Axode Synthesis of Cam-Follower Mechanisms with Hyperboloidal Rollers*

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The surface geometry of the roller-follower cam mechanisms are derived as the cam makes contact with the roller on their instantaneous screw axis. These surfaces, such as cam surface, follower surface, surface of action, and pitch surface, are ruled surfaces. With the restrictions of the geometric characteristics of a hyperboloidal roller, the relative locations of the axes of these cam mechanisms have to satisfy certain relation and the kinds of roller-follower cam mechanisms synthesized in axode contact are verified. The angular velocity of the roller and the screw pitch of the cam pair are derived, and the types of the contact surfaces, the pressure angle, and the pure-rolling cam pair are also discussed.

Key Words: Hyperboloidal Roller, Axode, Cam-Follower, Pure Rolling

1. Introduction

The fundamental purpose of a cam-follower system, in general, is to transmit motion from the cam to the follower. The cam surface is conjugate to the follower surface for 3-link cam mechanisms and to the roller surface for 4-link cam mechanisms. For a specified input-output relation, one of the two conjugate surfaces can be chosen as a generating surface to produce the other surface. Usually, the follower surface or the roller surface is used as a generating surface.

There are several approaches to synthesize the cam profile. The planar cam profiles can be constructed by the graphical method that is based on the principle of inversion. Raven⁽¹⁾ used the independent position equations to design disk cams with flat-faced follower or roller follower, and spatial cams with spherical follower. This method is based on the concept of vector loop and it is not convenient to

synthesize other spatial cam mechanisms with arbitrary roller surfaces. Davidson⁽²⁾ used instant centers to calculate planar cam profiles. Hanson and Churchill⁽³⁾ designed several planar cam profiles according to the theory of envelopes of one parameter in implicit form. Backhouse and Rees Jones⁽⁴⁾ used the same concept to derive the surface geometry of globoidal cam mechanisms. Tsay and Wei⁽⁵⁾ also designed cylindrical cams with translating conical followers by the envelope in parametric form. At the same time, Tsay and Hwang⁽⁶⁾ determined the camoid profiles with translating followers by the envelope of two parameters. On the other hand, Pandrea and Draganoiu⁽⁷⁾ derived spatial cam profiles by the orthogonality condition between the common normal of two conjugate surfaces and the relative velocity at any contact point. Dhande et al.⁽⁸⁾, and Chakraborty and Dhande⁽⁹⁾ also used the orthogonality condition to develop a generalized model of planar and spatial cam mechanisms for the cam profiles. We see that the cam profiles or the conjugate surfaces may be determined by graphical or analytical methods, and can be generated from the concepts of the envelopes and the reciprocal kinematic conditions. Basically, these methods are based on differential geometry and screw theory. Systematic studies of the conjugate surfaces were presented by Litvin^{(10),(11)}, Chen⁽¹²⁾, and Wu and Luo⁽¹³⁾.

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It is well known that there exists an instantaneous screw axis between any two rigid bodies if the relative motion of the two bodies is not stationary. For continuous motion, the instantaneous screw axis traces two axodes on the two bodies if the relative motion has no pure translation. The relative motion of the two bodies may be thought as a direct-contact motion between the two axodes. If two bodies contact each other on the instantaneous screw axis, the power loss and the relative sliding velocity are minimized. Gonzalez-Palacios and Angeles^{(14)–(16)} used the instantaneous screw axes to synthesize the cam surface and the follower surface for 3-link and 4-link cam mechanisms. The conjugate contact surfaces generated by these axodes can satisfy the equation of meshing since they are reciprocal to each other. Pure rolling occurs when the pitch of the instantaneous screw axis between the two bodies is zero. The surface geometry of conjugate surfaces with pure rolling contact have been studied by Ghosh and Yadav⁽¹⁷⁾, Gonzalez-Palacios and Angeles⁽¹⁸⁾, Wagner et al.⁽¹⁹⁾, and Chen⁽²⁰⁾.

In this paper, the contact surfaces and the pitch surface are synthesized for the roller-follower cam mechanisms in axode contact. The contact line is a straight line and lies along the instantaneous screw axis between the cam and the roller⁽¹⁴⁾. However, there are rarely a few cam mechanisms designed with cam pair in axode contact. Considering certain restrictions of a hyperboloidal roller, the cam pair with straight line contact or with pure rolling contact are found for these cam mechanisms.

2. Notes to Screw Axes

At any instant, it is well known that the instantaneous motion of a rigid body (labeled 2) relative to another body (labeled 1) can be expressed as a velocity screw

$$\hat{S}^{(21)} = \omega^{(21)} + \varepsilon \tau^{(21)}, \quad (1)$$

where $\omega^{(21)}$ and $\tau^{(21)}$ are Cartesian 3-vectors which represent the relative angular velocity and the relative velocity of a certain reference point, and ε is the Clifford screw operator with the property $\varepsilon^2=0$, also called the dual unity.

The unit vector parallel to the instantaneous screw axis (ISA for short) is given by

$$e = \frac{\omega^{(21)}}{\|\omega^{(21)}\|}. \quad (2)$$

And, the position vector from a reference point to the ISA with the shortest distance is⁽²¹⁾

$$p = \frac{\omega^{(21)} \times \tau^{(21)}}{\omega^{(21)} \cdot \omega^{(21)}}. \quad (3)$$

Equations (2) and (3) determine the orientation and location of the ISA, and satisfy $e \cdot e = 1$ and $e \cdot p = 0$. However, it is convenient that the reference

point is located at the origin of the coordinate system. Any point on the ISA is given by

$$r = p + ue. \quad (4)$$

The relative velocity between the two bodies at the contact point on the ISA is given by

$$v^{(21)} = \tau^{(21)} + \omega^{(21)} \times r = h_{21} \omega^{(21)}. \quad (5)$$

where h_{21} is the screw pitch which is defined as the ratio of the relative translational velocity to the relative angular velocity on the ISA and is given by

$$h_{21} = \frac{\omega^{(21)} \cdot \tau^{(21)}}{\omega^{(21)} \cdot \omega^{(21)}}. \quad (6)$$

Equation (4) represents a ruled surface that is generated by a straight-line and is known as a relative axode between body 1 and 2. The axode surface can be expressed as a function of two curvilinear coordinates, i.e., t and u . The unit normal of the axode surface given by Eq. (4) is derived as

$$n = \frac{r_t \times r_u}{\|r_t \times r_u\|} = \frac{(p' + ue') \times e}{\|(p' + ue') \times e\|}. \quad (7)$$

3. Four-Link Cam Mechanisms

A mechanism, whereby the motion or force is transmitted through contact between the driver and the follower, is called as direct-contact mechanism. The gear mechanisms and cam-follower mechanisms belong to this kind of mechanism. In general, a cam-follower mechanism consists of three bodies which are the frame fixed in space, the cam or driving link, and the follower or driven link. The cam couples the follower by means of a higher pair (also called cam pair) by point contact or line contact. Sometimes, it is hard to design a cam-follower mechanism on account of power loss, pressure angle, positive-return motion, and so on. Thus, an additional roller is introduced in the cam-follower mechanism so as to generate a four-link cam mechanism. Here, the roller is considered as a generalized meshing element that has a revolute pair incident to the follower and a higher pair with line contact incident to the cam. The geometric parameters describing the relative locations between the joints are limited in order for the roller meshing with the cam.

A four-link cam mechanism is shown in Fig. 1. For the generality of synthesizing cam profiles, both the cam (labeled 1) and the follower (labeled 2) have screw motions around their screw axes with respect to the frame (labeled 0), respectively; and the roller (labeled 3) rotates relative to the follower. However, the roller can have simultaneous rotation and translation relative to the follower, although it is hard to design such a roller in practice. The ISA_{ij} denotes herewith the instantaneous screw axis of body i relative to body j . There are six ISAs in a roller-follower cam mechanism. For a one-degree-of-freedom cam

Fig. 1 Four-link cam mechanism

mechanism and a specified input-output relation, the angular and translational displacements of the cam and the follower are expressed as functions of an independent variable. Usually, the motions of the cam and the follower relative to the frame are either rotational or translational and then the joints incident to the frame are, respectively, either revolute or prismatic pairs.

In Fig.1, there are five coordinates systems, namely, $S_i(x_i, y_i, z_i)$, $S_0(x_0, y_0, z_0)$, $S_1(x_1, y_1, z_1)$, $S_2(x_2, y_2, z_2)$, and $S_3(x_3, y_3, z_3)$. The coordinate systems S_i and S_0 are fixed to the frame, the coordinate system S_1 to the cam, and the coordinate systems S_2 and S_3 to the follower. Axes z_i and z_1 coincide with the input axis of the cam; axes z_0 and z_2 coincide with the output axis of the follower; and axis z_3 coincide with the axis of the roller. Axes x_i and x_0 lie along the common normal of axes z_i and z_0 from origins O_i and O_0 , while axes x_2 and x_3 lie along the common normal of axes z_2 and z_3 from origins O_2 and O_3 . The shortest distance and the twist angle from axis z_i to axis z_0 are a and α , respectively, and the shortest distance and the twist angle from axis z_2 to axis z_3 are b and β , respectively. The angular displacement and the translational displacement of the cam relative to the frame are ϕ_1 and s_1 , respectively, and those of the follower relative to the frame are ϕ_2 and s_2 . The angular displacement of the roller relative to the follower is ϕ_3 . The position vectors of the directrices of the cam surface and the roller surface are $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(3)}$, while the unit vectors parallel to their associated generatrices are $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(3)}$.

4. Axode Surfaces

The basic screw matrices from a coordinate system to another coordinate through a dual angle $\tilde{a}$ about axis x and axis z are given by, respectively

$$[\hat{R}_x(a, a)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \tilde{a} & -\sin \tilde{a} \\ 0 & \sin \tilde{a} & \cos \tilde{a} \end{bmatrix}, \quad (8)$$

$$[\hat{R}_z(a, a)] = \begin{bmatrix} \cos \tilde{a} & -\sin \tilde{a} & 0 \\ \sin \tilde{a} & \cos \tilde{a} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (9)$$

where

$$\tilde{a} = a + \varepsilon a.$$

Thus, the transformation matrices between the coordinate systems S_i , S_0 , S_1 , S_2 , and S_3 are represented by

$$[\hat{R}_{i1}] = [\hat{R}_z(\phi_1, s_1)], \quad [\hat{R}_{1i}] = [\hat{R}_z(-\phi_1, -s_1)], \quad (10)$$

$$[\hat{R}_{i0}] = [\hat{R}_x(a, a)], \quad [\hat{R}_{0i}] = [\hat{R}_x(-a, -a)], \quad (11)$$

$$[\hat{R}_{02}] = [\hat{R}_z(\phi_2, s_2)], \quad [\hat{R}_{20}] = [\hat{R}_z(-\phi_2, -s_2)], \quad (12)$$

$$[\hat{R}_{23}] = [\hat{R}_x(\beta, b)], \quad [\hat{R}_{32}] = [\hat{R}_x(-\beta, -b)]. \quad (13)$$

The ISA_{10} of the cam relative to the frame is given by

$$\hat{S}^{(10)} = \omega^{(10)} + \varepsilon \tau^{(10)} = (\dot{\phi}_1 + \varepsilon \dot{s}_1) \mathbf{k}_i, \quad (14)$$

where $\omega^{(10)}$ and $\tau^{(10)}$ are the angular and translational velocities of the cam with respect to the frame on the input axis, and $\mathbf{k}_i$ is the unit vector parallel to axis z_i .

Similarly, the ISA_{20} of the follower relative to the frame is given by

$$\hat{S}^{(20)} = \omega^{(20)} + \varepsilon \tau^{(20)} = (\dot{\phi}_2 + \varepsilon \dot{s}_2) \mathbf{k}_0, \quad (15)$$

where $\omega^{(20)}$ and $\tau^{(20)}$ are the angular and translational velocities of the follower with respect to the frame on the output axis, and $\mathbf{k}_0$ is the unit vector parallel to axis z_0 .

The ISA_{32} of the roller relative to the follower is given by

$$\hat{S}^{(32)} = \omega^{(32)} + \varepsilon \tau^{(32)} = \dot{\phi}_3 \mathbf{k}_3. \quad (16)$$

In Eqs.(14) to (16), the velocities $\dot{\phi}_1$, $\dot{s}_1$, $\dot{\phi}_2$, and $\dot{s}_2$ are known for the specified input and output functions but the angular velocity $\dot{\phi}_3$ is to be determined.

4.1 Pitch surface

The pitch surface is the locus of the center line of the roller on the reference frame attached to the cam. Thus, transforming the ISA_{32} in coordinate system S_1 leads to

$$[\hat{S}_1^{(32)}] = [\hat{R}_{13}][\hat{S}_3^{(32)}] = [\hat{R}_{1i}][\hat{R}_{i0}][\hat{R}_{02}][\hat{R}_{23}][\hat{S}_3^{(32)}]. \quad (17)$$

Substituting Eqs.(10) to (13) and Eq.(16) into Eq.(17), we have

$$\hat{S}^{(32)} = \dot{\phi}_3 (e_{x1}^{(32)} \mathbf{i}_1 + e_{y1}^{(32)} \mathbf{j}_1 + e_{z1}^{(32)} \mathbf{k}_1) + \varepsilon (\tau_{x1}^{(32)} \mathbf{i}_1 + \tau_{y1}^{(32)} \mathbf{j}_1 + \tau_{z1}^{(32)} \mathbf{k}_1), \quad (18)$$

with the components

$$\begin{aligned} e_{x1}^{(32)} &= -(SaCb + CaS\beta C\phi_2)S\phi_1 + S\beta C\phi_1 S\phi_2, \\ e_{y1}^{(32)} &= -(SaCb + CaS\beta C\phi_2)C\phi_1 - S\beta S\phi_1 S\phi_2, \\ e_{z1}^{(32)} &= -SaS\beta C\phi_2 + CaCb, \\ \tau_{x1}^{(32)} &= \dot{\phi}_3 \{ a(-CaCb + SaS\beta C\phi_2)S\phi_1 \\ &\quad + b((SaS\beta - CaCbC\phi_2)S\phi_1 + C\beta S\phi_2 C\phi_1) \\ &\quad + s_1(-(SaCb + CaS\beta C\phi_2)C\phi_1 - S\beta S\phi_2 S\phi_1) \} \end{aligned}$$

$$\begin{aligned}
& + s_2(S\beta C\phi_1 C\phi_2 + CaS\beta S\phi_1 S\phi_2)\}, \\
\tau_{y1}^{(32)} &= \dot{\phi}_3\{a(-CaC\beta + SaS\beta C\phi_2)C\phi_1 \\
& + b((SaS\beta - CaC\beta C\phi_2)C\phi_1 - C\beta S\phi_2 S\phi_1) \\
& + s_1((SaC\beta + CaS\beta C\phi_2)S\phi_1 - S\beta S\phi_2 C\phi_1) \\
& + s_2(-S\beta S\phi_1 C\phi_2 + CaS\beta C\phi_1 S\phi_2)\}, \\
\tau_{z1}^{(32)} &= \dot{\phi}_3\{a(-SaC\beta - CaS\beta C\phi_2) \\
& + b(-CaS\beta - SaC\beta C\phi_2) + s_2SaS\beta S\phi_2\}.
\end{aligned}$$

where sine and cosine are designated as S and C , respectively.

With the help of Eqs. (2) to (4) and using Eq. (18), the point coordinates of the pitch surface are derived as

$$\begin{aligned}
\mathbf{r}^{(32)} &= p_{x1}^{(32)} \mathbf{i}_1 + p_{y1}^{(32)} \mathbf{j}_1 + p_{z1}^{(32)} \mathbf{k}_1 + u(e_{x1}^{(32)} \mathbf{i}_1 + e_{y1}^{(32)} \mathbf{j}_1 \\
& + e_{z1}^{(32)} \mathbf{k}_1), \quad (19)
\end{aligned}$$

where

$$\begin{aligned}
p_{x1}^{(32)} &= x_i C\phi_1 + y_i S\phi_1, \\
p_{y1}^{(32)} &= -x_i S\phi_1 + y_i C\phi_1, \\
p_{z1}^{(32)} &= a(SaS\beta C\phi_2 - CaC\beta)S\beta S\phi_2 + bSaS\phi_2 \\
& - s_1((SaC\beta + CaS\beta C\phi_2)^2 + S^2\beta S^2\phi_2) + s_2(CaS^2\beta \\
& + SaS\beta C\beta C\phi_2), \\
x_i &= aS\beta S\phi_2(SaC\beta + CaS\beta C\phi_2) + bCaS\phi_2 \\
& + s_1(SaCa(S^2\beta C^2\phi_2 - C^2\beta) \\
& + S\beta C\beta C\phi_2(S^2\alpha - C^2\alpha)) \\
& + s_2(CaS\beta C\beta C\phi_2 - SaS^2\beta), \\
y_i &= a(C^2\beta + S^2\beta C^2\phi_2) + bC\phi_2 \\
& + s_1S\beta S\phi_2(CaC\beta - SaS\beta C\phi_2) - s_2S\beta C\beta S\phi_2.
\end{aligned}$$

4.2 Surface of action

The surface of action between the cam and the roller is the locus of the contact lines determined in the coordinate system fixed to the frame. Thus, the ISA_{31} of the roller relative to the cam in coordinate system S_i leads to

$$\begin{aligned}
[\hat{S}_i^{(31)}] &= [\hat{R}_{io}][\hat{R}_{o2}][\hat{R}_{23}][\hat{S}_3^{(32)}] + [\hat{R}_{io}][\hat{S}_2^{(20)}] \\
& - [\hat{S}_i^{(10)}]. \quad (20)
\end{aligned}$$

Substituting Eqs. (11) to (16) into Eq. (20), we have

$$\begin{aligned}
\mathbf{S}^{(31)} &= \omega_{xi}^{(31)} \mathbf{i}_i + \omega_{yi}^{(31)} \mathbf{j}_i + \omega_{zi}^{(31)} \mathbf{k}_i + \varepsilon(\tau_{xi}^{(31)} \mathbf{i}_i + \tau_{yi}^{(31)} \mathbf{j}_i \\
& + \tau_{zi}^{(31)} \mathbf{k}_i), \quad (21)
\end{aligned}$$

with the components

$$\begin{aligned}
\omega_{xi}^{(31)} &= \dot{\phi}_3 S\beta S\phi_2, \\
\omega_{yi}^{(31)} &= -\dot{\phi}_2 Sa + \dot{\phi}_3(-SaC\beta - CaS\beta C\phi_2), \\
\omega_{zi}^{(31)} &= (-\dot{\phi}_1 + \dot{\phi}_2 Ca) + \dot{\phi}_3(CaC\beta - SaS\beta C\phi_2), \\
\tau_{xi}^{(31)} &= \dot{\phi}_3(bC\beta S\phi_2 + s_2S\beta C\phi_2), \\
\tau_{yi}^{(31)} &= -\dot{s}_2 Sa - a\dot{\phi}_2 Ca \\
& + \dot{\phi}_3\{a(-CaC\beta + SaS\beta C\phi_2) + b(SaS\beta \\
& - CaC\beta C\phi_2) + s_2CaS\beta S\phi_2\}, \\
\tau_{zi}^{(31)} &= -\dot{s}_1 + \dot{s}_2 Ca - a\dot{\phi}_2 Sa \\
& + \dot{\phi}_3\{a(-SaC\beta - CaS\beta C\phi_2) + b(-CaS\beta \\
& - SaC\beta C\phi_2) + s_2SaS\beta S\phi_2\}.
\end{aligned}$$

Similarly, with the help of Eqs. (2) to (4) and using Eq. (21), the point coordinates of the surface of action are derived as

$$\begin{aligned}
\mathbf{r}^{(31)} &= x_i^{(31)} \mathbf{i}_i + y_i^{(31)} \mathbf{j}_i + z_i^{(31)} \mathbf{k}_i \\
& = p_{xi}^{(31)} \mathbf{i}_i + p_{yi}^{(31)} \mathbf{j}_i + p_{zi}^{(31)} \mathbf{k}_i + u(\omega_{xi}^{(31)} \mathbf{i}_i + \omega_{yi}^{(31)} \mathbf{j}_i
\end{aligned}$$

$$+ \omega_{zi}^{(31)} \mathbf{k}_i) / \omega^{(31)}, \quad (22)$$

where

$$\begin{aligned}
p_{xi}^{(31)}(\omega^{(31)})^2 &= a\dot{\phi}_2(\dot{\phi}_2 - \dot{\phi}_1 Ca) + (\dot{\phi}_2 \dot{s}_1 - \dot{\phi}_1 \dot{s}_2)Sa \\
& + \dot{\phi}_3\{\dot{\phi}_1(a(-CaC\beta + SaS\beta C\phi_2) + b(SaS\beta \\
& - CaC\beta C\phi_2) + s_2CaS\beta S\phi_2) \\
& + \dot{\phi}_2(2aC\beta + bC\beta C\phi_2 - s_2S\beta S\phi_2) \\
& + \dot{\phi}_3(a(1 - S^2\beta S^2\phi_2) + bC\phi_2 - s_2S\beta C\beta S\phi_2) \\
& + \dot{s}_1(SaC\beta + CaS\beta C\phi_2) - \dot{s}_2S\beta C\phi_2\}, \\
p_{yi}^{(31)}(\omega^{(31)})^2 &= \dot{\phi}_3\{-\dot{\phi}_1(bC\beta S\phi_2 + s_2S\beta C\phi_2) \\
& + \dot{\phi}_2(aSaS\beta S\phi_2 + bCaC\beta S\phi_2 + s_2CaS\beta C\phi_2) \\
& + \dot{\phi}_3(a(SaS\beta C\beta S\phi_2 + CaS^2\beta S\phi_2 C\phi_2) + bCaS\phi_2 \\
& + s_2(CaS\beta C\beta C\phi_2 - SaS^2\beta)) \\
& + \dot{s}_1S\beta S\phi_2 - \dot{s}_2CaS\beta S\phi_2\}, \\
p_{zi}^{(31)}(\omega^{(31)})^2 &= \dot{\phi}_3\{\dot{\phi}_2(-aCaS\beta S\phi_2 + bSaC\beta S\phi_2 \\
& + s_2SaS\beta C\phi_2) \\
& + \dot{\phi}_3(a(-CaS\beta C\beta S\phi_2 + SaS^2\beta S\phi_2 C\phi_2) + bSaS\phi_2 \\
& + s_2(SaS\beta C\beta C\phi_2 + CaS^2\beta)) \\
& - \dot{s}_2SaS\beta S\phi_2\}, \\
\omega^{(31)} &= (\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2 + 2(\dot{\phi}_1 \dot{\phi}_3(SaS\beta C\phi_2 - CaC\beta) \\
& - \dot{\phi}_1 \dot{\phi}_2 Ca + \dot{\phi}_2 \dot{\phi}_3 C\beta))^{1/2}. \quad (23)
\end{aligned}$$

Substituting Eq. (21) into Eq. (6), the pitch of the ISA_{31} is derived as

$$\begin{aligned}
h_{31} &= \{a\dot{\phi}_1 \dot{\phi}_2 Sa - (\dot{\phi}_1 \dot{s}_2 + \dot{\phi}_2 \dot{s}_1)Ca + (\dot{\phi}_1 \dot{s}_1 + \dot{\phi}_2 \dot{s}_2) \\
& + \dot{\phi}_3 \dot{\phi}_1(a(CaS\beta C\phi_2 + SaC\beta) + b(CaS\beta \\
& + SaC\beta C\phi_2) - s_2SaS\beta S\phi_2) \\
& - \dot{\phi}_3 \dot{\phi}_2 bS\beta + \dot{\phi}_3 \dot{s}_1(SaS\beta C\phi_2 - CaC\beta) \\
& + \dot{\phi}_3 \dot{s}_2 C\beta\} / (\omega^{(31)})^2. \quad (24)
\end{aligned}$$

4.3 Cam surface

Expressing the ISA_{31} in coordinate system S_1 leads to

$$[\hat{S}_1^{(31)}] = [\hat{R}_{1i}][\hat{S}_i^{(31)}]. \quad (25)$$

Likewise, to derive the surface of action, the cam surface can be obtained by Eqs. (2) to (4) and Eq. (25). However, the result will be complicated and then we use the point coordinates of the surface of action of Eq. (22) to derive the cam surface. By means of a suitable coordinate transformation, the cam surface is given by

$$\begin{aligned}
\mathbf{r}^{(31)} &= (x_i^{(31)} C\phi_1 + y_i^{(31)} S\phi_1) \mathbf{i}_1 + (-x_i^{(31)} S\phi_1 \\
& + y_i^{(31)} C\phi_1) \mathbf{j}_1 + (z_i^{(31)} - s_1) \mathbf{k}_1. \quad (26)
\end{aligned}$$

4.4 Roller surface

Expressing the ISA_{31} in coordinate system S_3 leads to

$$[\hat{S}_3^{(31)}] = [\hat{S}_3^{(32)}] + [\hat{R}_{32}][\hat{S}_2^{(20)}] - [\hat{R}_{2o}][\hat{R}_{oi}][\hat{S}_i^{(10)}]. \quad (27)$$

Substituting Eqs. (11) to (16) into Eq. (27), we have

$$\begin{aligned}
\mathbf{S}^{(31)} &= \omega_{x3}^{(31)} \mathbf{i}_3 + \omega_{y3}^{(31)} \mathbf{j}_3 + \omega_{z3}^{(31)} \mathbf{k}_3 + \varepsilon(\tau_{x3}^{(31)} \mathbf{i}_3 + \tau_{y3}^{(31)} \mathbf{j}_3 \\
& + \tau_{z3}^{(31)} \mathbf{k}_3). \quad (28)
\end{aligned}$$

with the components

$$\begin{aligned}
\omega_{x3}^{(31)} &= -\dot{\phi}_1 SaS\phi_2, \\
\omega_{y3}^{(31)} &= \dot{\phi}_1(-CaS\beta - SaC\beta C\phi_2) + \dot{\phi}_2 S\beta, \\
\omega_{z3}^{(31)} &= \dot{\phi}_1(-CaC\beta + SaS\beta C\phi_2) + \dot{\phi}_2 C\beta + \dot{\phi}_3, \\
\tau_{x3}^{(31)} &= -\dot{\phi}_1(aCaS\phi_2 + s_2SaC\phi_2) - \dot{s}_1 SaS\phi_2,
\end{aligned} \quad (29)$$

$$\begin{aligned}\tau_{y3}^{(31)} &= \dot{\phi}_1(a(S\alpha S\beta - CaC\beta C\phi_2) + b(-CaC\beta \\ &\quad + SaS\beta C\phi_2) + s_2SaC\beta S\phi_2) \\ &\quad + \dot{\phi}_2bC\beta - \dot{s}_1(CaS\beta + SaC\beta C\phi_2) + \dot{s}_2S\beta, \quad (30) \\ \tau_{z3}^{(31)} &= \dot{\phi}_1(a(SaC\beta + CaS\beta C\phi_2) + b(CaS\beta \\ &\quad + SaC\beta C\phi_2) - s_2SaS\beta S\phi_2) \\ &\quad - \dot{\phi}_2bS\beta - \dot{s}_1(CaC\beta - SaS\beta C\phi_2) + \dot{s}_2C\beta.\end{aligned}$$

Furthermore, with the aid of Eqs. (2) to (4) and using Eq. (28), the point coordinates of the roller surface are derived as

$$\mathbf{r}^{(31)} = p_{x3}^{(31)}\mathbf{i}_3 + p_{y3}^{(31)}\mathbf{j}_3 + p_{z3}^{(31)}\mathbf{k}_3 + u(e_{x3}^{(31)}\mathbf{i}_3 + e_{y3}^{(31)}\mathbf{j}_3 + e_{z3}^{(31)}\mathbf{k}_3), \quad (31)$$

where

$$\begin{aligned}e_{x3}^{(31)} &= \omega_{x3}^{(31)}/\omega^{(31)}, \quad e_{y3}^{(31)} = \omega_{y3}^{(31)}/\omega^{(31)}, \\ e_{z3}^{(31)} &= \omega_{z3}^{(31)}/\omega^{(31)},\end{aligned} \quad (32)$$

$$\begin{aligned}p_{x3}^{(31)}(\omega^{(31)})^2 &= (\dot{\phi}_2\dot{s}_1 - \dot{\phi}_1\dot{s}_2)SaC\phi_2 \\ &\quad + \dot{\phi}_1\dot{\phi}_2(aCaC\phi_2 + 2bCa - s_2SaS\phi_2) \\ &\quad + \dot{\phi}_1^2(-aC\phi_2 + b(S^2aS^2\phi_2 - 1) + s_2SaCaS\phi_2) \\ &\quad - \dot{\phi}_2^2b \\ &\quad + \dot{\phi}_3\dot{\phi}_1(a(-SaS\beta + CaC\beta C\phi_2) + b(CaC\beta \\ &\quad - SaS\beta C\phi_2) - s_2SaC\beta S\phi_2) - \dot{\phi}_3\dot{\phi}_2bC\beta \\ &\quad + \dot{\phi}_3(\dot{s}_1(CaS\beta + SaC\beta C\phi_2) - \dot{s}_2S\beta), \quad (33)\end{aligned}$$

$$\begin{aligned}p_{y3}^{(31)}(\omega^{(31)})^2 &= -(\dot{\phi}_2\dot{s}_1 - \dot{\phi}_1\dot{s}_2)SaC\beta S\phi_2 \\ &\quad - \dot{\phi}_1\dot{\phi}_2(aCaC\beta S\phi_2 + bSaS\beta S\phi_2 + s_2SaC\beta C\phi_2) \\ &\quad + \dot{\phi}_1^2(aC\beta S\phi_2 + bSaS\phi_2(CaS\beta + SaC\beta C\phi_2) \\ &\quad + s_2(-S^2aS\beta + SaCaC\beta C\phi_2)) \\ &\quad - \dot{\phi}_3\dot{\phi}_1(aCaS\phi_2 + s_2SaC\phi_2) - \dot{\phi}_3\dot{s}_1SaS\phi_2, \quad (34)\end{aligned}$$

$$\begin{aligned}p_{z3}^{(31)}(\omega^{(31)})^2 &= (\dot{\phi}_2\dot{s}_1 - \dot{\phi}_1\dot{s}_2)SaS\beta S\phi_2 \\ &\quad + \dot{\phi}_1\dot{\phi}_2(aCaS\beta S\phi_2 - bSaC\beta S\phi_2 + s_2SaS\beta C\phi_2) \\ &\quad + \dot{\phi}_1^2(-aS\beta S\phi_2 + bSaS\phi_2(CaC\beta - SaS\beta C\phi_2) \\ &\quad - s_2(S^2aC\beta + SaCaS\beta C\phi_2)). \quad (35)\end{aligned}$$

Note that the surface parameter u of Eq. (31) is different from that of Eq. (22) for the coincident contact point. Once the limitations of the geometric parameters are known for various special roller-follower cam mechanisms, we can substitute the values of a , α , b , and β , and the functions $\phi_1(t)$, $s_1(t)$, $\phi_2(t)$, and $s_2(t)$ into Eqs. (22), (26), and (31) to derive the surface of action and the contact surfaces.

5. Hyperboloidal Roller

Figure 2 shows the axode ruled surface. The position vector $\mathbf{p}$ is on the directrix of the ruled surface, located at the shortest distance to the origin, while the unit vector $\mathbf{e}$ is parallel to the direction of the generatrix. As mentioned above, we can use the vectors $\mathbf{p}$ and $\mathbf{e}$ to express the point coordinates of the ruled surface as follows

$$\mathbf{r} = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k} + u(e_x\mathbf{i} + e_y\mathbf{j} + e_z\mathbf{k}), \quad (36)$$

where

$$e_x^2 + e_y^2 + e_z^2 = 1, \quad (37)$$

$$p_x e_x + p_y e_y + p_z e_z = 0. \quad (38)$$

However, the straight line on the ruled surface is also expressed in parametric form as follows

$$\mathbf{r} = cC\theta\mathbf{i} + cS\theta\mathbf{j} + d\mathbf{k} + \nu(S\gamma S\theta\mathbf{i} - S\gamma C\theta\mathbf{j}$$

Fig. 2 Directrix of a ruled surface and its generating line

$$+ C\gamma\mathbf{k}), \quad (39)$$

where the parameters c , γ , d , and θ are functions of one independent surface coordinate for a ruled surface.

For a specified straight line, we have the following relations

$$S^2\gamma = e_x^2 + e_y^2, \quad C\gamma = e_z, \quad (40)$$

$$c = \begin{cases} \left(\frac{p_x^2 + p_y^2 - \frac{p_z^2 e_z^2}{e_x^2 + e_y^2}}{e_x^2 + e_y^2} \right)^{1/2} = \frac{p_y e_x - p_x e_y}{S\gamma} & \text{if } |e_z| \neq 1 \\ (p_x^2 + p_y^2)^{1/2} & \text{if } |e_z| = 1 \end{cases} \quad (41)$$

$$\tan \theta = \begin{cases} -e_x/e_y & \text{if } |e_z| \neq 1 \\ p_y/p_x & \text{if } |e_z| = 1 \end{cases} \quad (42)$$

$$d = \frac{p_z}{e_x^2 + e_y^2} \quad \text{if } |e_z| \neq 1. \quad (43)$$

where $c > 0$.

Observing Eqs. (22), (26), and (31), the angular velocity of the roller relative to the follower appears as well as the velocities of the cam and the follower in these equations. Thus, the cam surface of Eq. (26) is influenced by the relative velocity between the roller and the follower unless the roller has a surface of revolution, that is revolved about the axis of the roller. Usually, the kinematic condition of cam synthesis is limited to the input-output relation only. Without considering the relative motion between the roller and the follower, the roller surface must be a surface of revolution. Considering the roller with surface of revolution, the surface normal in Eq. (7) passes through the axis of the roller. Now, the contact lines are given by the various positions of the ISA of the roller relative to the cam. Thus, the surface of the roller, that is generated by a line, is a hyperboloid of revolution, that is named the hyperboloidal roller. If we let the axis z of the reference frame be aligned with the axis of the hyperboloidal roller, then the following conditions are satisfied

$$\gamma = \text{const}, \quad c = \text{const}, \quad d = \text{const}. \quad (44)$$

Substituting Eq. (44) into Eqs. (40), (41), and (43), we have

$$e_z = \text{const}, \quad p_x^2 + p_y^2 = \text{const}, \quad p_z = \text{const}. \quad (45)$$

6. Cam Mechanisms with Hyperboloidal Roller

In this section, we will study the possible cam mechanisms with hyperboloidal roller in axode contact. The joints incident to the frame are either revolute or prismatic for these cam mechanisms. In what follows, the symbols R, P, and A designate revolute, prismatic, and cam pairs, respectively. For example, a RARP mechanism expresses a roller-follower cam mechanism in which the kinematic pair of the input axis is R and that of the output axis P. In order to synthesize these cam mechanisms, we consider

(a) The cam makes contact with the roller on their ISA continuously if the parameters γ , c , and d are constant, i.e., the conditions of Eq.(44) or (45) are always satisfied.

(b) If two equalities of Eq.(44) or (45) are satisfied, the third equality is used to determine the angular velocity of the roller relative to the follower.

(c) If $d=0$ or $S\gamma=0$, Eq.(43) yields

$$p_{z3}^{(31)}=0. \quad (46)$$

(d) If $S\gamma=0$, combining Eqs.(32) and (40) leads to

$$\omega_{x3}^{(31)}=0, \quad \omega_{y3}^{(31)}=0, \quad (47)$$

for arbitrary functions $\phi_1(t)$ and $\phi_2(t)$.

The pitch of the ISA_{31} is simplified as

$$h_{31}=\omega_{z3}^{(31)}\tau_{z3}^{(31)}/(\omega^{(31)})^2. \quad (48)$$

For the cam mechanisms with $S\gamma=0$, the geometric parameters γ and $p_{z3}^{(31)}$ are constant and the parameter d is uncertain. Then we need to determine the conditions for $S\gamma=0$ only.

(e) If $S\gamma \neq 0$, combining Eqs.(32), (35), (40), and (43) leads to

$$d((\omega_{x3}^{(31)})^2 + (\omega_{y3}^{(31)})^2) = p_{z3}^{(31)}(\omega^{(31)})^2. \quad (49)$$

Equation (49) is always satisfied for arbitrary functions $\phi_1(t)$, $\phi_2(t)$, $s_1(t)$, and $s_2(t)$.

6.1 RARR mechanism

For a RARR cam mechanism, both the kinematic pairs of the input axis and the output axis are revolute pairs such that $s_1=\text{const}$, $s_2=\text{const}$, $\dot{s}_1=0$, $\dot{s}_2=0$, and we let $\phi'_2=d\phi_2/d\phi_1$, $\phi'_3 \equiv d\phi_3/d\phi_1$, and $\dot{\phi}_1 > 0$.

(1) If $S\gamma=0$, from Eq.(47), we have

$$-SaS\phi_2=0, \quad -CaS\beta - SaC\beta C\phi_2 + \phi'_2S\beta=0. \quad (50)$$

For an arbitrary function $\phi_2(t)$, Eqs.(50) are identities if the following conditions are satisfied

$$Sa=0, \quad S\beta=0.$$

Therefore, Eq.(48) yields

$$h_{31}=0. \quad (51)$$

Now, we apply the equations related to the parameter c to determine the angular velocity ϕ'_3 . Substituting Eqs.(33) and (34) into Eq.(41), we have

$$c^2M^2=((aC\phi_2+b)Ca-\phi'_2b)^2+(aS\phi_2)^2,$$

$$M=|-CaC\beta+\phi'_2C\beta+\phi'_3|.$$

For arbitrary c , this leads to

$$\phi'_3=MC\gamma+CaC\beta-\phi'_2C\beta,$$

$$M=((aC\phi_2+b)Ca-\phi'_2b)^2+(aS\phi_2)^2/c. \quad (52)$$

Since all of the ISAs are parallel to each other and the input and output axes have pure rotation motion, the cam mechanism is a planar one. From Eq.(51), the cam pair always has pure rolling contact. The parameters d and s_2 bear no relation to the cam mechanism.

If $S\gamma \neq 0$, applying Eq.(49), we have

$$\begin{aligned} &\phi'_2dS^2\beta - \phi'_2[2dCaS^2\beta + (aCaS\beta - bSaC\beta)S\phi_2 \\ &\quad + (2dC\beta + s_2)SaS\beta C\phi_2] \\ &\quad + S^2a(dS^2\phi_2 + bS\beta S\phi_2 C\phi_2 + dC^2\beta C^2\phi_2) + (aS\beta \\ &\quad - bSaCaC\beta)S\phi_2 \\ &\quad + (s_2 + 2dC\beta)SaCaS\beta C\phi_2 + dC^2aS^2\beta + s_2S^2aC\beta \\ &= 0, \end{aligned} \quad (53)$$

where $Sa \neq 0$ or $S\beta \neq 0$.

For an arbitrary function $\phi_2(t)$, the first term of Eq.(53) leads to $dS\beta=0$.

If $d=0$, Eq.(53) becomes

$$\begin{aligned} &\phi'_2[(aCaS\beta - bSaC\beta)S\phi_2 + s_2SaS\beta C\phi_2] + (-aS\beta \\ &\quad + bSaCaC\beta)S\phi_2 \\ &\quad - bS^2aS\beta S\phi_2 C\phi_2 - s_2SaCaS\beta C\phi_2 - s_2S^2aC\beta = 0. \end{aligned} \quad (54)$$

If $S\beta=0$, Eq.(53) becomes

$$Sa(Sa(d+s_2C\beta) + (\phi'_2-Ca)bC\beta S\phi_2)=0. \quad (55)$$

For an arbitrary function $\phi_2(t)$ and $Sa \neq 0$ or $S\beta \neq 0$, either Eq.(54) or (55) is an identity if the geometric parameters satisfy one of the following conditions

- (a) $a=0$, $d=0$, $Sa=0$, $S\beta \neq 0$.
- (b) $a=0$, $b=0$, $d=0$, $s_2=0$, $Sa \neq 0$ or $S\beta \neq 0$.
- (c) $b=0$, $s_2=-dC\beta$, $S\beta=0$, $Sa \neq 0$.

In what follows, we consider the cam mechanisms that satisfy $S\gamma \neq 0$.

(2) If $a=0$, $d=0$, $Sa=0$, $S\beta \neq 0$, $S\gamma \neq 0$, combining Eqs.(32) and (40) yields

$$\phi'_2C\beta + \phi'_3=MC\gamma + CaC\beta, \quad (56)$$

$$M^2S^2\gamma=(Ca-\phi'_2)^2S^2\beta, \quad (57)$$

where

$$M^2=1+\phi'_2{}^2+\phi'_3{}^2+2(-\phi'_2Ca-\phi'_3CaC\beta+\phi'_2\phi'_3C\beta). \quad (58)$$

For known β and γ , M is determined by Eq.(57) and then the angular-velocity ratio ϕ'_3 is derived by Eq.(56).

Using Eqs.(33) and (34) and applying Eq.(56), we obtain

$$p_{z3}^{(31)}=b(C\gamma C\beta(Ca-\phi'_2)/M-S^2\gamma), \quad p_{y3}^{(31)}=0. \quad (59)$$

Substituting Eqs.(57) and (59) into (41), we obtain

$$c=|bS^2\gamma(1 \pm \cot \gamma \cot \beta)|. \quad (60)$$

For known β , γ , and b , the parameter c is determined by Eq.(60). Using Eq.(24), the screw pitch of

the ISA_{31} is given by

$$h_{31} = b(C\gamma S\beta(C\alpha - \phi'_2)/M + \cot \beta S^2\gamma) = \text{const.} \quad (61)$$

We observe from Eq. (59) that the location of the ISA_{31} on the roller is constant. If $b=0$, the pitch of the ISA_{31} is zero and this cam mechanism has a cam pair with pure rolling. Note that the input axis is coaxial to the output axis, and the surface normal of the roller passes through the axis of the roller and the output axis. The contact force transmitted to the follower passes through the output axis. Thus, the cosine of the pressure angle is zero, and so, this cam mechanism is incapable of force or motion transmission. We can show that both the surface of action and the cam surface are hyperboloids of revolution.

(3) If $a=0$, $b=0$, $d=0$, $s_2=0$, $S\alpha \neq 0$ or $S\beta \neq 0$, we have

$$p_{x3}^{(31)}=0, \quad p_{y3}^{(31)}=0, \quad c=0, \quad h_{31}=0. \quad (62)$$

Combining Eqs. (29) and (40), we have

$$M^2 S^2 \gamma = (SaS\phi_2)^2 + (-CaS\beta - SaC\beta C\phi_2 + \phi'_2 S\beta)^2, \\ MC\gamma = -CaC\beta + SaS\beta C\phi_2 + \phi'_2 C\beta + \phi'_3. \quad (63)$$

where

$$M^2 = 1 + \phi'_2{}^2 + \phi'_3{}^2 + 2(-\phi'_2 Ca + \phi'_3(-CaC\beta + SaS\beta C\phi_2) + \phi'_2 \phi'_3 C\beta). \quad (64)$$

For given α , β , and γ , the angular velocity ϕ'_3 can be determined by the following equation

$$A\phi'_3{}^2 + B\phi'_3 + C = 0, \quad (65)$$

where

$$A = S^2\gamma, \\ B = 2S^2\gamma(-CaC\beta + SaS\beta C\phi_2 + \phi'_2 C\beta), \\ C = S^2\gamma(1 - 2\phi'_2 Ca + \phi'_2{}^2) - (SaS\phi_2)^2 - (-CaS\beta - SaC\beta C\phi_2 + \phi'_2 S\beta)^2.$$

Equation (65) gives two values for the angular-velocity ratio ϕ'_3 , since the roller couples with the cam at two lines instantaneously. In practice, the roller velocity depends on the location of the contact line and has a unique value. Since all the ISAs intersect at a common point and the input and output axes have pure rotation, the cam mechanism is spherical. In this case, the cam pair always has pure rolling contact.

(4) If $b=0$, $s_2 = -dC\beta$, $S\beta=0$, $S\alpha \neq 0$, $S\gamma \neq 0$, combining Eqs. (29) and (40) yields

$$M^2 S^2 \gamma = S^2 \alpha, \quad (66)$$

$$MC\gamma = -CaC\beta + \phi'_2 C\beta + \phi'_3, \quad (67)$$

where

$$M^2 = S^2 \alpha + (-CaC\beta + \phi'_2 C\beta + \phi'_3)^2. \quad (68)$$

From Eq. (66), M must be constant for given α and γ . The angular velocity ϕ'_3 is determined by Eq. (67).

Applying Eq. (66) to Eqs. (33), (34), and (35), we obtain

$$p_x M^2 = (MC\alpha C\beta C\gamma - S^2 \alpha) a C\phi_2 + dMS\alpha C\gamma S\phi_2, \\ p_y M^2 = -(MC\alpha C\beta C\gamma - S^2 \alpha) a C\beta S\phi_2 \\ + dMS\alpha C\beta C\gamma C\phi_2, \\ p_z M^2 = dS^2 \alpha. \quad (69)$$

Substituting Eqs. (66) to (69) into (41), we obtain

$$c = |a S^2 \gamma (\pm \cot \alpha \cot \gamma - C\beta)|. \quad (70)$$

Using Eq. (24), the screw pitch of the ISA_{31} is given by

$$h_{31} = aSa(C\alpha + MC\beta C\gamma)/M^2 = \text{const.} \quad (71)$$

Since the axis of the roller is coaxial to the output axis, it follows that the cosine of the pressure angle is zero. The surface of action is a straight line and the cam surface is a hyperboloid of revolution.

6.2 RARP mechanism

For a RARP cam mechanism, $s_1 = \text{const}$, $\phi_2 = \text{const}$, $\dot{s}_1 = 0$, $\dot{\phi}_2 = 0$ and we let $s'_2 = ds_2/d\phi_1$, $\phi'_3 = d\phi_3/d\phi_1$, and $\dot{\phi}_1 > 0$.

For $S\gamma = 0$, the roller surface is cylindrical and Eq. (47) yields

$$SaS\phi_2 = 0, \quad CaS\beta + SaC\beta C\phi_2 = 0. \\ \text{Furthermore, we obtain} \\ S\alpha = 0, \quad S\beta = 0, \quad (72)$$

or

$$S\phi_2 = 0, \quad S(\alpha + \beta C\phi_2) = 0. \quad (73)$$

(5) If $S\alpha = 0$, $S\beta = 0$, and $S\gamma = 0$, we let M be

$$M = \omega^{(31)}/\dot{\phi}_1 = |-CaC\beta + \phi'_3|. \quad (74)$$

Using Eqs. (33) and (34), we have

$$p_{x3}^{(31)} M^2 = (aC\phi_2 + b)(-CaC\beta + \phi'_3)CaC\beta, \\ p_{y3}^{(31)} M^2 = -a(-CaC\beta + \phi'_3)CaS\phi_2. \quad (75)$$

Substituting Eq. (75) into Eq. (41), we have

$$c^2 = (a^2 + 2abC\phi_2 + b^2)/M^2. \quad (76)$$

For a given c , we obtain

$$M = (a^2 + 2abC\phi_2 + b^2)^{1/2}/c = \text{const}, \\ \phi'_3 = CaC\beta + C\gamma M = \text{const.} \quad (77)$$

Using Eq. (24), the screw pitch of the ISA_{31} is given by

$$h_{31} = s'_2 C\beta C\gamma / M. \quad (78)$$

The cam mechanism in this case does not generate pure rolling contact always. Since the ISA_{32} is parallel to the ISA_{20} and the contact force acting on the roller is normal to the axis of the roller, the cosine of the pressure angle is zero. The surface of action is a straight line and the cam surface is cylindrical.

(6) If $S\gamma = 0$, $S\phi_2 = 0$, and $CaS\beta + SaC\beta C\phi_2 = 0$, we let M be

$$M = \omega^{(31)}/\dot{\phi}_1 = |-C(\alpha + \beta C\phi_2) + \phi'_3|. \quad (79)$$

Using Eqs. (33) and (34), we have

$$p_{x3}^{(31)} M^2 = -s'_2(SaC\phi_2 + \phi'_3 S\beta) + (aC\phi_2 + b)(\phi'_3 C(\alpha + \beta C\phi_2) - 1), \\ p_{y3}^{(31)} M^2 = -s_2 SaC\phi_2(-C(\alpha + \beta C\phi_2) + \phi'_3). \quad (80)$$

Substituting Eq. (80) into Eq. (41), we have

$$c^2 M^2 = ((aC\phi_2 + b)C(\alpha + \beta C\phi_2) - s'_2 S\beta)^2 + (s_2 Sa)^2.$$

For a given c , this leads to

$$M = \{((aC\phi_2 + b)C(\alpha + \beta C\phi_2) - s'_2 S\beta)^2 + (s_2 Sa)^2\}^{1/2}/c, \quad (81)$$

$$\phi'_3 = C(\alpha + \beta C\phi_2) + MC\gamma.$$

Using Eq. (24), the pitch of the ISA_{31} is given by

$$h_{31} = s'_2 C\beta(MC\gamma + (1 - C\phi_2)SaS\beta)/M^2. \quad (82)$$

If $C\alpha=0$ and $C\beta=0$, the pitch of the ISA_{31} is zero and this cam mechanism has a cam pair with pure rolling contact. It is a disk cam mechanism.

(7) For $S\gamma \neq 0$, we let M be

$$M = \omega^{(31)} / \dot{\phi}_1 = (1 + \phi_3^2 + 2\phi_3(-CaC\beta + SaS\beta C\phi_2))^{1/2}. \quad (83)$$

Using Eq. (40), we have

$$M^2 S^2 \gamma = S^2 \alpha S^2 \phi_2 + C^2 \alpha S^2 \beta + S^2 \alpha C^2 \beta C^2 \phi_2 + 2 SaCaS\beta C\beta C\phi_2 = \text{const}, \quad (84)$$

$$MC\gamma = -CaC\beta + SaS\beta C\phi_2 + \phi_3'. \quad (85)$$

For a constant twist angle γ , M and ϕ_3' of Eqs. (84) and (85) must be constant. The parameter M is to be determined afterwards.

Using Eq. (35), we have

$$p_{23}^{(31)} M^2 = -s_2' SaS\beta S\phi_2 - s_2 Sa(SaC\beta + CaS\beta C\phi_2) - aS\beta S\phi_2 + bSaS\phi_2(CaC\beta - SaS\beta C\phi_2). \quad (86)$$

Considering $S\gamma \neq 0$ and arbitrary function $s_2(t)$, Eq. (86) is an identity if the following conditions are satisfied

$$Sa=0, \quad S\beta \neq 0. \quad (87)$$

Furthermore, we have

$$p_{x3}^{(31)} M^2 = -s_2' \phi_3' S\beta + (\phi_3' CaC\beta - 1)(aC\phi_2 + b),$$

$$p_{y3}^{(31)} M^2 = -\phi_3' aCaS\phi_2 + aC\beta S\phi_2 = \text{const}, \quad (88)$$

$$p_{z3}^{(31)} M^2 = -aS\beta S\phi_2 = \text{const}.$$

For arbitrary function $s_2(t)$, Eqs. (88), except for the first equation, are identities; we then obtain

$$\phi_3' = 0. \quad (89)$$

Substituting Eqs. (84), (85), (88), and (89) into Eq. (41), we obtain

$$c = |aC\phi_2 + b|.$$

Moreover, we can obtain

$$\tan \gamma = -\tan \beta, \quad h_{31} = -s_2' Ca. \quad (90)$$

It can be verified that the ISA_{31} is coaxial to the ISA_{10} and both the surface and action and the cam surface degenerate into straight lines.

6.3 PARR mechanism

For a PARR cam mechanism, $\phi_1 = \text{const}$, $s_2 = \text{const}$, $\dot{\phi}_1 = 0$, and $\dot{s}_2 = 0$, we let $\phi_2' = d\phi_2/ds_1$, $\phi_3' = d\phi_3/ds_1$, and $\dot{s}_1 > 0$.

We let M be

$$M = \omega^{(31)} / \dot{s}_1 = (\phi_2'^2 + \phi_3'^2 + 2\phi_2'\phi_3'C\beta)^{1/2}. \quad (91)$$

(8) If $S\gamma = 0$, Eq. (47) becomes

$$S\beta = 0. \quad (92)$$

Thus, Eq. (29) leads to

$$MC\gamma = \phi_2' C\beta + \phi_3'. \quad (93)$$

Using Eqs. (33) and (34), we have

$$p_{x3}^{(31)} M^2 = MC\beta C\gamma (SaC\phi_2 - \phi_2' b),$$

$$p_{y3}^{(31)} M^2 = -MC\gamma SaS\phi_2. \quad (94)$$

For a given c , substituting Eq. (94) into Eq. (41), we have

$$M = (S^2 \alpha + \phi_2'^2 b^2 - 2\phi_2' b SaC\phi_2)^{1/2} / c, \quad c > 0. \quad (95)$$

Using Eqs. (93) and (95), the angular velocity ϕ_3' can be determined. Using Eq. (24), the screw pitch of the ISA_{31} is given by

$$h_{31} = -CaC\beta C\gamma / M. \quad (96)$$

If $C\alpha = 0$, this cam mechanism is planar. Additionally, the pitch h_{31} is zero and the cam couples to the roller with pure rolling.

(9) If $S\gamma \neq 0$, $S\beta \neq 0$, combining Eqs. (29) and (40), we obtain

$$MS\gamma = -\phi_2' S\beta, \quad (97)$$

$$\phi_3' = S\beta \phi_2' (\pm \cot \gamma - \cot \beta). \quad (98)$$

Using Eq. (35), we have

$$p_{23}^{(31)} = SaS\phi_2 S^2 \gamma / (\phi_2' S\beta). \quad (99)$$

For arbitrary function $\phi_2(t)$, Eq. (99) is an identity if the following condition is satisfied

$$Sa = 0.$$

Moreover, this leads to

$$p_{23}^{(31)} = 0, \quad p_{y3}^{(31)} = 0, \quad d = 0. \quad (100)$$

Using Eqs. (33) and (41), we have

$$p_{x3}^{(31)} M^2 = -\phi_2'^2 b - \phi_3' \phi_2' b C\beta + \phi_3' CaS\beta, \quad (101)$$

$$c = |p_{x3}^{(31)}|. \quad (102)$$

For an arbitrary function $\phi_2(t)$ and constant parameter c , the coordinate $p_{x3}^{(31)}$ must be constant. Substituting Eqs. (97) and (98) into Eqs. (101) and (102), we obtain

$$\cot \beta = \pm \cot \gamma, \quad p_{x3}^{(31)} = -b, \quad c = |b|, \quad \phi_3' = 0, \quad M = |b|. \quad (103)$$

It can be verified that the ISA_{31} is coaxial to the ISA_{20} and then the cosine of the pressure angle is zero. Both the surface of action and the cam surface degenerate into straight lines.

6.4 PARP mechanism

For a PARP cam mechanism, $\phi_1 = \text{const}$, $\phi_2 = \text{const}$, $\dot{\phi}_1 = 0$, and $\dot{\phi}_2 = 0$, and we let $s_2' = ds_2/ds_1$ and $\phi_3' = d\phi_3/ds_1$.

(10) Using Eqs. (33) to (35), we have

$$S\gamma = 0, \quad (104)$$

$$p_{23}^{(31)} = (CaS\beta + SaC\beta C\phi_2 - s_2' S\beta) / \phi_3', \quad (105)$$

$$p_{y3}^{(31)} = -SaS\phi_2 / \phi_3', \quad (106)$$

$$p_{z3}^{(31)} = 0. \quad (107)$$

We see that γ and $p_{23}^{(31)}$ are constant for this situation, and the roller has a cylindrical surface. Thus, the radius c of the roller surface can be arbitrary and the angular velocity of the roller is to be determined.

Substituting Eqs. (105) and (106) into Eq. (41), we obtain

$$\phi_3' = C\gamma ((CaS\beta + SaC\beta C\phi_2 - s_2' S\beta)^2 + (SaS\phi_2)^2)^{1/2} / c. \quad (108)$$

Using Eq. (24), the screw pitch of the ISA_{31} is given by

$$h_{31} = (-CaC\beta + SaS\beta C\phi_2 + s_2' C\beta) / \phi_3'. \quad (109)$$

We see that the parameters a and b have no influence on the cam mechanism. Pure rolling cam mechanism generates for $C\beta = 0$ and $SaC\phi_2 = 0$.

6.5 Design procedure

The design procedure of the cam-follower

Fig. 3 Planar PARR Mechanism

Fig. 4 Spherical RARR Mechanism

Fig. 5 RARR Mechanism

mechanism with hyperboloidal roller is described below.

(1) According to the types of the joints incident

Fig. 6 PARR Mechanism

Fig. 7 PARP Mechanism

to the frame, select an available cam mechanism.

(2) Under the restriction conditions of the relative locations of the axes, assign the parameters α , a , β , b , γ , c and d .

(3) Set the motion curve to specify the input-output relation. For example, the function $\phi_2(\phi_1)$ is given and the parameters s_1 and s_2 are fixed for RARR mechanism.

(4) Use Eq.(19) to synthesize the pitch surface, Eq.(22) to the surface of action, Eq.(26) to the cam surface, and Eq.(31) to the contact line on the roller surface. The geometry of the hyperboloidal roller is also determined by the parameters γ , c and d . The screw pitch of the cam pair and the angular velocity of the roller are determined by the equations given for each case, e.g., Eqs.(108) and (109) for PARR mechanism.

To demonstrate the design equations, the available cam mechanisms are synthesized and the numerical results are shown in Figs. 3-7. In these figures, the pitch surface, surface of action, cam surface, and roller surface are labeled as Σ_p , Σ_a , Σ_1 , and Σ_3 .

7. Conclusion

Based on the concept of instantaneous screw axis, the axode surfaces of roller-follower cam mechanisms

are presented. These axode surfaces include the pitch surface, the surface of action, the cam surface, and the roller surface. The roller has a surface of revolution and the surface becomes a hyperboloid of revolution if the roller makes contact with the cam by a straight line at any instant. According to the geometric characteristics of the hyperboloidal roller, the possible cam mechanisms with hyperboloidal roller in axode contact are provided for their joints incident to the frame restricted to R-pair or P-pair. Here, the conditions of axode contact between the roller and the cam are verified. Among these cam mechanisms, there are five practical cam mechanisms with pure rolling cam pair, and these planar mechanisms with cylindrical roller or spherical mechanisms with conical roller. For the specified roller-follower cam mechanisms in axode contact, the angular velocity of the roller and the screw pitch of the cam pair can be derived.

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