Numerical Algorithm for Dynamic Problems Involving Fractional Operators*

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This paper presents the numerical method for dynamic problems involving fractional operators. Based on Liouville-Riemann's definition of fractional derivatives, the algorithm is developed by incorporating one-step schemes of well-known Newmark types into our own formula. The scheme has the merit of excellent operational characteristics and the simplicity of application. Numerical experiments were given to show the superiority and effectiveness of the algorithm.

Key Words: Vibration, Fractional Operator, Viscoelasticity, Newmark.

1. Introduction

The concept of fractional operators in engineering applications is now increasingly attractive in the formulation of the constitutive law for some viscoelastic materials. The merits of modeling the viscoelastic material's behavior with fractional operator mainly lies in two aspects:

(1) The frequency dependence of some viscoelastic materials is often difficult to be described with classical viscoelasticity of integer-order derivative operators with which too complicated strain-stress relations are often resulted in;

(2) The model with fractional-derivative operator on viscoelastic materials has successfully been fitted to experimental data over a broad frequency range by using only few parameters required to accurately describe the dynamic properties.

Recalling the development of viscoelasticity with the fractional operator models, P. G. Nutting (1921),

A. Gemant (1936), and Scott-Blair (1949) were thought to be the first to use this concept in the constitutive equations⁽¹⁾, whereas Bagley and Torvik (1983)^{(2),(3)} not only applied the fractional operator model to certain types of materials, but also showed their attempts to explain the physical implication of the seemingly abstract concept of fractional calculus. Recent years Shimizu⁽⁴⁾ has successfully applied so-called Generalized-Voigt model with fractional derivative to the experimental data from newly developed viscoelastic material based on Silicone gel, which shows the astonishing property of shock absorbing. The list of the authors on the field of fractional calculus and its application can be long as we can point out the contribution of K. S. Miller, B. Ross, K. B. Oldham, J. Spanier, and R. C. Koeller etc.. And the fractional operators indeed give rise to a richer variety of functional families with the solid background.

In spite of the advantages of fractional operators for modeling some viscoelastic materials, the use of fractional operators often involves tedious and complex analysis, although for some simple cases the Laplace or Fourier transformation method has often been used to carry out the analysis as did by Bagley (1983) and recently Sakakibara (1997)⁽⁵⁾. Accordingly, the numerical solution for the problem is our logical choice. Padovan (1987)⁽⁶⁾, based on the Grunwald's definition for the fractional operators, outlined the numerical schemes of the implicit,

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explicit and predictor-corrector types to this end. Bearing on the same base, Koh and Kelly (1990)⁽⁷⁾, Eldred, Baker and Palazotto (1995)⁽⁸⁾ developed the numerical algorithms to solve their own problems from the viscoelastic materials.

Although Padovan in his paper pointed out the equivalence of Grunwald's definition and Liouville-Riemann's for fractional derivatives, the numerical algorithm for the fractional operator of Liouville-Riemann's definition does pose a question. In fact, K. S. Miller⁽⁹⁾ has indicated that if the fractional number q is in the range of $(0, 1)$, the Liouville-Riemann's formula exists as an improper integral. This difficulty can be bypassed by taking advantage of the definition with infinite sum types such as Grunwald's definition or the Laplace and Fourier transformation method. Although Oldham⁽¹⁰⁾ derived R-algorithms from Liouville-Riemann's formula for $q < 0$, his treating the derivative of integrand as a constant within the interval $[t_i, t_{i+1}]$ in his L1-algorithms sometimes too simplified. Koh, being aware of the problem, developed what he called Quadratic L1-algorithm. But his whole numerical scheme was based on the finite difference method.

Based on Liouville-Riemann's fractional integral formula, this paper presents single-step numerical scheme for the dynamic problems of viscoelastic materials. As the Newmark-type numerical methods have the excellent operational characteristics, the efforts have been made in the research to incorporate the Newmark numerical method into our own algorithm to solve the fractional differential equation. In order to compare the numerical results with the exact solution of dynamic systems, the simplest single-degree-of-freedom system was taken as our example. Although the simple case was studied in this paper, our goal to develop the numerical scheme for fractional differential equations is to deal with more complicated and even nonlinear dynamic problems for viscoelastic structures in our future studies. The numerical approximation is compared with exact solution of the dynamic system, especially its transient behavior. And the numerical experiments were given to show the superiority and efficiency of the scheme, the error analysis is also briefly discussed.

2. Numerical Algorithm of Dynamic System

2.1 Equation of motion involving fractional operator

Shimizu fit his experimental data from the viscoelastic material silicone gel by adopting so-called "Generalized" Voigt model.

$$\sigma(t) = G_0 \varepsilon(t) + \mu D^q \varepsilon(t), \quad 0 < q < 1 \quad (1)$$

where $\sigma(t)$ is the shear stress, $\varepsilon(t)$ the shear strain, G_0

and μ the experimental parameters. D denotes the ordinary time derivative d/dt , whereas D^q represents the fractional derivative of order q .

Consider now the oscillator of the viscoelastic body with above constitutive relationship. The equation of motion for the simplest single degree of freedom system then takes the form

$$mD^2x(t) + cD^q x(t) + kx(t) = F(t) \quad (2)$$

where m is the mass of the oscillator, $c \propto \mu$ the damping of the system, $k \propto G_0$ the spring constant, $x(t)$ the displacement of oscillator, and $F(t)$ the external force applied to the system.

Rewrite above differential equation of motion as follows

$$D^2x(t) + aD^q x(t) + bx(t) = f(t) \quad (3)$$

where $a = c/m$ indicates the damping coefficient, $b = k/m$ the natural frequency of the system and $f(t) = F(t)/m$ the external excitation.

2.2 Numerical Scheme

Let's first focus our attention on the fractional operator of Eq. (3), namely, $D^q x(t)$.

The Liouville-Riemann's fractional integral is defined as

$$D^{-u}x(t) = \frac{1}{\Gamma(u)} \int_0^t \frac{x(\tau) d\tau}{(t-\tau)^{1-u}}, \quad 0 < u < 1 \quad (4)$$

where $\Gamma(u)$ is the gamma function.

Then the fractional derivative of Liouville-Riemann's definition for $D^q x(t)$ takes the form

$$\begin{aligned} D^q x(t) &= D[D^{-u}x(t)] \\ &= \frac{1}{\Gamma(u)} \frac{d}{dt} \int_0^t \frac{x(\tau) d\tau}{(t-\tau)^{1-u}} \end{aligned} \quad (5)$$

where $u = 1 - q$, $0 < u < 1$.

It can be seen immediately from Eq. (5) that at upper limit, $\tau = t$, the integrand becomes infinite. It is also noted that the result of the integration is to be differentiated with respect to t , but if we differentiate it before integration, the resulting integral will diverge. These difficulties may ruin the numerical algorithms chosen, and require us to take particular cares. On the other hand, as Liouville-Riemann's definition for fractional derivatives has the obvious physical insight and connection with the theory of viscoelasticity⁽¹¹⁾, and has some simplicity for the mathematical analysis⁽¹²⁾, we prefer to choose it as our understanding the behaviors of some viscoelastic materials fitted to the fractional operator model.

In order to make use of Liouville-Riemann's formula to deduce our numerical scheme and to prevent from the problems mentioned above, we first apply the composition rule to $D^q x(t)$, that is⁽⁹⁾

$$\begin{aligned} D^q x(t) &= D[D^{-u}x(t)] \\ &= \frac{x(0)}{\Gamma(u)} t^{u-1} + D^{-u} \dot{x}(t), \quad u = 1 - q. \end{aligned} \quad (6)$$

where $\dot{x}(t) = Dx(t)$ represents the velocity of the

oscillator, and $x(0)$ is the value of displacement at $t=0$ and is often given as an initial condition.

We can see that there is singular point at $t=0$ in Eq. (6). But if we just want to investigate the system response at time t except at zero, it doesn't make trouble. It, on the other hand, does rise the problem when the instantaneous response of system at $t=0$ is required. But, this problem is closely related with the modeling of materials. The more detailed discussion can be referred to the paper⁽¹¹⁾.

And the numerical algorithm to calculate the fractional derivative $D^q x(t)$ at $t=t_n$ of Eq. (6) is

$$\begin{aligned} D^q x(t_n) &= \frac{x(0)}{\Gamma(1-q)} t_n^{-q} + D^{q-1} \dot{x}(t_n) \\ &= \frac{x(0)}{\Gamma(1-q)} t_n^{-q} + \frac{1}{\Gamma(1-q)} \int_0^{t_n} \frac{\dot{x}(\tau) d\tau}{(t_n - \tau)^q} \\ &= \frac{1}{\Gamma(1-q)} \frac{x(0)}{t_n^q} + \frac{1}{\Gamma(1-q)} \left[\int_0^{t_{n-1}} \frac{\dot{x}(\tau) d\tau}{(t_n - \tau)^q} \right. \\ &\quad \left. + \int_{t_{n-1}}^{t_n} \frac{\dot{x}(\tau) d\tau}{(t_n - \tau)^q} \right] \\ &= \frac{1}{\Gamma(1-q)} I_0 + \frac{1}{\Gamma(1-q)} I_{n-1} + \frac{1}{\Gamma(1-q)} \Delta I_n \end{aligned} \quad (7)$$

where we denote

$$I_0 = \frac{x(0)}{t_n^q} \quad (8)$$

$$I_{n-1} = \int_0^{t_{n-1}} \frac{\dot{x}(\tau) d\tau}{(t_n - \tau)^q} \quad (9)$$

$$\Delta I_n = \int_{t_{n-1}}^{t_n} \frac{\dot{x}(\tau) d\tau}{(t_n - \tau)^q} \quad (10)$$

If we write $I = D^q x(t_n)$, Eq. (7) can be expressed as

$$I = \frac{1}{\Gamma(1-q)} (I_0 + I_{n-1} + \Delta I_n) \quad (11)$$

Now, suppose at time t_n the equation of motion of the oscillator has the form

$$D^2 x(t_n) + a D^q x(t_n) + b x(t_n) = f(t_n) \quad (12)$$

where the $D^2 x(t_n) = \ddot{x}(t_n)$ and $\dot{x}(t_n)$ with subscript n denote the displacement and acceleration at time t_n respectively.

Let's find the approximation formula for ΔI_n of Eq. (10) first. The integrand $\dot{x}(t)$ under integral sign describes substantially the strain rate of the constitutive law in Eq. (1) and is supposed to be sensitive to time changes. To our experience, it can not be treated as a constant for most dynamic problems even if the integration interval in Eq. (10) could be taken very small. Notice should also be made to this integral that it is an improper integral. Keeping all of these in mind, we assume that

$$\dot{x}(\tau) \Delta \dot{x}_{n-1} + \eta(\tau) (\dot{x}_n - \dot{x}_{n-1}) \quad (13)$$

that is, $\dot{x}(\tau)$ changes within $[t_{n-1}, t_n]$, where we write $\dot{x}(t_n)$ at $t=t_n$ as $\dot{x}_n$ to denote the approximate value of it, and by analogy. If we choose

$$\eta(\tau) = \frac{1}{\Delta t} (\tau - t_{n-1}), \quad \Delta t = t_n - t_{n-1}, \quad t_{n-1} \leq \tau \leq t_n \quad (14)$$

and by substituting Eq. (13), (14) into Eq. (10), we can obtain

$$\Delta I_n = \int_{t_{n-1}}^{t_n} \frac{\dot{x}_{n-1} d\tau}{(t_n - \tau)^q} + \int_{t_{n-1}}^{t_n} \frac{\dot{x}_n - \dot{x}_{n-1}}{\Delta t} \cdot \frac{\tau - t_{n-1}}{(t_n - \tau)^q} d\tau \quad (15)$$

And we can note that the integrals in Eq. (15) are ordinary definite integrals when $0 < q < 1$, and can be easily solved.

Then, by incorporating the formula

$$\dot{x}_n \Delta \dot{x}_{n-1} + (1-\alpha) \Delta t \ddot{x}_{n-1} + \alpha \Delta t \ddot{x}_n$$

of Newmark-type numerical schemes into Eq. (15) and integrating ΔI_n , we obtain

$$\begin{aligned} \Delta I_n &\approx \frac{\Delta t^{1-q}}{1-q} \dot{x}_{n-1} + (1-\alpha) \frac{\Delta t^{2-q}}{(1-q)(2-q)} \ddot{x}_{n-1} \\ &\quad + \alpha \frac{\Delta t^{2-q}}{(1-q)(2-q)} \ddot{x}_n \end{aligned} \quad (16)$$

where α is the parameter of Newmark-type methods.

Now we discuss the integral I_{n-1} of Eq. (9). We'll see later that the integral I_{n-1} describes the deformation history of the viscoelastic body. It is the convolution type integral, and for our one-step algorithm this definite integral can be approximated by trapezoid numerical integration as follows

$$\begin{aligned} I_{n-1} &\approx \frac{\Delta t}{2} \left[\frac{\dot{x}_0}{t_n^q} + \frac{\dot{x}_{n-1}}{\Delta t^q} + 2 \sum_{i=1}^{n-2} \frac{\dot{x}(i\Delta t)}{(t_n - i\Delta t)^q} \right] \\ n &\geq 2 \end{aligned} \quad (17)$$

Our numerical experiments show that Eq. (17) is the most time-consuming part of the scheme because we have to keep the values of $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ at each time step and to calculate the time history from $t=0$ to each time concerned (i.e. $t_i = i\Delta t$) in order to get the value of I_{n-1} at t_n . But if we want to study the time history of the dynamic system, it is not always necessarily the burden of the computer.

By making use of the Newmark-type iteration formula for acceleration $\ddot{x}(t_n)$, i.e.

$$\ddot{x}_n \Delta - \frac{1}{\beta \Delta t^2} (x_n - x_{n-1}) - \frac{1}{\beta \Delta t} \dot{x}_{n-1} - \left(\frac{1}{2\beta} - 1 \right) \ddot{x}_{n-1}$$

where β is the free parameter, and substituting Eq. (11) together with Eq. (16) and Eq. (17) into Eq. (12), we can finally obtain our one-step numerical scheme as

$$\begin{aligned} &\left[\frac{1}{\beta \Delta t^2} + \alpha \frac{\Delta t^{-q}}{\beta(1-q)(2-q)\Gamma(1-q)} + b \right] x_n \\ &= f(t_n) - \frac{\alpha}{\Gamma(1-q)} I_0 - \frac{\alpha}{\Gamma(1-q)} I_{n-1} \\ &\quad + \left[\frac{1}{\beta \Delta t^2} x_{n-1} + \frac{1}{\beta \Delta t} \dot{x}_{n-1} + \left(\frac{1}{2\beta} - 1 \right) \ddot{x}_{n-1} \right] \\ &\quad + \alpha \frac{\Delta t^{1-q}}{(1-q)\Gamma(1-q)} \left[\frac{\alpha}{\beta \Delta t} x_{n-1} \right. \\ &\quad \left. + \left(q - 2 + \frac{\alpha}{\beta} \right) \dot{x}_{n-1} + \left(\frac{\alpha}{2\beta} - 1 \right) \Delta t \ddot{x}_{n-1} \right] \end{aligned} \quad (18)$$

and together with

$$\begin{cases} \ddot{x}_n = \frac{1}{\beta \Delta t^2} x_n - \left[\frac{1}{\beta \Delta t^2} x_{n-1} + \frac{1}{\beta \Delta t} \dot{x}_{n-1} \right. \\ \left. + \left(\frac{1}{2\beta} - 1 \right) \ddot{x}_{n-1} \right] \\ \dot{x}_n = \alpha \Delta t \ddot{x}_n + [\dot{x}_{n-1} + (1-\alpha) \Delta t \ddot{x}_{n-1}] \end{cases} \quad (19)$$

Here we suppose that initial conditions to above formulas $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$ are given.

Note that the integral I_{n-1} records the time history of the system since the integrand $\dot{x}(\tau)$ in I_{n-1} is now the known value in Eq.(18). And the feature of I_{n-1} is that the data of $\dot{x}(i\Delta t)$ toward t_n have more influence than its earlier ones on I_{n-1} . As we implied before, much work is left for this integral to improve the efficiency of the scheme. The attempts on it can be referred to Padovan's suggestions and the works Koh and Eldred have done.

2. 3 Theoretical solution of the oscillator with semifractal derivative

To compare the numerical results of our scheme with the exact solution of the dynamic system involving fractional operator, we now consider the free vibration of the oscillator system with semifractal derivative, which is supposed to be able to obtain the theoretical solution. Furthermore, if we take the transient response of the oscillator as our example, such that

$$\begin{cases} D^2 x(t) + aD^{1/2}x(t) + bx(t) = 0 \\ x(0) = 0, \quad \dot{x}(0) = \dot{x}_0 \end{cases} \quad (20)$$

where the initial conditions are $x(0)=0$, $\dot{x}(0)=\dot{x}_0$ which is correspondent to the transient response of the oscillator under the impulsive excitation.

The exact solution of Eq.(20) can be found by means of Laplace transformation method⁽⁹⁾. And the solution referring to paper^{(4),(5)} is

$$x(t) = \dot{x}_0 x_1(t) \quad (21)$$

where

$$x_1(t) = \sum_{i=1}^4 \frac{z_i}{P'(z_i)} e^{z_i t} [1 + \operatorname{erf}(z_i \sqrt{t})], \quad (22)$$

and $z_i (i=1, 2, 3, 4)$ the roots of the characteristic equation of Eq.(20) and

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} d\xi$$

is the error function, and

$$P'(z_i) = \prod_{j \neq i} (z_i - z_j), \quad i, j = 1, 2, 3, 4$$

The value of $\ddot{x}(0)$ to Eq.(20), which serves an initial condition for our numerical algorithm, can be proved to be⁽¹³⁾:

$$\ddot{x}(0) = 0.$$

2. 4 Errors caused by the numerical scheme

The truncation error of our numerical algorithm mainly originates from two aspects.

One is from the well-known Newmark-type algorithms. The properties of the algorithms for the single-degree-of-freedom system without damping

Table 1 The stability and accuracy of some Newmark-type algorithms

algorithm	α	β	Stability limit $\omega_0 \Delta t$	Amplitude error	Periodic error $\Delta T / T$
Newmark- β	$\frac{1}{2}$	$\frac{1}{4}$	∞	0	$\frac{\omega_0^2 \Delta t}{12}$
linear acceleration	$\frac{1}{2}$	$\frac{1}{6}$	3.46	0	$\frac{\omega_0^2 \Delta t}{24}$
Fox & Goodwin	$\frac{1}{2}$	$\frac{1}{12}$	2.45	0	$O(\Delta t^3)$

ω_0 : natural frequency without damping, Δt : time step

are cited in Table 1⁽¹⁴⁾.

Another error comes from the numerical integration for the convolution type integral I_{n-1} and the linearization of our scheme for strain rate in ΔI_n . Following Hamming's formula for error evaluation of trapezoid numerical integration⁽¹⁵⁾,

$$|error| = \frac{(b-a)\Delta t^2}{12} |f^{(2)}(\theta t)| \quad 0 < \theta < 1 \quad (23)$$

where a and b are the lower and upper limits, Δt the time step and $f(\tau)$ the integrand, and since in our case

$$\begin{aligned} f^{(2)}(t) &= \frac{d^2}{dt^2} \left[\frac{\dot{x}(t)}{(t_n - t)^q} \right] \\ &= \frac{x^{(3)}(t)}{(t_n - t)^q} + 2q \frac{x^{(2)}(t)}{(t_n - t)^{q+1}} + q(q+1) \frac{\dot{x}(t)}{(t_n - t)^{q+2}} \\ &0 \leq t < t_n \end{aligned}$$

we may roughly evaluate the error as

$$\begin{aligned} |error| &\leq \frac{t_n \Delta t^2}{12} \left| \frac{x^{(3)}(\theta t)}{(t_n - t)^q} + 2q \frac{x^{(2)}(\theta t)}{(t_n - t)^{q+1}} \right. \\ &\quad \left. + q(q+1) \frac{\dot{x}(\theta t)}{(t_n - t)^{q+2}} \right|, \\ &0 < t < t_n, \quad 0 < \theta < 1 \end{aligned} \quad (24)$$

Although the smaller time step can reduce the truncation errors, it will cost more computer time, and may have the danger to increase the roundoff error. For the linear Voigt model with fractional operators it can be seen later that the errors are under control in our numerical algorithm. The main concern for the choice of time step will depends on the accuracy required.

3. Numerical Examples and Discussion

The shock response of the oscillator with semi fractional operator is taken as our numerical examples. The FORTRAN code was developed to implement the numerical algorithm, while the *MATHEMATICA* program was made to obtain the closed form solution and to be used as the comparison with numerical results.

Figure 1 shows the comparison between the numerical and theoretical results of the oscillator when the coefficients of semifractal derivative, the system damping, are taken as $a=1$ and $a=0.25$ respectively. It depicts that for both cases the numerical solutions

(a) System damping $a=0.25$ (b) System damping $a=1.0$
 Fig. 1 Comparison of the exact solution with the numerical results ($a=0.5$, $\beta=0.25$)

Fig. 2 Comparison of the exact solution with the numerical results ($a=0.5$)

are coincided fairly well with the exact solutions when the time step Δt was taken as 0.3(s), but the difference between them tends to be large when $\Delta t=0.5$ (s).

Figure 2 presents the additional data for the comparison between the exact solution and the results of the numerical algorithm when different system damping a , Newmark-type parameter β and time step Δt are chosen. It can be seen that the error between the exact and numerical results obviously decreases when the time step is smaller, or the value of β decreases, which is correspondent to more accurate Newmark-type algorithms. And these results also hold for the different time duration illustrated in Fig. 2. Table 2

further presents the data of absolute and relative errors to illustrate the difference and their changes quantitatively.

The data expose the feature of the periodic error of numerical results, that is, the errors change periodically. This suggests the effects of the period error caused by Newmark-type algorithm indicated in Table 1. From Table 1 we can see that for all the Newmark-type algorithms listed, the amplitude error is zero, but periodic error presents. This kind of features of Newmark-methods are supposed to be the dominate reason that the error of our algorithm appears periodic. But even for the longer elapsed time

Table 2 Comparison between the exact solution and numerical results

time(s)	Disp(x)	Disp($\tilde{x}$)	$ x - \tilde{x} $	$\frac{ x - \tilde{x} }{x}$
0.6	0.4895	0.4977	0.0082	0.0168
0.9	0.5985	0.6153	0.0168	0.0281
1.2	0.6059	0.6325	0.0266	0.0439
1.5	0.5268	0.5595	0.0326	0.0619
1.8	0.3886	0.4231	0.0345	0.0888
2.1	0.2223	0.2546	0.0323	0.1453
2.4	0.0587	0.0848	0.0261	0.4446
2.7	-0.0779	-0.0608	0.0171	0.2195
3.0	-0.1716	-0.1649	0.0067	0.0390
3.3	-0.2182	-0.2196	0.0034	0.0157
3.6	-0.2149	-0.2260	0.0111	0.0157
3.9	-0.1751	-0.1924	0.0173	0.0988
4.2	-0.1122	-0.1316	0.0194	0.1729
4.5	-0.0401	-0.0583	0.0182	0.4538
4.8	0.0279	0.0147	0.0132	0.4731
5.1	0.0816	0.0734	0.0082	0.1005
5.4	0.1151	0.1136	0.0015	0.0130
5.7	0.1267	0.1315	0.0048	0.0379
6.0	0.1185	0.1284	0.0099	0.0835
6.3	0.0953	0.1083	0.0130	0.1364
6.6	0.0635	0.0774	0.0139	0.2189
6.9	0.0295	0.0423	0.0128	0.4339
7.5	-0.0238	-0.0177	0.0061	0.2563

system damping $a=1.0$; time step: $\Delta t=0.3$; Newmark parameter: $\alpha=0.5$, $\beta=0.25$; x , $\tilde{x}$: exact and numerical data of displacement

Fig. 3 Comparison among different numerical schemes ($a=0.25$, $\Delta t=0.5s$) Fig. 4 Comparison between different numerical schemes ($a=0.25$, $\alpha=0.5$)

as shown in Fig. 2, both absolute and relative errors don't show any tendency of enlargement. This implies that the error is under the better control in our scheme for the oscillator. This conclusion can be further proved by adopting different β parameters of New-

Table 3 Comparison of errors between numerical results with different time steps

time(s)	Disp(x)	Disp($\tilde{x}_1$)	Disp($\tilde{x}_2$)	$ x - \tilde{x}_1 $	$ x - \tilde{x}_2 $	$\frac{ x - \tilde{x}_1 }{x}$	$\frac{ x - \tilde{x}_2 }{x}$
		$(\Delta t = 0.3s)$		$(\Delta t = 0.5s)$			
1.5	0.5269	0.5595	0.6030	0.0326	0.0761	0.0439	0.1443
3.0	-0.1716	-0.1649	-0.1514	0.0067	0.0202	0.0390	0.1177
4.5	-0.0401	-0.0583	-0.1931	0.0182	0.0530	0.4538	1.3200
6.0	0.1185	0.1284	0.1435	0.0099	0.0250	0.0835	0.2110
7.5	-0.0238	-0.0177	-0.0055	0.0061	0.0183	0.2563	0.7689

system damping: $a=1.0$, Newmark parameter: $\alpha=0.5$, $\beta=0.25$; x : exact solution; $\tilde{x}_1$, $\tilde{x}_2$: numerical results

mark-type algorithm.

Figure 3, together with Fig. 2, demonstrates the results of the numerical scheme with different β parameters of Newmark-type algorithms, namely, Newmark- β , linear acceleration and Fox & Goodwin methods respectively. The data shows that the relative error decreases with more accurate Newmark-type algorithms adopted. Referring to Table 1 we know that the accurate algorithm of Newmark-type means less periodic error incurred, therefore the decrease of relative error shown in Fig. 1 and Fig. 2 is mainly due to the decrease of periodic error. On the other hand, the fairly good coincidence between numerical and theoretical results for longer elapsed time interval also suggests that the roundoff error is under control too in our scheme.

For further approval, Fig. 4 illustrates the numerical results of the dynamic response by choosing Newmark- β algorithm with $\Delta t=0.1(s)$ and Fox & Goodwin with $\Delta t=0.5$ respectively. Instead of the increasing of the error for larger time step, no obvious difference between these two algorithms can be observed in our scheme, that is, the numerical approximation can be obtained by better Newmark type algorithms with quite favorable and large time step. Moreover, by comparing Fig. 4 with Fig. 2 we can note that still no obvious errors between exact and numerical results can be seen even for the longer time interval and the smaller time step. Referring to Eq. 24, we know that the error changes with the time in at least $O(t^{-q})$ order of magnitude. This character is favorably accordant with our numerical results. This again indicates the well-controlled roundoff error of the scheme. And we can reach the conclusion that the error related with time duration is caused mainly from the periodic error incurred by using Newmark-type algorithms, and the accuracy of the numerical results can be improved by choosing either more accurate algorithm or smaller time step, or both.

Table 3 quantitatively presents the data for comparison of the exact solution with the numerical results of different time steps. As expected, the smaller is the time step chosen, the better the numerical solution obtained. This conclusion can also be observed

by other data and pictures presented here. While the stability region of our scheme is determined by Newmark-type algorithm, we again reach the similar conclusion that the choice of the time step will mainly depends on the numerical accuracy required.

On the other hand, we should point out, the well-controlled errors of both truncation and roundoff in our numerical scheme are also attributed to our algorithm, particularly to the deduction of one-step scheme ΔI_n , where only part of the integrand is approximated and the others are in fact the exact values. The more accurate results can be expected if the function $\eta(\tau)$ of Eq.(13) could be replaced by the higher order polynomial instead of the linear function of our scheme.

4. Conclusions

From the above discussion we may reach the following conclusions:

(1) One-step numerical algorithm presented here for dynamic problems involving fractional operator is effective and successful. The numerical results obtained from the scheme are coincided very well with theoretical ones. And the numerical scheme possess the merit of Newmark-type methods;

(2) The errors of the scheme are mainly caused by the periodic error introduced from Newmark-type algorithms and the linearization of the part of integrand in the integration subinterval, and can be effectively reduced by properly choosing Newmark-type algorithm and/or time step. And the roundoff error is also well controlled in our numerical algorithm;

(3) The most time consuming part of the scheme is from the numerical algorithm of the convolution-type integral recording the deformation history of the material. The improvement of the numerical integration for this part may to some extent save the computing time and therefore increase the effectiveness of the scheme although in our examples of the simple cases only a few seconds were taken.

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