S_{1o}^2 + \Gamma_2 S_{2o}^2 \\
D_{3o} &= (H_1 S_{1o})^2 + (H_2 S_{2o})^2, \quad D_{4o} = H_1 \Gamma_1 S_{1o}^2 + H_2 \Gamma_2 S_{2o}^2 \\
D_{5o} &= (\Gamma_1 S_{1o})^2 + (\Gamma_2 S_{2o})^2 \\
D_{3o}^* &= (H_1 S_{1o})^2 + H_1 H_2 S_{2o}^2 \\
D_{4o}^* &= H_1 \Gamma_1 S_{1o}^2 + (H_1 \Gamma_2 + H_2 \Gamma_1) S_{2o}^2 / 2 \\
D_{5o}^* &= (\Gamma_1 S_{1o})^2 + \Gamma_1 \Gamma_2 S_{2o}^2.
\end{aligned}$$

By nondimensionalizing and linearizing the boundary conditions Eqs. (8)–(10), the solutions of $\Delta \bar{Q}$, $\Delta \bar{G}$ and $\Delta \bar{P}$ in Eq. (21) are obtained by Galerkin's method, and Legendre's polynomial is used in the basic function. From the solutions, the dimensionless moments $\Delta \bar{M}_y$ and so forth are obtained as follows.

$$\begin{aligned}
\Delta \bar{M}_y &= -\frac{R_{qo}^2(-1)}{2} \int_{-1}^1 \left[L_{ez} \left\{ R_{1o} \{ \Delta \bar{P} + \Delta (\bar{T}_{r\theta})_1 \} \right. \right. \\
&\quad \left. \left. + \frac{H_1 R_{1o} \Delta (\bar{T}_{rz})_1}{L_o} + \frac{(T_{rz})_{1o} \{ R_{1o} L_o + H_1 L_{ez} \} \Delta \bar{\alpha}}{L_o} \right\} \right] dZ \\
&= \{ \Lambda_{m1}(\Omega) + j \Lambda_{m2}(\Omega) \} \Delta \bar{\alpha}, \quad (22)
\end{aligned}$$

where

$$\begin{aligned}
\Delta M_y &= \Delta \bar{M}_y \exp(j\Omega T) = R_{qo}^2(-1) \Delta m_y / (2\pi l_s r_{1c}^2 \rho c o) \\
\Delta M_x &= \Delta M_z = 0, \quad L_{ez} = L_{es} + L_o Z \\
\Delta \bar{Q} &= \{ Y_{11}(Z) + j Y_{12}(Z) \} \Delta \bar{\alpha} \\
\Delta \bar{G} &= \{ Y_{21}(Z) + j Y_{22}(Z) \} \Delta \bar{\alpha} \\
\Delta \bar{P} &= \{ Y_{31}(Z) + j Y_{32}(Z) \} \Delta \bar{\alpha} \\
Y_{mk}(Z) &= \sum_{n=0}^N (C_{mk})_n P_n(Z).
\end{aligned}$$

3. Calculated Results

The conditions for calculation are shown in Table 1. Taking into consideration the turbulent flow in the leakage passage, Reynolds number at the passage inlet is about 30 000, where the typical length is 4 m (m : the hydraulic mean depth at the inlet) and the typical velocity is the mean velocity averaged over the cross-sectional area of the inlet. The maximum degree of Legendre's polynomial, which is used as the basic function of Galerkin's method, is $N=15$.

3.1 Fluid-dynamic coefficients

Using $\Lambda_{m1}(\Omega)$ and $\Lambda_{m2}(\Omega)$ in Eq. (22), the fluid-dynamic moment of inertia I_a , damping C_{aa} , and stiffness K_{aa} are determined as follows.

$$\left. \begin{aligned} I_a &= \{ \Lambda_{m1}(\Omega) - \Lambda_{m1}(0) \} / \Omega^2 \\ C_{aa} &= -\Lambda_{m2}(\Omega) / \Omega \\ K_{aa} &= -\Lambda_{m1}(0) \end{aligned} \right\}, \quad (23)$$

where the subscript s expresses the fluid-dynamic coefficients independent of Ω .

Changing the pivot location l_e , the fluid-dynamic coefficients in Eq. (23) for the area ratio of the exit to the inlet A_r are shown in Figs. 2(a)–(c). In Fig. 2(a), the pivot location l_e is equal to $-l_s$. That is, the pivot is located one-half total length of the passage above the inlet. In Fig. 2(b), the pivot is situated at the inlet of the passage. The pivot locations shown in Figs. 2(c) and (a) are symmetrical with respect to the passage inlet. In these figures, the value of A_r

Table 1 Calculated conditions

Nomenclature	Value
l_s	0.0175 m
q_o	$3 \times 10^{-4} \text{ m}^3/(\text{rad} \cdot \text{s})$
ρ	999 kg/m ³
r_{1c}	0.015 m
r_{2c}	0.0158 m
ξ_{in}	1.0
ξ_{ex}	0.0

(a) $l_e = -l_s$

(b) $l_e = 0$

(c) $l_e = l_s$

Fig. 2 Fluid-dynamic coefficients for I_a as a function of Ω

ranges from 0.1 to 1. As the angle of the wall surface of the outer cylinder to the z -axis ν_2 is zero, the angle of the wall surface of the center body ν_1 for each A_r is determined. Since the pressure loss coefficient of the passage $\beta\{=2C_f l_s/(r_{2c}-r_{1c})\}$ is equal to 1, the value of the coefficient of friction C_f is about 0.023. The value of Ω is chosen as 0.1, 1, and furthermore, the fluid-dynamic coefficients are also calculated at $\Omega=10^5$ in the case of very large Ω .

In Fig. 2(a), C_{aa} at $\Omega=0.1$ is negative over an A_r range of not less than 4 in the case of the divergent passage. In Fig. 2(b), C_{aa} values shown by solid lines for $\Omega=0.1, 1$ and 10^5 are almost identical and are not negative throughout the range of A_r . On the other hand, Fig. 2(c) shows that C_{aa} values in both cases of $\Omega=0.1$ and 1 are negative over an A_r range of greater than 2. Therefore, in the case of the rotational vibration, the pivot location exerts a significant influence on the stability of the system. That is, when the pivot is located in the order of the inlet, upstream and downstream of the inlet, the system is gradually unstable in that order. When the pivot is situated downstream of the inlet, intense dynamic instability occurs in the system when the passage is divergent.

As shown in Figs. 2(a)-(c), I_a and C_{aa} at $\Omega=10^5$ decrease monotonically as A_r increases. When the value of Ω is greater than 10^5 , I_a and C_{aa} become constants independent of Ω in all cases of the pivot location. That is, the equations of I_a and C_{aa} may be written as

$$\left. \begin{aligned} I_{as} &= \lim_{\Omega \rightarrow \infty} \{ \Lambda_{m1}(\Omega)/\Omega^2 \} \\ C_{aas} &= \lim_{\Omega \rightarrow \infty} \{ -\Lambda_{m2}(\Omega)/\Omega \} \end{aligned} \right\} \quad (24)$$

When the value of Ω approaches infinity, I_{as} and C_{aas} in Eq. (24) express the fluid-dynamic moment of inertia and damping in the fluid at rest, respectively, because the velocity and acceleration are very large compared with the other quantities.

Considering the small vibration in the fluid at rest, I_{as} is analytically calculated in special cases. The governing equations of $\Delta\bar{Q}$, $\Delta\bar{G}$ and $\Delta\bar{P}$, which are used in calculation of I_{as} , are given by considering infinite Ω and $C_f=0$ in Eq. (21):

$$\begin{bmatrix} d/dZ & 1 & 0 \\ 0 & j\Omega & K_{40}/K_{20} \\ j\Omega & 0 & 2(K_{60}/K_{20})d/dZ \end{bmatrix} \times \begin{bmatrix} \Delta\bar{Q} \\ \Delta\bar{G} \\ \Delta\bar{P} \end{bmatrix} = \begin{bmatrix} j\Omega L_{ez} K_{10} \\ 0 \\ 0 \end{bmatrix} \Delta\bar{a}. \quad (25)$$

(1) Analytical solution of I_{as} in the case of the parallel passage

The solution of $\Delta\bar{P}(Z)$ is obtained by eliminating $\Delta\bar{Q}$ and $\Delta\bar{G}$ in Eq. (25) and considering the boundary conditions $\Delta\bar{P}(\pm 1)=0$. Then, the analytical solution

of I_{as} is given by integrating $\Delta\bar{P}(Z)$ from the inlet to the exit along the wall surface of the center body:

$$I_{as} = -\frac{1}{L_R^3 R_{q0}(-1)} \left[-\frac{(L_{es}^2 - L_o^2)}{\sinh(2L_R)} + \frac{(L_{es}^2 + L_o^2)}{\tanh(2L_R)} - \left\{ L_{es}^2 + \frac{1}{3}L_o^2 \right\} L_R - \frac{L_o^2}{L_R} \right], \quad (26)$$

where

$$L_R = 2L_o/(R_{2c}+1), \quad B_s = 3B_p^2 B_m/4.$$

(2) The generalized method (Galerkin's method)

$$I_{as} = -2 \left\{ H_1 L_o \left(\frac{C_{a0}}{3} + \frac{2C_{a2}}{15} \right) + \frac{C_{a1}}{3} (H_1 L_{es} + \Gamma_1 L_o) + C_{a0} \Gamma_1 L_{es} \right\}, \quad (27)$$

where

$$\Delta\bar{P}(Z) = \frac{2\Omega^2}{R_{q0}^2(-1)} \left[\sum_{n=0}^{15} C_{an} P_n(Z) \right] \Delta\bar{a}.$$

Considering the consistency of each fluid-dynamic coefficient, these are determined as follows. The fluid-dynamic moment of inertia is defined as I_{as} and the stiffness K_{aa} is given by the other real part of $\Lambda_{m1}(\Omega)$ which is a function of Ω . In order to examine the effect of the steady flow q_o , the fluid-dynamic damping C_{aa} is given by the sum C_{aas} , which is independent of Ω , and $C_{aa\Omega}(\Omega)$, which is a function of Ω . That is, these equations are as follows:

$$\left. \begin{aligned} K_{aa} &= -\Lambda_{m1}(\Omega) + I_{as}\Omega^2 \\ C_{aa} &= C_{aas} + C_{aa\Omega}(\Omega) \end{aligned} \right\}, \quad (28)$$

where

$$C_{aa\Omega}(\Omega) = -\{\Lambda_{m2}(\Omega)/\Omega + C_{aas}\}.$$

Similar to Fig. 2, the fluid-dynamic coefficients for $\Omega=0.1$ and 1 are shown in Figs. 3(a) and (b). In Fig. 3(a), the pivot is situated upstream of the passage inlet and K_{aa} at $\Omega=1$ shows negative stiffness over an A_r range of not less than about 2.6. In Fig. 3(b), the pivot is situated downstream of the passage inlet and K_{aa} in both cases of $\Omega=0.1$ and 1 shows negative stiffness over an A_r range of not less than about 0.7. That is, K_{aa} is negative even for the convergent passage. The figure, in which the pivot is located at the inlet, is omitted because K_{aa} is not negative throughout the range of A_r . Like the dynamic instability, intense static instability occurs when the pivot is situated downstream of the inlet. In this case, static instability occurs even for the convergent passage, where the instability is generally produced in the case of the divergent passage.

Since C_{aas} in Eq. (24) expresses the fluid-dynamic damping with respect to viscous resistance, $C_{aa\Omega}$ at small Ω is the damping which has the effect of the steady flow q_o in the z direction. That is, the effect of q_o increases with the decrease in Ω , and $C_{aa\Omega}$ at $\Omega=0.1$ is approximately equal to limit value as Ω approaches zero. $C_{aa\Omega}$ in Figs. 3(a) and (b) is approximately negative throughout the range of A_r .

Fig. 3 Fluid-dynamic coefficients for I_{as} independent of Ω Fig. 4 Region of negative damping on the plane ν_1 - ν_2

On the other hand, C_{aas} on the viscous resistance is positive throughout the range of A_r . From the above discussion, it is found that the negative damping is caused by the effect of the steady flow q_o .

As shown in Figs. 3(a) and (b), the differences in K_{aa} and $C_{aa\Omega}$ between $\Omega=0.1$ and $\Omega=1$ gradually decrease with the decrease in A_r . The reason for this is that K_{aa} and $C_{aa\Omega}$ at which A_r is smaller than 1 are approximately independent of Ω because Bode diagram for the fluid-dynamic moment at $A_r=0.1$ is approximately expressed as of a second-order system, whereas Bode diagrams in the case of the rotational vibration are similar to the figures in Ref.(1).

3.2 Region of the negative fluid-dynamic damping

Since the angles ν_2 shown in the preceding paragraph are all equal to zero, the regions of the negative damping on the plane ν_1 - ν_2 are shown in Figs. 4(a) and (b). The solid line at which C_{aa} is equal to zero is calculated by Regula-Falsi method and the dashed lines show the area ratio A_r .

In Fig. 4(a), the pivot is situated upstream of the inlet and the solid line shows only the boundary line at $\Omega=0.1$. As shown in Fig. 2(a), C_{aa} at $\Omega=1$ does not decrease to the negative damping even though the values of ν_1 and ν_2 are varied with each other in the range of -15° to 30° . On the other hand, Fig. 4(b) shows that the solid line at $\Omega=1$ is in the vicinity of the solid line at $\Omega=0.1$ on the plane ν_1 - ν_2 and these are approximately expressed by the straight line $\nu_1 = \nu_2$, when the pivot is situated downstream of the inlet. From the dashed lines, the values of A_r for the solid lines shown in Fig. 4(b) are smaller than the values in Fig. 4(a). Therefore, when the pivot is situated downstream of the inlet, the region of the dynamic instability including Ω is very large compared to the other pivot locations.

4. Conclusions

We have analyzed the moments of the fluid-dynamic forces acting on an axisymmetric body which vibrates with rotational motion in the narrow

annular and tapered passage. The moments have been divided into three elements which are proportional to the angular acceleration, velocity, and displacement, respectively. Using these three elements, we have defined the fluid-dynamic moment of inertia, damping, and stiffness. When the pivots are located at the inlet of the passage, one-half total length of the passage above and below the inlet, respectively, the characteristics of those coefficients are evaluated by changing the area ratio of the exit to the inlet and so forth. It has been shown that :

(1) In the case of the rotational vibration, the stability of the system is greatly governed by the pivot location. That is, when the pivots are located at the passage inlet, upstream, and downstream of the inlet, respectively, the system is gradually unstable in that order. When the pivot is situated downstream of the inlet, intense dynamic instability due to negative damping and static instability due to negative stiffness occur in the system and static instability occurs even for the convergent passage, where the instability is generally produced in the case of the divergent passage.

(2) When the nondimensional circular frequency approaches infinity, the fluid-dynamic coefficients correspond to the moment of inertia and damping within the fluid at rest independent of the pivot location. Taking that moment of inertia into consideration, we defined each fluid-dynamic coefficient.

(3) The governing equations of the nondimensional flow rate in the axial direction, circumferential flow rate and pressure, which are analytically calculated for the fluid-dynamic moment of inertia defined as in (2), can be derived. In the case of the parallel passage, this moment is given by an analytical solution.

(4) It is found that negative damping is caused by the effect of the steady flow rate in the axial direction, whereby the fluid-dynamic damping is expressed by the sum of the element independent of the nondimensional circular frequency and the element which is a function of the frequency.

(5) When the angles of the wall surfaces of the center body and the outer cylinder to the axis are varied with each other, the boundary for negative damping is given by a straight line in the plane of both angles. If the pivot is situated downstream of the inlet, the boundary is expressed by the straight line where the angle is approximately equal to the other angle.

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