Discrete Adaptive State Feedback Control Based on a Strict Positive Realness and Its Application to the Liquid Container Transfer System*

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This paper deals with a discrete adaptive state feedback control method based on the concept of strict positive realness and its application to a liquid transfer system. First, we propose a basic construction of the adaptive control system taking into account the stability of the system and feasibility of the adaptive algorithm. Second, the proposed control method is applied to a liquid transfer system control problem. Control performances are examine from the point of view of experiments. It is shown that the adaptive state feedback control is valid for the liquid transfer system receiving the change of liquid mass.

Key Words: Adaptive Control, Strict Positive Realness, State Feedback, Sloshing, Liquid Container Transfer System

1. Introduction

Simple Adaptive Control (SAC)\(^{(1)}\) is a direct model reference adaptive control of which the structure is very simple compared with conventional adaptive control schemes. It has attractive characteristics such as robustness with regard to disturbances, unmodelled dynamics and non-linearities\(^{(2)}\). The SAC is a control strategy based on the almost strictly positive realness (ASPR-ness) of the control plant. A plant is said to be ASPR if there exists a static output feedback such that the resulting closed loop transfer function is strictly positive real (SPR). In the SAC system, the output feedback gain which ensures the stability of the control system can be adaptively adjusted by using the ASPR-ness of the plant.

On the other hand, there exists a significant constraint that the SAC can only be applied to the plant with the same number of inputs and outputs. Hence SAC cannot be applied to a general m input r output plant. To resolve such a problem, Kawasaki et al.\(^{(3)}\) has introduced an m input m output ASPR virtual plant for such systems and applied the adaptive stabilizing control method to such an ASPR virtual plant. However, the discrete time case has not been yet treated. In a discrete case, the relative degree of the virtual plant transfer function is required to be zero in order to satisfy the ASPR condition\(^{(4)}\). It means that the straightforward construction of the adaptive control algorithm destroys the causality\(^{(5)}\) because the output signal \(y(k)\) contains the control input signal \(u(k)\) which should be determined from the information concerning \(y(k)\).

In this paper, we deal with a discrete adaptive state feedback control system based on ASPR-ness of a virtual plant, and propose a discrete-time adaptive control law taking into account the above-mentioned feasibility. Furthermore, in order to confirm the control performance, the proposed adaptive control method is applied to a controlling and sloshing suppression control of liquid container transfer system which can be approximated as single input and double output pendulum type sloshing model\(^{(6)}\). Liquid container transfer systems have gradually received attention from the view point of factory automation.
Especially, they are in demand as a way of transference of melted steel in the steel industry. The concrete design procedure of the proposed adaptive control system is shown through this example. For a wide rage of plant parameter variations, an adaptive control scheme is one of the promising control methods. An application of the adaptive control method using an adaptive observer to the liquid container transfer system with an U-type Tube has been considered by Fukuda et al.(9). In comparison with the control system with adaptive observer, the proposed method can realize the simple structure of adaptive control system. The effectiveness of the proposed adaptive control scheme for the change of liquid level is evaluated by simulations and laboratory experiments.

2. Discrete Adaptive State Feedback Control System

Consider the controllable and observable linear time invariant multi-input multi-output plant described by the following equations

\[ x(k+1) = Ax(k) + Bu(k) \] (1)
\[ y(k) = Cx(k) \] (2)

where \( x \in \mathbb{R}^n \) is a state vector, \( y \in \mathbb{R}^r \) is a controlled output vector, \( u \in \mathbb{R}^m \) is a control input vector and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{r \times n} \) are matrices with plant parameters.

By introducing an \( m \times n \) constant matrix \( Q \) and an \( m \times m \) constant non-singular matrix \( D \), we obtain the low order virtual output of the form:

\[ \hat{y}(k) = Qx(k) + Du(k) \] (3)

Then we can obtain following the m-inputs/m-outputs virtual plant described by Eqs. (1) and (3).

\[ x(k+1) = Ax(k) + Bu(k) \] (4)
\[ \hat{y}(k) = Qx(k) + Du(k) \]

Now, we make the following assumptions for the virtual plant (4).

[Assumption 1]

The virtual plant (4) is controllable. Further, there exists a matrix \( Q \) such that the virtual plant (4) is observable.

[Assumption 2]

Let the following \( m \times m \) transfer function matrix for the virtual plant (4) be

\[ G^*(z) = \tilde{C}(zI - A - BK^*_z)^{-1}B + D \] (5)
\[ \tilde{C} = Q + DK^*_z \]

Then, there exist matrix \( Q, m \times n \) matrix \( K^*_z \) and \( m \times m \) matrix \( D \) such that the above transfer function \( G^*(z) \) is SPR (strictly positive real).

[Assumption 3]

State vector \( x(k) \) is available.

The assumption 2 is known as the ASPR (almost SPR) condition. For the single input and single output system, \( G(z) = Q(zI - A)^{-1}B + D \) is ASPR if and only if the following conditions hold:

[Condition 1]

(1) \( G(z) \) is inversely stable.
(2) The relative degree of \( G(z) \) is zero.
(3) The leading coefficient is positive.

where \( Q \in \mathbb{R}^{r \times r} \) and \( D \in \mathbb{R}^{r} \).

Under the assumptions 1 to 3, we can obtain the following theorem.

[Theorem 1]

Consider the following control input for the virtual plant (4),

\[ u(k) = \theta(k)x(k) \] (6)

Here, \( m \times n \) feedback gain matrix \( \theta(k) \) is adaptively adjusted by the following adaptive adjusting law:

\[ \theta(k) = \theta(k) + \sigma \theta(k-1) - \hat{y}(k)x(k)^T \Gamma \] (7)
\[ \hat{y}(k) = \hat{y}(k)x(k)^T \Gamma \] (8)
\[ \Gamma = \Gamma^* > 0, \Gamma = \sigma \Gamma^* > 0, 0 < \sigma < 1 \] (9)

Then, all the signals in the control system are uniformly ultimate bounded (UUB) under the assumption 1 to 3.

(Proof) Substituting the state feedback (6) to virtual plant (4) results in the following closed-loop system:

\[ x(k+1) = \tilde{A}x(k) + B\hat{u}(k) \]
\[ \hat{y}(k) = Cx(k) + D\hat{u}(k) \]

where

\[ \tilde{A} = A + BK^*_z \]
\[ \hat{u}(k) = \hat{\theta}(k)x(k) \]
\[ \hat{\theta}(k) = \theta(k) - K^*_z \]

and \( Q \) and \( K^*_z \) are matrices satisfying assumption 2. Then it follows from assumption 2 and the discrete type Kalman-Yakubovich lemma(10) that there exist \( n \times n \) positive symmetric matrices \( P \) and \( Q \) and matrices \( L \) and \( W \) satisfying the following equations:

\[ \tilde{A}^TPA = -PP^T \]
\[ \tilde{A}^TPB = \tilde{C}^TW^L \]
\[ W^TW = D + D^T - B^TPB \]

Take the positive function:

\[ V(k) = x(k)^TPx(k) + tr[\hat{\theta}(k-1)\Gamma^{-1}\hat{\theta}(k-1)^T] \]

(10)

\[ P = \Gamma > 0, \hat{\theta}(k) = \theta(k) - K^*_z \]

and define

\[ \Delta V(k) = V(k+1) - V(k) \]

Then, from Eqs. (4) and (6), the following equation is obtained.

\[ \Delta V(k) = -[x(k)^TL + x(k)^T\hat{\theta}(k)W^T]^2 - x(k)^TQx(k) + 2x(k)^T\hat{\theta}(k)x(k) + tr[\sigma^2\hat{\theta}(k-1)\Gamma^*\hat{\theta}(k-1)^T] + x(k)^T\Gamma x(k) + x(k)^T\hat{y}(k)x(k)^T + 2\sigma\hat{y}(k)x(k)^T\hat{\theta}(k-1) + 2\sigma\Gamma x(k)^T\hat{\theta}(k-1) + \sigma^2\Gamma^* \]


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From Eqs. (18), (19), (20), (23) and (25), it is apparent that the actual control input $u(k)$ given by Eq. (18) is equal to $\Theta(k)x(k)$ which is the control input $u(k)$ defined in Eq. (6). Since all signals of the right-hand side of Eq. (18) are available in the present time $k$ and $\Theta(k)$ is non-singular matrix for all $k (>0)$, the adaptive control algorithm $u(k)$ expressed by Eqs. (18) to (20) gives actually feasible control algorithm.

3. Application to a Liquid Container Transfer System

3.1 Mathematical model of liquid container transfer system

In this section, the effectiveness of the adaptive control method presented in the preceding section is examined through the application to a positioning control of a container and a sloshing suppression control for a liquid container transfer system(69)-(70). Here, we consider the case that the container with a thin liquid tank is transferred along the straight course, and we consider the 1st sloshing mode only so that its dynamics is approximated as a simplified pendulum model (Fig. 1)(69). Based on this model, the state space representation of the liquid container transfer system is described as follows:

$$
\dot{x}(t) = Ax(t) + bu(t) 
$$
$$
y(t) = Cx(t)
$$

Where $A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -1/T_m & -c/m & 0 \\
K_m/T_m & 0 & 0 & 0
\end{bmatrix}$

$$
b_c = \begin{bmatrix}
0 \\
K_m/T_m \\
C_c = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & L & 0 & 0
\end{bmatrix}
\end{bmatrix}
$$

where $x(t)$ is a state vector and $y(t)$ is an output vector such as:

$$
x(t) = [r(t) \ \dot{r}(t) \ \dot{\theta}(t) \ \dot{\theta}(t)]^T
$$
$$
y(t) = [r(t) \ h(t)]^T
$$

and $u(t)$ is an input which is command voltage for DC motor.

![Fig. 1](image)

**Fig. 1** Simplified model of liquid container transfer system
servo motor. The liquid level \( h \) measured at the neighboring tank wall is expressed as \( h = L \tan \theta \). Hence, for a sufficiently small angle \( \theta \), \( h \) is approximately equal to \( L \theta \). \( T_n \) and \( K_n \) are parameters of the servo system approximated as the following equation:

\[
\dot{y}(t) = -\frac{1}{T_n} \dot{y}(t) + \frac{K_n}{T_n} u(t)
\]  

(29)

The other symbols are defined as follows.

\( l \) : length of equivalent pendulum [m]
\( 2L \) : width of liquid tank [m]
\( c \) : viscous friction coefficient applied to equivalent pendulum [Ns/rad]
\( g \) : acceleration of gravity [m/s²]
\( m \) : mass of transferred liquid [kg]

The liquid level \( h \) is measured as a DC voltage of a laser sensor. The position of container is measured as a DC voltage of a potentiometer. These voltage signals are converted to the digital signals by an AD converter. Their digital signals are utilized to calculate the control input signal by the digital computer. Because the calculated control input signal is a digital signal, it should be converted to an analog signal suitable for command voltage of DC servo motor by a DA converter.

Then, parameters of the liquid container transfer system described by Eqs. (1) and (2) are given as follows:

\[
A = e^{-\lambda t}, \quad B = \int_0^t e^{-\lambda t} \, dt b_c, \quad C = C_c
\]  

(30)

where \( T \) is a sampling period of the AD converter and the DA converter.

### 3.2 Design of virtual plant parameters

Design parameters of the proposed control system, \( \Omega \) and \( D \), should be chosen so as to satisfy the assumption 1 and assumption 2 in section 2. The assumption 1 is satisfied if and only if the following conditions hold(11).

[Condition 2]

(1) \((A_c, B_c)\) is controllable and \((\Omega, A_c)\) is observable.

(2) The sampling interval \( T \) satisfies the relationship:

\[
\lambda_p - \lambda_q + \frac{2\pi k}{p} \quad p, q = 1, \ldots, n; p \neq q
\]  

(31)

where \(\{\lambda_p\} \) are the eigenvalues of the matrix \( A_c \) and \( k \) is a non-zero integer.

Let us examine the above mentioned condition 2 concretely. The controllability matrix \( N \) for \((A_c, B_c)\) and observability matrix \( U \) for \((\Omega, D)\) are obtained as follows:

\[
N = \begin{bmatrix}
0 & \frac{K_n}{T_n} & \frac{K_n}{T_n^2} & \frac{K_n}{T_n^3} \\
0 & \frac{K_n}{T_n} & \frac{K_n}{T_n^2} & \frac{K_n}{T_n^3} \\
\frac{K_n}{T_n^2} & \frac{K_n}{T_n^3} & a_1 & a_2 \\
\frac{K_n}{T_n^3} & \frac{K_n}{T_n^4} & a_1 & a_2 & a_3
\end{bmatrix}
\]

(32)

where

\[
\begin{align*}
a_1 &= \frac{K_n}{T_n} \left( \frac{1}{T_n} + \frac{c}{m} \right) \\
a_2 &= \frac{K_n}{T_n} \left( \frac{g}{T_n} \right) \frac{1}{m} \frac{1}{T_n m} - \frac{c^2}{m^2} \\
a_3 &= \frac{K_n}{T_n} \left( \frac{g}{T_n} + \frac{2gc}{m} \right) \frac{1}{T_n m} + \frac{c^2}{m^2}
\end{align*}
\]

and

\[
U = \begin{bmatrix}
\omega_1 & 0 & 0 & 0 \\
\omega_2 & \beta_2 & \beta_3 & \beta_4 \\
\omega_3 & \beta_2 & \beta_3 & \beta_4 \\
\omega_4 & \beta_2 & \beta_3 & \beta_4
\end{bmatrix}
\]

(33)

where

\[
\begin{align*}
\omega &= \begin{bmatrix} \omega_1, \omega_2, \omega_3, \omega_4 \end{bmatrix} \\
\beta_2 &= \omega_1, \quad \beta_3 = -\frac{g}{T_n} \omega_1, \quad \beta_4 = -\frac{g}{T_n} \omega_2, \\
\beta_2 &= \omega_1 + \frac{\omega_1}{l}, \quad \beta_3 = \frac{\beta_2}{T_n}, \quad \beta_4 = \frac{\beta_3}{T_n}, \\
\beta_2 &= \omega_1 - \frac{c \omega_1}{m}, \quad \beta_3 = \beta_2 - \frac{c \beta_2}{m} \\
\beta_4 &= \beta_2 - \frac{c \beta_2}{m}
\end{align*}
\]

Since matrix \( N \) is independent of \( \Omega \), it can be easily shown that \( N \) has full rank in regard to parameters of a simplified model of liquid container transfer system. On the contrary, matrix \( U \) depends on \( \Omega \), so that the matrix \( \Omega \) should be chosen so as to satisfy the non-singularity of \( U \).

Next, condition 2(2) is examined. The eigenvalues of matrix \( A_c \) are obtained as follows:

\[
\lambda_1 = 0, \quad \lambda_2 = -\frac{1}{T_n}, \quad \lambda_3, \lambda_4 = -\frac{c + \frac{4g}{m} \frac{c^2}{l}}{2}
\]

(34)

For \( T = 0.01 \), it is also shown that condition 2(2) is satisfied in regard to parameters of a simplified model of liquid container transfer system. Therefore, assumption 1 is satisfied when rank \( U = 4 \) holds.

As mentioned in section 2, for the SISO virtual plant, the assumption 2 is satisfied if and only if the condition 1 holds. Thus, let us examine the condition 1 concretely. It can be easily shown that the condition 1(2) and (3) are satisfied by choosing as \( D > 0 \).
Finally, we can see that $Q$ and $D$ should be chosen so as to hold both rank $U=4$ and the condition $1(1)$.

4. Numerical Simulation and Experiment

The schematic diagram of experimental equipment is shown in Fig. 2. Table 1 shows the nominal values of plant parameters. In the following simulations, a distance of the container transference is taken as $0.2\ [\text{m}]$. Furthermore, the control system is designed such that (1) settling time of container response is less than $2\ [\text{s}]$ and (2) maximum amplitude of sloshing is less than $15\ [\text{mm}]$.

4.1 Numerical simulation

The design parameters $Q$ and $D$ are selected as shown in Table 2 such that both rank $U=4$ and condition $1(1)$ hold. By using these values of $Q$ and $D$, these conditions are satisfied for some class of the plant parameter variations. In Fig. 3, an example of the region of permissible variations of $m$ and $l$, 'ASPR Region', is shown. It tells us that the stability of the adaptive control system can be robustly maintained for a wider range of variation of liquid mass.

Simulation results are shown in Fig. 4. In that case, $T_t$, $T_p$ and $\sigma$ are shown in Table 2. Dotted lines show the simulation result by container positioning control with constant gain container position feedback. Solid lines show the simulation result by the proposed adaptive control method. Comparing them, we can see that the proposed adaptive control system has good performance. Figure 5 shows the simulation result when $m$ changes from 0.8 [kg] to 1.6 [kg] while

![Diagram](image-url)

**Fig. 2** Experimental device

![Diagram](image-url)

**Fig. 3** Robust stability for plant parameters variation

![Diagram](image-url)

**Fig. 4** Simulation result ($m=0.8\ [\text{kg}]$)

**Table 1** Parameters of plant

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_m$</td>
<td>0.152 [s]</td>
</tr>
<tr>
<td>$K_m$</td>
<td>0.580</td>
</tr>
<tr>
<td>$l$</td>
<td>0.1425 [m]</td>
</tr>
<tr>
<td>$c$</td>
<td>0.481 [N s/m]</td>
</tr>
<tr>
<td>$m$</td>
<td>0.849 [kg]</td>
</tr>
<tr>
<td>$L$</td>
<td>0.2 [m]</td>
</tr>
<tr>
<td>$g$</td>
<td>9.8 [m/s$^2$]</td>
</tr>
<tr>
<td>$T$</td>
<td>0.01 [s]</td>
</tr>
</tbody>
</table>

**Table 2** Design parameters of SAC system for numerical simulation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega$</td>
<td>$[2, 1, 0.1, 0.05]$</td>
</tr>
<tr>
<td>$D$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\Gamma_P$</td>
<td>$\text{diag} [2, 43.5, 43.5, 2]$</td>
</tr>
<tr>
<td>$\Gamma_I$</td>
<td>$\text{diag} [2, 2, 2, 2]$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.01</td>
</tr>
</tbody>
</table>
keeping the same design parameters as Fig. 4. Through these simulations we can conclude that the good response of container is obtained and the sloshing is also sufficiently suppressed.

4.2 Experimental result

In the experiments, the signals $\theta(kT)$, $\dot{r}(kT)$ and $\dot{\theta}(kT)$, the elements of state vector $\mathbf{x}(k)$, can not be directly measured. These are approximated as follows:

$$
\theta(kT) = \frac{h(kT)}{L}
$$

$$
\dot{r}(kT) = \frac{r(kT) - r((k-1)T)}{T}
$$

$$
\dot{\theta}(kT) = \frac{h(kT) - h((k-1)T)}{LT}
$$

Note that $h(kT)$ can be directly measured by the laser sensor and $r(kT)$ can be also directly measured by the potentiometer (Fig. 2). Figure 6 shows the experimental result for $m \approx 0.8$ [kg] using the design parameters shown in Table 3. It is clear that the proposed method achieves good performances of both the container positioning control and the sloshing suppression control. Figure 7 shows the experimental result for $m \approx 1.6$ [kg]. The design parameters are also shown in Table 3. As predicted via simulation results, we can see that the proposed adaptive control method can achieve good performances of both the container positioning control and sloshing suppressing control for variety of transferred liquid mass.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega$</td>
<td>[2, 1, 0.1, 0.05]</td>
</tr>
<tr>
<td>$D$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\Gamma_R$</td>
<td>diag [22, 40, 40, 2]</td>
</tr>
<tr>
<td>$\Gamma_I$</td>
<td>diag [2, 2, 2, 2]</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 3 Design parameters of SAC system for experiments
5. Conclusion

In this report, a discrete-time adaptive state feedback control method using a strictly positive realness and its application to the liquid container transfer system have been presented.

First, we consider a basic discrete-time adaptive state feedback control system and its stability. It is shown that the basic adaptive control law can be rewritten in the feasible form. Second, we confirm the effectiveness of proposed adaptive control method through the application to a liquid container transfer system. By considering the 1st mode of sloshing, the liquid container transfer system can be described as the simplified pendulum model. Then the adaptive control system is designed based on the simplified pendulum model in order to achieve both the container positioning control and the sloshing suppressing control. Through the numerical simulations and experiments, we can confirm that the proposed method achieves good performance for a wide range of plant parameter variations.

References