Kinematic Analysis and Branch Identification of RSCR Spatial Four-Link Mechanisms*

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The formulas for displacement and velocity analyses are derived for the RSCR spatial four-link mechanism and the velocity of the coupler link is geometrically determined by instantaneous screw axes. The condition for limit positions of the driving-link motion is obtained on the basis of the force transmissibility from the coupler link to the driven link and the determinant, which is composed of the coefficients of simultaneous equations for velocity analyses. By geometrically investigating the conditional function, a quantity is proposed in order to evaluate motion and force transmission characteristics. The parts of the functional relationship between input and output angles divided by the limit positions of the driving link, which are called branches and correspond to the composition loops of the mechanism, are identified by means of two newly established rules.

Key Words: Kinematics, Spatial Four-Link Mechanism, Instantaneous Screw Axis, Motion Transmissibility, Limit Position, Branch, Circuit

1. Introduction

The spatial four-link mechanism with two revolute pairs, a spherical pair and a cylindrical pair (RSCR four-link mechanism), has potential as function- or spatial path-generating mechanism.

Funabashi and Ogawa⁽¹⁾ investigated the RSCR four-link mechanism and Funabashi et al.⁽²⁾ presented a systematic procedure for displacement analyses. Tsai and Lee modified the transmission index defined by Sutherland and Roth⁽³⁾. Hunt qualitatively investigated the limit position of the driving link⁽⁴⁾ and Ogawa et al. gave a procedure for calculating the input angle at those positions⁽⁵⁾. Thompson et al.⁽⁶⁾ and Ananthasuresh and Kramer⁽⁷⁾ developed kinematic analyses for the purpose of synthesis of mechanisms. Shrivastava and Hunt presented a procedure for synthesizing dwell motion mechanisms⁽⁸⁾

and Balaji Rao and Lakshminarayana investigated the oscillating range of the driving link and the property of the transmission angle of the RSCR four-link mechanism⁽⁹⁾. Chiang et al. clarified the condition of the rigid body guidance mechanism passing through four positions⁽¹⁰⁾ and Kohli et al. applied the relationship between the roots of the equation of fourth degree and its coefficients to the classification of motion ranges of driving link⁽¹¹⁾.

Nevertheless, the functional relationship between input and output angles of the RSCR four-link mechanism is prescribed by an equation of fourth degree, so the practical index of motion and force transmissibility and the procedure of branch identification, which are essential for the optimal design of mechanisms, have not yet been developed.

In this paper, we clarify the mutual relationship among six instantaneous screw axes of the RSCR four-link mechanism and apply them to the geometrical velocity analysis of the coupler link. Moreover, we deduce the condition for limit positions of the driving-link motion and establish a procedure for identifying the branches of the RSCR four-link mechanism.

2. Kinematic Analysis

We deduce from the closed loop equations of the RSCR spatial four-link mechanism that the stationary

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link and two adjacent links are connected by revolute pairs(R), the driving link and the coupler link are connected by a spherical pair(S), and the coupler link and the driven link are connected by a cylindrical pair(C).

2.1 Kinematic constants and variables

Let s_1 , s_4 and s_3 denote two axes of revolute pairs of the driving link and the driven link and the axis of the cylindrical pair, respectively. Let A and H denote two feet of the perpendicular from center B of the spherical pair to axis s_1 of the revolute pair and axis s_3 of the cylindrical pair, respectively. Let C and D denote two intersections of the common perpendicular between axes s_3 and s_4 and these axes, respectively.

Then, the kinematic constants of the RSCR four-link mechanism are two lengths a_1 , a_2 of driving and coupler links, two sets of a length and a twist angle (a_3, α_3) , (a_4, α_4) of driven and stationary links and two distances s_1 , s_4 from the common perpendicular EF of two axes s_1 and s_4 to two representative points A and D of stationary revolute pairs.

Let θ_1 , θ_2 , ξ_3 , θ_3 , s_3 and θ_4 denote the relative displacements of two adjacent links as shown in Fig. 1 in the stationary rectangular coordinate system D-xyz such that the origin is point D and x-axis and z-axis coincide respectively with the axis of rotation s_4 and the straight line passing through point D and parallel to the common perpendicular EF.

2.2 Closed loop equations

Three unit vectors s_1 , s_3 and s_4 , which indicate directions and senses of the axes of two revolute pairs and the cylindrical pair respectively, and four unit vectors a_1 , a_2 , a_3 and a_4 , which indicate directions and senses of the representative straight lines of four links, respectively, are expressed by basic unit vectors i , j and k as follows.

Fig. 1 RSCR spatial four-link mechanism

$$\left. \begin{aligned} s_1 &= \cos \alpha_4 i + \sin \alpha_4 j \\ s_3 &= \cos \alpha_3 i - \sin \alpha_3 \cos \theta_4 j + \sin \alpha_3 \sin \theta_4 k \\ s_4 &= i \\ a_1 &= \sin \alpha_4 \sin \theta_1 i - \cos \alpha_4 \sin \theta_1 j + \cos \theta_1 k \\ a_2 &= \sin \alpha_3 \sin \theta_3 i \\ &\quad + (\cos \theta_3 \sin \theta_4 + \cos \alpha_3 \sin \theta_3 \cos \theta_4) j \\ &\quad + (\cos \theta_3 \cos \theta_4 - \cos \alpha_3 \sin \theta_3 \sin \theta_4) k \\ a_3 &= \sin \theta_4 j + \cos \theta_4 k \\ a_4 &= k \end{aligned} \right\} \quad (1)$$

The position vector of center B of the spherical pair, which exists on the closed loop D-F-E-A-B-H-C-D passing through origin D, is expressed in two manners.

$$\begin{aligned} \vec{DB} &= s_4 s_4 + a_4 a_4 + s_1 s_1 + a_1 a_1 \\ &= a_3 a_3 - s_3 s_3 - a_2 a_2 \end{aligned} \quad (2)$$

Equating the components of x-, y- and z-axes on both sides of the vectorial closed loop equation (2), we have the closed loop equations of the RSCR four-link mechanism as follows.

$$-s_3 \cos \alpha_3 - a_2 \sin \alpha_3 \sin \theta_3 = x_B \quad (3)$$

$$\begin{aligned} a_3 \sin \theta_4 + s_3 \sin \alpha_3 \cos \theta_4 \\ - a_2 (\cos \theta_3 \sin \theta_4 + \cos \alpha_3 \sin \theta_3 \cos \theta_4) = y_B \end{aligned} \quad (4)$$

$$\begin{aligned} a_3 \cos \theta_4 - s_3 \sin \alpha_3 \sin \theta_4 \\ - a_2 (\cos \theta_3 \cos \theta_4 - \cos \alpha_3 \sin \theta_3 \sin \theta_4) = z_B \end{aligned} \quad (5)$$

Here (x_B, y_B, z_B) are the coordinates of center B of the spherical pair. Their values are given by the following equations.

$$\left. \begin{aligned} x_B &= s_4 + s_1 \cos \alpha_4 + a_1 \sin \alpha_4 \sin \theta_1 \\ y_B &= s_1 \sin \alpha_4 - a_1 \cos \alpha_4 \sin \theta_1 \\ z_B &= a_4 + a_1 \cos \theta_1 \end{aligned} \right\} \quad (6)$$

2.3 Displacement and velocity analyses

When the value of the input angle θ_1 is given, the closed loop equations (3), (4) and (5) become simultaneous equations for three relative displacements of adjacent links, namely, θ_3 , s_3 and θ_4 .

Eliminating θ_3 and s_3 from these equations, the conditional equation with respect to the output angle θ_4 is obtained as follows.

$$\begin{aligned} (y_B \sin \theta_4 + z_B \cos \theta_4 - a_3)^2 \\ + \{x_B \sin \alpha_3 + (y_B \cos \theta_4 - z_B \sin \theta_4) \cos \alpha_3\}^2 = a_2^2 \end{aligned} \quad (7)$$

By substituting $t = \tan^{-1}(\theta_4/2)$, Eq.(7) can be written as the following equation of fourth degree.

$$q_1 t^4 + q_2 t^3 + q_3 t^2 + q_4 t + q_5 = 0 \quad (8)$$

where

$$\begin{aligned} q_1 &= p_1 - p_4 + p_6, \quad q_2 = -2(p_3 - p_5) \\ q_3 &= -2(p_1 - 2p_2 - p_6), \quad q_4 = 2(p_3 + p_5) \\ q_5 &= p_1 + p_4 + p_6 \\ p_1 &= y_B^2 \cos^2 \alpha_3 + z_B^2, \quad p_2 = y_B^2 + z_B^2 \cos^2 \alpha_3 \\ p_3 &= 2y_B z_B \sin^2 \alpha_3 \\ p_4 &= 2(x_B y_B \cos \alpha_3 \sin \alpha_3 - a_3 z_B) \\ p_5 &= -2(x_B z_B \cos \alpha_3 \sin \alpha_3 + a_3 y_B) \\ p_6 &= x_B^2 \sin^2 \alpha_3 - a_2^2 + a_3^2 \end{aligned}$$

Three unknowns θ_4 , θ_3 and s_3 are determined for

each of the two or four real roots of Eq. (8) by the following equations.

$$\theta_4 = 2 \tan^{-1}(t_\theta) \quad (9)$$

$$\left. \begin{aligned} \cos \theta_3 &= (a_3 - y_B \sin \theta_4 - z_B \cos \theta_4) / a_2 \\ \sin \theta_3 &= -\{x_B \sin \alpha_3 + (y_B \cos \theta_4 - z_B \sin \theta_4) \cos \alpha_3\} / a_2 \end{aligned} \right\} \quad (10)$$

$$s_3 = -x_B \cos \alpha_3 + (y_B \cos \theta_4 - z_B \sin \theta_4) \sin \alpha_3 \quad (11)$$

Differential coefficients of the coordinates of center B of the spherical pair with respect to the input angle θ_1 are given by the following equations.

$$\left. \begin{aligned} x_B^{(1)} &= a_1 \sin \alpha_4 \cos \theta_1 \\ y_B^{(1)} &= -a_1 \cos \alpha_4 \cos \theta_1 \\ z_B^{(1)} &= -a_1 \sin \theta_1 \end{aligned} \right\} \quad (12)$$

Rearranging resultant equations obtained by differentiating Eqs. (7), (10) and (11) with respect to θ_1 , differential coefficients of three relative displacement θ_4 , θ_3 and s_3 are derived as follows.

$$\theta_4^{(1)} = f_n / f_d \quad (13)$$

$$\begin{aligned} f_n &= x_B^{(1)} \sin \alpha_3 \sin \theta_3 \\ &\quad + y_B^{(1)} (\cos \theta_3 \sin \theta_4 + \cos \alpha_3 \sin \theta_3 \cos \theta_4) \\ &\quad + z_B^{(1)} (\cos \theta_3 \cos \theta_4 - \cos \alpha_3 \sin \theta_3 \sin \theta_4) \end{aligned}$$

$$\begin{aligned} f_d &= a_3 \cos \alpha_3 \sin \theta_3 - s_3 \sin \alpha_3 \cos \theta_3 \\ \theta_3^{(1)} &= \{y_B^{(1)} \sin \theta_4 + z_B^{(1)} \cos \theta_4 \\ &\quad + \theta_4^{(1)} (y_B \cos \theta_4 - z_B \sin \theta_4)\} / a_2 \sin \theta_3 \end{aligned} \quad (14.a)$$

$$\begin{aligned} \theta_3^{(1)} &= -\{x_B^{(1)} \sin \alpha_3 + (y_B^{(1)} \cos \theta_4 - z_B^{(1)} \sin \theta_4) \cos \alpha_3 \\ &\quad - \theta_4^{(1)} (y_B \sin \theta_4 + z_B \cos \theta_4) \cos \alpha_3\} / a_2 \cos \theta_3 \end{aligned} \quad (14.b)$$

$$\begin{aligned} s_3^{(1)} &= -x_B^{(1)} \cos \alpha_3 + (y_B^{(1)} \cos \theta_4 - z_B^{(1)} \sin \theta_4) \sin \alpha_3 \\ &\quad - \theta_4^{(1)} (y_B \sin \theta_4 + z_B \cos \theta_4) \sin \alpha_3 \end{aligned} \quad (15)$$

2.4 Angular velocity

Let the relative displacement of the coupler link BH, spherically paired with the driving link, be expressed by two angles (θ_2 , ξ_2) of spherical coordinates of the straight line BH and the rotational angle ψ_2 about this line in the rectangular coordinate system such that the origin coincides with point B and three directions of axes are given by unit vectors $\mathbf{s}_1$, $\mathbf{a}_1$ and $\mathbf{a}_1 \times \mathbf{s}_1$ as shown in Fig. 2. Here, $0 \leq \theta_2 \leq \pi$, $0 \leq \xi_2 \leq \pi$ and $0 \leq \psi_2 \leq 2\pi$. Three angles (ξ_2 , θ_2 , ψ_2) compose Euler's angles and are calculated by the following equations.

$$\begin{aligned} \cos \theta_2 &= \cos \theta_1 (\cos \theta_3 \cos \theta_4 - \cos \alpha_3 \sin \theta_3 \sin \theta_4) \\ &\quad + \sin \theta_1 \{ \sin \alpha_3 \sin \alpha_4 \sin \theta_3 \\ &\quad - \cos \alpha_4 (\cos \theta_3 \sin \theta_4 + \cos \alpha_3 \sin \theta_3 \cos \theta_4) \} \end{aligned} \quad (16)$$

$$\left. \begin{aligned} \sin \theta_2 \cos \xi_2 &= \sin \alpha_3 \cos \alpha_4 \sin \theta_3 \\ &\quad + \sin \alpha_4 (\cos \theta_3 \sin \theta_4 + \cos \alpha_3 \sin \theta_3 \cos \theta_4) \\ \sin \theta_2 \sin \xi_2 &= -\cos \theta_1 \{ \sin \alpha_3 \sin \alpha_4 \sin \theta_3 \\ &\quad - \cos \alpha_4 (\cos \theta_3 \sin \theta_4 + \cos \alpha_3 \sin \theta_3 \cos \theta_4) \} \\ &\quad + \sin \theta_1 (\cos \theta_3 \cos \theta_4 - \cos \alpha_3 \sin \theta_3 \sin \theta_4) \end{aligned} \right\} \quad (17)$$

$$\begin{aligned} \cos \psi_2 &= -\cos \alpha_3 \{ \sin \alpha_4 \sin \theta_1 \sin \theta_2 \\ &\quad - \cos \theta_2 (\cos \alpha_4 \cos \xi_2 - \sin \alpha_4 \cos \theta_1 \sin \xi_2) \} \\ &\quad - \sin \alpha_3 \cos \theta_4 \{ \cos \alpha_4 \sin \theta_1 \sin \theta_2 \\ &\quad + \cos \theta_2 (\sin \alpha_4 \cos \xi_2 + \cos \alpha_4 \cos \theta_1 \sin \xi_2) \} \\ &\quad - \sin \alpha_3 \sin \theta_4 (\cos \theta_1 \sin \theta_2 - \sin \theta_1 \cos \theta_2 \sin \xi_2) \end{aligned}$$

Fig. 2 Displacement of the coupler link relative to the driving link

(18)

The resultant equations obtained by differentiating both sides of Eqs. (16), (17) and (18) yield the differential coefficients ($\xi_2^{(1)}$, $\theta_2^{(1)}$, $\psi_2^{(1)}$). Then, the angular velocity of the coupler link BH relative to the driving link AB can be expressed as follows.

$$\omega_{12} = \dot{\xi}_2 \mathbf{a}_1 + \dot{\theta}_2 \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\sin \theta_2} + \dot{\psi}_2 \mathbf{a}_2 \quad (19)$$

2.5 Instantaneous screw axes

We designate the driving link AB, the coupler link BH, the driving link DC and the stationary link of the RSCR four-link mechanism as links 1, 2, 3 and 4, respectively. Furthermore, we let S_{12} , S_{23} , S_{34} , S_{41} , S_{13} and S_{24} denote six instantaneous screw axes of any two of the four links.

Then, three instantaneous screw axes S_{41} , S_{23} and S_{34} of two links connected by the revolute or cylindrical pair coincide with their axes of rotation EA, CH and FD, respectively.

The instantaneous screw axis S_{12} of link 2 relative to link 1 coincides with the oriented straight line, that passes through point B and has the same direction as the angular velocity vector ω_{12} .

The instantaneous screw axis S_{42} ($-S_{24}$) of link 2 relative to link 4 is obtained by applying the Kennedy-Aronhold theorem⁽¹²⁾ to three links 4, 1 and 2 or three links 4, 3 and 2.

Moreover, when instantaneous screw axes S_{41} , S_{12} and S_{23} , S_{34} are known, S_{24} is determined as the common perpendicular between two common perpendiculars between S_{41} , S_{12} and S_{23} , S_{34} , respectively.

The instantaneous screw axis S_{31} ($-S_{13}$) of link 1 relative to the link 3 is obtained in a similar manner. An example of six instantaneous screw axes of the RSCR four-link mechanism is shown in Fig. 3, whose

Fig. 3 Example of instantaneous screw axes of the RSCR mechanism

kinematic constants are as follows.

$$a_1=5, \quad a_2=3, \quad a_3=5, \quad a_4=-7$$

$$s_1=5, \quad s_4=-5, \quad \alpha_3=217^\circ, \quad \alpha_4=350^\circ$$

The instantaneous screw axis S_{24} and the angular velocity vector ω_{24} are determined from (S_{34}, s_4) and (S_{23}, s_3) by using the Kennedy-Aronhold theorem. Then, (S_{12}, ω_{12}) is determined from (S_{24}, ω_{24}) and (S_{41}, ω_{41}) in the same manner. Consequently, values of components ξ_2 , θ_2 and ψ_2 of the angular velocity of the coupler link relative to the driving link are calculated as the scalar products of the angular velocity vector ω_{12} and the unit vectors a_1 , $a_1 \times a_2 / \sin \theta_2$ and a_2 , respectively.

3. Limit Positions of the Driving Link Motion

The positions where the driving link cannot move very far in one direction are called limit positions of the link mechanism. Because limit positions are the lower and upper limit of the domain of motion of the mechanism, their existence and non-existence correspond to whether the driving link can make a rotation or not. This is one of the important kinematic characteristics which may be applicable to the classification of input-output relationships of the RSCR four-link mechanism.

3.1 Investigation on the basis of force transmission

The force to be transmitted to the driven link from the coupler link passes through center B of the spherical pair and is perpendicular to the axis of rotation of the cylindrical pair. Consequently, letting the transmitted force be $F(\theta_1)$ as shown in Fig. 4, moment $M(\theta_1)$ due to this force about the axis of rotation FD of the driven link is derived as.

$$M(\theta_1) = a_3 F(\theta_1) \cos \alpha_3^* \cos \theta_3^* + s_3 F(\theta_1) \sin \alpha_3^* \sin \theta_3^*$$

$$= -F(\theta_1) (a_3 \cos \alpha_3 \sin \theta_3 - s_3 \sin \alpha_3 \cos \theta_3) \quad (20)$$

In such a position where moment $M(\theta_1)$ becomes

Fig. 4 Moment about the axis of rotation of the driven link

zero, the driving link cannot rotate very far in the direction of the moment acting on the link. Therefore, the limit position of the RSCR four-link mechanism is the position in which the following condition is satisfied.

$$a_3 \cos \alpha_3 \sin \theta_3 - s_3 \sin \alpha_3 \cos \theta_3 = 0 \quad (21)$$

Besides, the position where force $F(\theta_1)$ becomes zero is the end point of the oscillating motion where the sense of force acting on the driving link reverses.

3.2 Investigation on the basis of velocity analyses

Formulas for calculating differential coefficients $\theta_4^{(1)}$, $\theta_3^{(1)}$ and $s_3^{(1)}$ of the relative displacement of two adjacent links with respect to the input angle contain the following expression as a common denominator.

$$D(\theta_1) = a_2 (a_3 \cos \alpha_3 \sin \theta_3 - s_3 \sin \alpha_3 \cos \theta_3) \quad (22)$$

Since the expression in parentheses () on the right-hand side of Eq.(22) coincides with that on the left-hand side of Eq.(21), the condition such that $D(\theta_1)=0$ holds is equivalent to the condition of limit positions.

In such positions that $D(\theta_1)=0$ holds, differential coefficients $\theta_4^{(1)}$, $\theta_3^{(1)}$ and $s_3^{(1)}$ become simultaneously infinite, so the slope of the tangent to the relationship curve between the input and output angles takes a value of 90° or 270° . These positions of link mechanisms are generally called stationary configurations⁽¹³⁾.

3.3 Characteristic quantity of motion transmission

Letting $\bar{a}_2$ denote the moment of unit vector a_2 existing on the representative straight line BH of the coupler link with respect to origin D of the stationary rectangular coordinate system D-xyz, the oriented straight line BH is expressed by means of a unit dual vector as follows.

$$\bar{a}_2 = a_2 + \epsilon \bar{a}_2 \quad (23)$$

$$\begin{aligned}\bar{a}_2 = & -(a_3 \cos \alpha_3 \sin \theta_3 - s_3 \sin \alpha_3 \cos \theta_3) \mathbf{i} \\ & + \{a_3 \sin \alpha_3 \sin \theta_3 \cos \theta_4 \\ & - s_3 (\sin \theta_3 \sin \theta_4 - \cos \alpha_3 \cos \theta_3 \cos \theta_4)\} \mathbf{j} \\ & - \{a_3 \sin \alpha_3 \sin \theta_3 \sin \theta_4 \\ & + s_3 (\sin \theta_3 \cos \theta_4 + \cos \alpha_3 \cos \theta_3 \sin \theta_4)\} \mathbf{k} \quad (24)\end{aligned}$$

Here, ϵ is a unit quantity such that the operation rule $\epsilon^n = 0 (n \geq 2)$ is satisfied.

The perpendicular distance $d(\theta_1)$ between two oriented lines, the axis of rotation of the driven link ($\bar{s}_4 = \mathbf{i} + \epsilon \times \mathbf{0}$) and the straight line of force transmission ($\bar{a}_2$), is obtained by the following equation as

$$\begin{aligned}d(\theta_1) &= \frac{|\mathbf{i} \cdot \bar{a}_2|}{\sqrt{1 - (\mathbf{i} \cdot \bar{a}_2)^2}} \\ &= \frac{|a_3 \cos \alpha_3 \sin \theta_3 - s_3 \sin \alpha_3 \cos \theta_3|}{\sqrt{1 - \sin^2 \alpha_3 \sin^2 \theta_3}} \quad (25)\end{aligned}$$

The expression of Eq. (25) in the absolute sign $| \quad |$ on the right-hand side coincides with that on the left-hand side of Eq. (21). Therefore, it is concluded that the necessary and sufficient condition of the limit position is for the representative straight line BH of the coupler link to intersect the axis of rotation FD of the driven link.

3.4 Limit position in the case of general configuration

The vectorial closed loop equation of the general RSRC four-link mechanism (Fig. 5) is expressed as

$$\begin{aligned}\bar{D}\bar{B} &= s_4 \bar{s}_4 + a_4 \bar{a}_4 + s_1 \bar{s}_1 + a_1 \bar{a}_1 \\ &= a_3 \bar{a}_3 - s_3 \bar{s}_3 - a_2 \bar{a}_2 - s_2 \bar{s}_2 \quad (26)\end{aligned}$$

The common denominator of formulas for calculating differential coefficients $\theta_4^{(1)}$, $\theta_3^{(1)}$ and $s_3^{(1)}$ of the relative displacement of two adjacent links with respect to the input angle θ_1 is deduced in the same manner as in the section 3.2 as follows.

$$D(\theta_1) = a_3 \cos \alpha_3 (a_2 \sin \theta_3 - s_2 \sin \alpha_2 \cos \theta_3)$$

Fig. 5 RSCR mechanism with the general-form coupler link

$$-(s_2 \cos \alpha_2 + s_3) \sin \alpha_3 (a_2 \cos \theta_3 + s_2 \sin \alpha_2 \sin \theta_3) \quad (27)$$

Letting a_2^* and γ denote the length of the perpendicular BH* to the axis of rotation CH of the cylindrical pair from center B of the spherical pair and the twist angle between two straight lines HG and H*B, these constants are obtained by the following equations as

$$\left. \begin{aligned}a_2^* &= \sqrt{a_2^2 + s_2^2 \sin^2 \alpha_2} \\ \cos \gamma &= a_2 / a_2^* \\ \sin \gamma &= -s_2 \sin \alpha_2 / a_2^*\end{aligned} \right\} \quad (28)$$

Therefore, introducing new variables

$$\left. \begin{aligned}\theta_3^* &= \theta_3 + \gamma \\ s_3^* &= s_3 + s_2 \cos \alpha_2\end{aligned} \right\} \quad (29)$$

Equation (23) is rewritten in the same form as Eq. (22).

$$D(\theta_1) = a_2^* (a_3 \cos \alpha_3 \sin \theta_3^* - s_3^* \sin \alpha_3 \cos \theta_3^*) \quad (30)$$

Furthermore, as shown in Fig. 6, letting Q denote the foot of the perpendicular to the x -axis from the intersection P of the straight line BH* and the xz' -plane, the length of segment PQ is expressed as

$$\overline{PQ} = a_3 - \frac{a_2 \cos \theta_3 + s_2 \sin \alpha_2 \sin \theta_3}{a_2 \sin \theta_3 - s_2 \sin \alpha_2 \cos \theta_3} (s_2 \cos \alpha_3 + s_3) \tan \alpha_3. \quad (31)$$

Therefore, it is geometrically shown that the condition of the limit position, that $D(\theta_1) = 0$ holds, is equivalent to the fact that the perpendicular to the axis of rotation CH of the cylindrical pair from center B of the spherical pair intersects the axis of rotation of the driven link, in such a geometrical case that the following condition holds.

$$\cos \alpha_3 (a_2 \sin \theta_3 - s_2 \sin \alpha_2 \cos \theta_3) \neq 0 \quad (32)$$

3.5 Determination of limit positions

Substitution of Eq. (3) into the resultant equation obtained by eliminating θ_4 from Eqs. (4) and (5) yields:

Fig. 6 Geometrical investigation of the condition of the limit position

$$x_B^2 + y_B^2 + z_B^2 = a_2^2 + a_3^2 - 2a_2a_3 \cos \theta_3 + s_3^2 \quad (33)$$

Elimination of θ_1 of three equations in Eq. (6) yields:

$$4a_4^2(x_B - s_1 \cos \alpha_4 - s_4)^2 + \sin^2 \alpha_4(x_B^2 + y_B^2 + z_B^2 - 2s_4x_B - a_1^2 - a_4^2 - s_1^2 + s_4^2)^2$$

$$-4a_1^2a_4^2\sin^2\alpha_4=0 \quad (34)$$

The condition of limit position: $D(\theta_1)=0$ is expressed as follows.

$$s_3 = a_3 \cot \alpha_3 \tan \theta_3 \quad (35)$$

Then, substituting Eqs. (3), (33) and (35) for Eq. (34), the equation of twelfth degree of $\tan(\theta_3/2)$ for

Fig. 7 Examples (a)~(o) of three functional relationships between θ_1 and θ_3, s_3, θ_4 [θ_4 : bold line, θ_3 : normal line, s_3 : fine line]

determining the limit position is obtained as

$$\begin{aligned} &h_1 \cos^6 \theta_3 + h_2 \cos^5 \theta_3 + h_3 \cos^4 \theta_3 + h_4 \cos^3 \theta_3 \\ &+ h_5 \cos^2 \theta_3 \\ &+ \sin \theta_3 (h_6 \cos^5 \theta_3 + h_7 \cos^4 \theta_3 + h_8 \cos^3 \theta_3 \\ &+ h_9 \cos^2 \theta_3 + h_{10} \cos \theta_3) + h_{11} = 0 \end{aligned} \quad (36)$$

4. Branch Identification

The closed loop mechanism with one degree of freedom may be constructed in more than two configurations (composition loops) from the same set of kinematic constants where functional relationships of input and output angles are mutually different. This kinematic property of mechanisms makes automatic tracing of a specified functional relationship impossible.

4.1 Branch and motion domain

The parts of the functional relationship between input and output angles divided by the limit positions of the driving link are called branches and correspond to the composition loops of the mechanism. In this paper, we call the domain of the input angle that corresponds to the branch as the motion domain of the driving link.

Watanabe and coworkers^{(14),(15)} and Chase and Mirth^{(16),(17)} established procedures for the branch identification of such Stephenson six-link mechanisms that the displacement analysis is performed through solution of an equation of sixth degree with respect to the output angle.

The RSCR four-link mechanism that the displacement analysis is performed through solution of an equation of fourth degree with respect to the output angle, namely, two or four values of the output angle are obtained for any value of the input angle, has a similar difficulty in branch identification as in the above-mentioned six-link mechanisms.

4.2 Input-output relationship

The classification of the functional relationship between input and output angles of the RSCR four-link mechanism is not yet presented. Therefore, the establishment of a general procedure for identifying the branches of this mechanism is difficult. Then, we characterize many input-output relationships obtained in this paper (Figs. 7(a)-(o)) and present two rules in order to identify the branches of these mechanisms.

The bold line, the normal line and the fine line in Fig. 7 are the θ_1 - θ_4 curve, the θ_1 - θ_3 curve and the θ_1 - s_3 curve, respectively. Figures 1, 2, 3, ..., 8 assigned to each curve are used to discriminate the branches. Kinematic constants of these mechanisms are shown in Table 1.

4.3 Identification rule 1

The common denominator of the formulas of

Table 1 Kinematic constants of mechanisms used as examples

	a_1	a_2	a_3	a_4	s_1	s_4	α_3	α_4
(a)	1.0	2.0	0.5	2.0	6.0	-5.0	80°	30°
(b)	5.0	7.0	3.0	3.5	4.5	3.5	180°	90°
(c)	3.0	4.0	2.0	3.0	7.0	-5.0	60°	30°
(d)	1.0	2.0	0.5	2.0	4.0	-3.0	60°	30°
(e)	5.0	2.0	3.0	2.5	1.5	-2.5	75°	15°
(f)	1.0	2.0	0.5	2.0	2.0	-1.5	160°	20°
(g)	1.0	2.0	0.5	2.0	2.0	-1.5	125°	35°
(h)	7.0	5.0	1.0	4.0	2.5	2.0	290°	150°
(i)	5.0	2.0	3.0	2.5	7.0	-5.0	80°	60°
(j)	5.0	2.0	3.0	2.5	4.0	-4.0	90°	10°
(k)	5.0	2.0	3.0	2.5	7.0	-6.0	80°	315°
(l)	5.0	2.0	3.0	-2.5	7.0	-5.0	80°	15°
(m)	5.0	2.0	3.0	2.5	4.0	-4.0	90°	30°
(n)	1.0	2.0	0.5	2.0	2.0	-1.5	125°	30°
(o)	5.0	2.0	3.0	-2.5	7.0	-6.0	80°	20°

velocity analysis, that is, the determinant $D(\theta_1)$, which is composed of the coefficients of simultaneous equations for three differential coefficients $\theta_3^{(1)}$, $s_3^{(1)}$ and $\theta_4^{(1)}$, becomes zero at the limit position of the driving link.

The limit position of the driving link coincides with the common upper or lower limit of two adjacent branches and the signs of $D(\theta_1)$ to be calculated in these branches are mutually different. Therefore, the sign of determinant $D(\theta_1)$ can be used as a criterion for discriminating two adjacent branches.

Figure 8 shows the input-output relationship and the curve of determinant $D(\theta_1)$ of case (n) in Fig. 7.

4.4 Identification rule 2

The representative point H of the cylindrical pair to connect the coupler link and the driven link of the RSCR four-link mechanism moves on the hyperboloid of one sheet whose centerline and generating line are the axis of rotation s_4 of the driving link and axis s_3 of the cylindrical pair, respectively.

This hyperboloid of one sheet is divided into two parts bordered by the locus of point H such that $s_3 = 0$ holds, namely, the circle of minimum radius as shown in Fig. 9. These two parts are discriminated by means of the sign of s_3 .

In the domain of input angle θ_1 such that the number of real roots of output angle θ_4 is two, branches corresponding to these values are discriminated by the sign of $D(\theta_1)$.

In the domain of input angle θ_1 such that the number of real roots of output angle θ_4 is four, namely, there exist four composition loops, the relationship between input angle θ_1 and the relative displacement of the coupler link to the driven link s_3 is classified into four cases as shown in Figs. 10(a), (b), (c) and (d).

The four θ_1 - s_3 curves correspond to four branches

Fig. 8 Input-output relationship of mechanism (n) and coefficient determinant $D(\theta_1)$ of the velocity analysis equations

Fig. 9 Locus of the representative point of the cylindric pair

in Fig. 6 and two signs of $D(\theta_1)$ are positive while the rest are negative. In cases (a) and (b), the driving link can make a complete rotation at all the four composition loops and at two composition loops, respectively.

In case (c), the driving link cannot make a complete rotation at all the four composition loops. Here, the representative point H of the cylindrical pair does not pass through the circle of the minimum radius of the hyperboloid of one sheet. In case (d), the driving link either cannot make a complete rota-

Fig. 10 Four cases of relationships between θ_1 and s_3 for branch identification

tion at all the four composition loops, or can make a complete rotation at two composition loops, and point H passes through the circle of the minimum radius.

The branches of the mechanism in the above-mentioned cases have the following properties.

(i) In cases (a) and (c) of Fig. 10, the sign of s_3 is positive at two branches and negative at the other two. Besides, the sign of $D(\theta_1)$ is positive and negative respectively at such two branches that the sign of s_3 is positive or negative.

(ii) In case (b), let θ_{1l} , θ_{1u} and s_{3l} , s_{3u} respectively denote lower and upper values of θ_1 at limit positions and values of s_3 in such a composition loop that the limit of driving link motion arises. Moreover, let θ_{1l}^* and θ_{1u}^* respectively denote lower and upper values of θ_1 such that $s_3=0$ holds and let θ_{1m} denote any value in the range $[\theta_{1l}^*, \theta_{1u}^*]$.

Then, substituting $s_b=s_{3l}$ or $s_b=s_{3u}$ according to $\theta_{1l} \leq \theta_1 \leq \theta_{1m}$ or $\theta_{1m} \leq \theta_1 \leq \theta_{1u}$, respectively, the value of s_3-s_b becomes positive at two branches and negative at the other two. Besides, the sign of $D(\theta_1)$ is positive and negative respectively at such two branches that the sign of s_3 is positive or negative.

(iii) In case (d), let, θ_{1l} , θ_{1u} and θ_{3l} , θ_{3u} respectively denote lower and upper values of θ_1 at limit positions and values of s_3 similar to case (b). Moreover, let s_b denote one of s_{3l} and s_{3u} such that its absolute value is larger than the other. The signs of s_3-s_b and $D(\theta_1)$ correspond to two sets of two branches and their individual branches, respectively, similar to case (b).

Consequently, four branches, to which four values

of the output angle obtained by solving the equation of fourth degree belong, can be identified by means of two signs of $D(\theta_1)$ and s_3 or $s_3 - s_b$.

For example, in the range $[-108.263^\circ, 8.761^\circ]$ of case (n) of Fig. 11, at two branches 2, 5 and 1, 6, the signs of $s_3 - s_b$ become positive and negative, respectively, but the signs of $D(\theta_1)$ at their individual branches are different, such as negative, positive, and positive, negative, respectively. Therefore, we can identify four branches.

4.5 Algorithm for branch identification

The conditional equation of input angle θ_1 for such positions that $s_3 = 0$ holds is obtained by rearranging resultant equations of Eqs. (3), (4) and (5) in this case as follows.

$$\begin{aligned} \sin^2 \alpha_3 \{ & a_1 a_4 \cos \theta_1 + a_1 s_4 \sin \alpha_4 \sin \theta_1 + s_1 s_4 \cos \alpha_4 \\ & + (a_1^2 - a_2^2 - a_3^2 + a_4^2 + s_1^2 + s_4^2)/2 \}^2 \\ & + a_3^2 (a_1 \sin \alpha_4 \sin \theta_1 + s_1 \cos \alpha_4 + s_4)^2 \\ & - a_2^2 a_3^2 \sin^2 \alpha_3 = 0 \end{aligned} \quad (37)$$

The number and the values of limit positions of the driving link motion and those of positions where $s_3 = 0$ are determined from Eqs. (36) and (37), respectively. By investigating three functional relationships between θ_1 and θ_3 , s_3 , θ_4 of examples (a) ~ (o) in Fig. 7, we obtained twelve models of branches and circuits of the RSCR four-link mechanism as shown in Fig. 12.

Fig. 11 One to one correspondence among four branches in the θ_1 - θ_4 and θ_1 - s_3 relationships

Here, the longitudinal axis is a representative value of output angle θ_4 in each branch.

In our original paper (in Japanese), we have established an algorithm for confirming the branch-circuit model for any set of kinematic constants and for identifying the branches to which the two or four values of the output angle obtained for any value of the input angle belong.

5. Classification of Mechanisms

Depending on whether the driving link can make a complete rotation or not and whether the driven link

Fig. 12 Basic examples of identified branches

Fig. 13 Sixteen kinds of schematic input-output relationships of RSCR four-link mechanisms

can make a complete rotation or not, and depending on whether the mechanism is of the Grashof type or the non-Grashof type, the RSCR four-link mechanism is classified into sixteen kinds as shown in Fig. 13.

[Grashof type] (a) rotational - oscillating type (b) rotational - two - oscillating type (c) recipro - rotational - oscillating type (d) two - recipro - rotational - oscillating type (e) double rotational type (f) double recipro - rotational type (g) oscillating - rotational type (h) two - oscillating - rotational type (i) two - oscillating - recipro - rotational type (j) double oscillating type (k) double recipro - oscillating type

[Non-Grashof type] (l) double oscillating type (m) recipro - oscillating - oscillating type (n) double recipro - oscillating type (o) two - recipro - oscillating - oscillating type (p) recipro - oscillating - two - recipro - oscillating type

6. Conclusion

We performed kinematic analyses of the RSCR four-link mechanism, and investigated the condition for limit positions of the driving link motion and the procedure for identifying branches. The results obtained are summarized as follows.

(1) The instantaneous axis of the coupler link relative to the stationary link is obtained in three ways by means of the Kennedy-Aronhold theorem. Consequently, its angular velocity vector is geometrically calculated.

(2) The condition of limit positions is deduced as the equation of twelfth degree with respect to the relative displacement of the coupler link to the driven link.

(3) The length of the perpendicular from the center of the spherical pair to the axis of the cylindrical pair is found as the common factor in the denominators of the transmitted force to the driven link and the velocity of the output link. Therefore, this quantity is a useful index for evaluating motion and force transmissibility.

(4) In the domain of input angle θ_1 such that real roots of output angle θ_4 are two and four, the branches are identified by means of the sign of $D(\theta_1)$ and two signs of $D(\theta_1)$ and $s_3 - s_6$, respectively.

(5) The relationship between input and output angles is classified into sixteen kinds depending on whether the driving link and the driven link can make a complete rotation or not.

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