Dynamics of a Pendulum with Feedforward and Feedback Control*

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We consider a pendulum subjected to feedforward and feedback control with periodic desired motions. The pendulum is assumed to be driven by a servo-motor with a small time constant. Here we concentrate on the case in which the desired motion is globally or locally asymptotically stable when its amplitude is zero. Using the averaging method and Melnikov's technique, we show that the desired motion becomes unstable at several bifurcations and chaotic motions occur for the nonzero amplitude case. Numerical simulations are also given and compared with the theoretical predictions for periodic and chaotic motions.

Key Words: Bifurcation, Chaos, Control System, Averaging, Melnikov Method, Stability, Pendulum

1. Introduction

Dynamical systems theory has made great progress and nonlinear behavior such as bifurcations and chaos has been studied in many fields of natural sciences and engineering⁽¹⁾. It was also reported that several nonlinear phenomena occur in some control systems. See Ref.(2) and references therein. In particular, the nonlinear dynamics of a pendulum subjected to linear feedback control with periodic desired motions, which is considered as a simple model for robot manipulators, were theoretically, numerically and experimentally investigated, and it was shown that some bifurcations and chaos occur when sufficient feedback gain is not given and the desired motion is unstable^{(2),(3)}.

In this paper we also consider a similar controlled pendulum but treat a more realistic case from control engineering point of view. First, the pendulum is subjected to not only feedback control but also feedforward control. Second, the feedback gain is sufficiently large so that the center position of the desired motion is globally or at least locally

asymptotically stable when it is controlled to rest there. We theoretically show that the desired motion can become unstable and several bifurcations and chaos occur even in this situation, and demonstrate the theoretical predictions by numerical simulations.

This paper is organized as follows. In section 2 an analytical model for the pendulum with feedforward and feedback control is presented. In section 3 the constant desired motion case is considered. Equilibrium states are obtained and their stability are analyzed. Especially, the stability type of the desired position is described. We discuss the periodic desired motion case in sections 4 and 5. In section 4 the stability of the desired motion and bifurcations of periodic orbits are analyzed by using the second-order averaging method⁽⁴⁾ and center manifold theory^{(5),(6)} when the primary or second-order subharmonic resonance occurs near the desired motion. In section 5 we consider a case in which multiple equilibria exist when the amplitude of the desired motion is zero. It is shown by using Melnikov's method^{(5),(6)} that chaotic dynamics may occur in certain parameter regions. In section 6 numerical simulation results are given and compared with the theoretical results. Finally, we conclude with a summary and some comments in section 7.

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2. An Analytical Model

The control system considered here is shown in Fig. 1. A similar model without feedforward control was previously studied^{(2),(3)}. A physical pendulum with mass m and inertial moment J is driven by a servo-motor with negligible inertia. The position of the pendulum is denoted by θ , and the torque which the servo-motor supplies to the pendulum is denoted by T . The pendulum is also assumed to be subjected to damping force proportional to the velocity, $D\dot{\theta}$, resulting from servo-motor friction and so on. Equation of motion for the pendulum is then given by

$$J\ddot{\theta} + D\dot{\theta} + mgl \sin \theta = T, \quad (1)$$

where g is the gravitational acceleration and l the distance between the rotational axis and the center of gravity of the pendulum. If $\theta \approx 0$ and $D = T = 0$, then the natural angular frequency of Eq. (1) is given by $\omega_n = \sqrt{mgl/J}$.

The electric current I of the servo-motor satisfies

$$L\dot{I} + RI = E - K_0\dot{\theta}, \quad (2)$$

where R , L and K_0 represent the resistance, inductance, and back-electromotive constant of the servo-motor, respectively, and E represents electromotive force provided to the servo-motor. We assume that $\varepsilon = \omega_n L/R \ll 1$. This assumption means that the time constant L/R of the servo-motor is much smaller than the period $2\pi/\omega_n$ for small oscillations of the pendulum. We also have the relationship

$$T = K_T I, \quad (3)$$

where K_T is the torque constant of the servo-motor.

We consider the case in which the desired motion

of the pendulum is given by

$$\theta_d(t) = a_1 \cos \Omega t + a_0.$$

Without loss of generality, we can set $a_1 \geq 0$ and $-\pi < a_0 \leq \pi$. The pendulum is thus desired to swing about $\theta = a_0$ with an amplitude a_1 and angular frequency Ω . We apply feedforward control so that the pendulum can perform the prescribed motion. Using Eq. (1), we see that the required output torque of the servo-motor is given by

$$\begin{aligned} T_0(t) &= J\ddot{\theta}_d(t) + D\dot{\theta}_d(t) + mgl \sin \theta_d(t) \\ &= -\Omega^2 J a_1 \cos \Omega t - \Omega D a_1 \sin \Omega t \\ &\quad + mgl \sin(a_1 \cos \Omega t + a_0), \end{aligned} \quad (5)$$

which we can realize by supplying adequate electromotive force E_0 to the servo-motor (see Eqs. (2) and (3)). Furthermore, we apply feedback control so that electromotive force

$$K_a(\theta_d(t) - \theta) + K'_a(\dot{\theta}_d(t) - \dot{\theta}) \quad (6)$$

is supplied to the servo-motor, where K_a and K'_a are some constants.

Change the time variable $t \rightarrow \omega_n t$ and let

$$x = \theta, y = \dot{\theta}, z = \frac{K_T \Delta I}{mgl} \quad (7)$$

where $\Delta I = I - I_0$ with I_0 the electric current of the servo-motor for electromotive force E_0 . Then Eqs. (1) and (2) are written as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\sin x - \delta_0 y + z + z_0(t), \\ \varepsilon \dot{z} &= -z + \alpha(a_1 \cos \omega t + a_0 - x) \\ &\quad - \delta_1(y - \omega a_1 \sin \omega t), \end{aligned} \quad (8)$$

where

$$\begin{aligned} \alpha &= \frac{K_T K_a}{mglR}, \quad \omega = \frac{\Omega}{\omega_n}, \quad \delta_0 = \frac{D}{\omega_n J}, \\ \delta_1 &= \frac{\omega_n K_T (K_0 + K'_a)}{mglR} \end{aligned} \quad (9)$$

and

$$\begin{aligned} z_0(t) &= \frac{T_0(t)}{mgl} = -\omega^2 a_1 \cos \omega t - \delta_0 \omega a_1 \sin \omega t \\ &\quad + \sin(a_1 \cos \omega t + a_0). \end{aligned} \quad (10)$$

Since $\varepsilon \ll 1$, the singular perturbation theory⁽⁷⁾ implies that solutions of Eq. (8) are well approximated by those of the corresponding slow system for $t = O(1)$. Hence, the pendulum motion is described by a two-dimensional non-autonomous system

$$\dot{x} = y, \quad \dot{y} = -\sin x - \alpha x - \bar{\delta} y + f(t), \quad (11)$$

where $\bar{\delta} = \delta_0 + \delta_1$ and

$$\begin{aligned} f(t) &= (\alpha - \omega^2) a_1 \cos \omega t - \bar{\delta} \omega a_1 \sin \omega t \\ &\quad + \sin(a_1 \cos \omega t + a_0) + \alpha a_0 \end{aligned} \quad (12)$$

(cf. Ref. (2)). Note that α and $\bar{\delta}$ are positive by definition. In the following sections we analyze Eq. (11).

3. The Constant Prescribed Motion

When $a_1 = 0$, i.e., the prescribed motion of the pendulum is constant, $\theta_d(t) \equiv a_0$, Eq. (11) becomes

Fig. 1 The control system. (a) The pendulum; (b) the block diagram

$$\dot{x} = y, \quad \dot{y} = -\sin x - \alpha x - \bar{\delta}y + f_0, \quad (13)$$

where $f_0 = \sin a_0 + \alpha a_0$. The phase portraits and bifurcations in the undamped version of Eq. (13), i.e., the case of $\bar{\delta} = 0$, were previously discussed⁽²⁾. By slightly modifying the arguments given there, we can describe the dynamics of Eq. (13) when $\bar{\delta} \neq 0$.

Let $x = x_0$ be a root of the transcendent equation $-\sin x - \alpha x + f_0 = 0$. (14)

Then $(x, y) = (x_0, 0)$ is a fixed point of Eq. (13). Moreover, the fixed point is a sink if $\cos x_0 + \alpha > 0$, and a hyperbolic saddle if $\cos x_0 + \alpha < 0$. In particular, the desired state $(x, y) = (a_0, 0)$ is a fixed point of Eq. (13), which is stable if $\cos a_0 + \alpha > 0$ and unstable if $\cos a_0 + \alpha < 0$. Any small disturbance prevents the pendulum from remaining at unstable states. Hence, the gain α has to be chosen such that $\cos a_0 + \alpha > 0$ in order that the pendulum can actually rest at the desired position. Using Bendixson's criterion^{(5),(6)}, we also see that Eq. (13) has no periodic orbit in the phase plane. The situation is different for the case of $\alpha = 0$, i.e., when no feedback control applies. See Section 4.6 of Ref. (5).

Suppose that Eq. (14) has no root except $x = a_0$. We easily see that this is true for $\alpha > 1$. Then the left hand side of Eq. (14) is negative for $x > a_0$ and positive for $x < a_0$, and $(a_0, 0)$ is the unique fixed point of Eq. (13). For $\bar{\delta} = 0$, Eq. (13) becomes a Hamiltonian system with Hamiltonian

$$H(x, y) = \frac{1}{2}y^2 - \cos x + \frac{1}{2}\alpha x^2 - f_0x + H_0, \quad (15)$$

where $H_0 = \cos a_0 - \alpha a_0^2/2 - f_0a_0$. Note that $H(a_0, 0) = 0$, $H(x, y) > 0$ for $(x, y) \neq (a_0, 0)$ and $H(x, y)$ is constant along solution curves of Eq. (13) when $\bar{\delta} = 0$. Let

$$V(x, y) = H(x, y) + \beta \left[(x - a_0)y + \frac{1}{2}\bar{\delta}(x - a_0)^2 \right]. \quad (16)$$

Again, $V(a_0, 0) = 0$. Moreover, the derivative of V along solution curves of Eq. (13) is given by

$$\dot{V}(x, y) = \beta(x - a_0)(-\sin x - \alpha x + f_0) - (\bar{\delta} - \beta)y^2. \quad (17)$$

We choose $\beta > 0$ sufficiently small, so that $V(x, y) > 0$ and $\dot{V}(x, y) < 0$ for $(x, y) \neq (a_0, 0)$. Thus, $V(x, y)$ is the Liapunov function^{(8),(9)} for Eq. (13) so that the fixed point $(a_0, 0)$ is globally asymptotically stable. Hence, the pendulum finally rests at the desired position for any initial state. A similar analysis for the pendulum with different feedback control was also performed by Salam⁽¹⁰⁾.

Next, suppose that $\alpha \leq 1$ and let $\bar{x} = \pi - \arccos \alpha$. Then there may exist a root of Eq. (14) except $x = a_0$. We note that the left hand side of Eq. (14) has a local minimum at $x = \bar{x} + 2\pi n (= -\arccos \alpha + (2n+1)\pi)$, $n = 0, \pm 1, \dots$, and a local maximum at $x = -\bar{x} + 2\pi n (= \arccos \alpha + (2n-1)\pi)$, $n = 0, \pm 1, \dots$, where the range of the arccosine function is $[0, \pi]$. For $-\bar{x} < a_0 < \bar{x}$, we

Fig. 2 Stability regions of the desired position

easily see that $x = a_0$ is the unique root of Eq. (14) if and only if

$$-\sqrt{1-\alpha^2} + \alpha[\pi + \arccos \alpha] + \sin a_0 + \alpha a_0 > 0 \quad (18)$$

and

$$\sqrt{1-\alpha^2} - \alpha[\pi + \arccos \alpha] + \sin a_0 + \alpha a_0 < 0. \quad (19)$$

For $-\pi < a_0 < -\bar{x}$ and for $\bar{x} < a_0 < \pi$, Eq. (14) has multiple roots. Hence, if $\alpha \leq 1$, then Eqs. (18) and (19) along with

$$\arccos \alpha - \pi < a_0 < \pi - \arccos \alpha \quad (20)$$

represent a condition for the desired position $(a_0, 0)$ is globally asymptotically stable.

Figure 2 shows the stability regions for the desired state. Thus, by setting the gain sufficiently large, we can control the pendulum to finally rest at the prescribed position for any initial condition. Figure 3 shows numerical simulation results of phase plane motions by a computer software called Dynamics⁽¹¹⁾ for the indicated regions of Fig. 2. The desired state is stable in Figs. 3(a) and (b), and unstable in Fig. 3(c).

4. Dynamics near the Prescribed Motion for Sufficient Gain

We turn to the general case of (11). We choose the gain α so large that the desired state $(a_0, 0)$ in (13) is asymptotically stable. One may then expect that the prescribed motion is also asymptotically stable in (11). However, this is fault in some cases as we will see below.

We consider small perturbations of the prescribed motion. Let

$$\begin{aligned} \xi &= x - \theta_d(t) = x - (a_1 \cos \omega t + a_0), \\ \eta &= y - \dot{\theta}_d(t) = y + \omega a_1 \sin \omega t. \end{aligned} \quad (21)$$

Then Eq. (11) becomes

$$\begin{aligned} \dot{\xi} &= \eta, \\ \dot{\eta} &= -\sin(\xi + a_1 \cos \omega t + a_0) \\ &\quad + \sin(a_1 \cos \omega t + a_0) - \alpha \xi - \bar{\delta} \eta. \end{aligned} \quad (22)$$

Note that Eq. (22) always has a trivial equilibrium $(\xi, \eta) = (0, 0)$, which corresponds to the prescribed motion in Eq. (11).

Fig. 3 Numerically computed orbits of equation (13) for $\bar{\delta}=0.12$ and $a_0=2$. (a) $\alpha=1.2$; (b) $\alpha=0.7$; (c) $\alpha=0.3$. The desired position is asymptotically stable in Figs. (a) and (b) but unstable in Fig. (c)

When $\bar{\delta}=a_1=0$, Eq. (22) has a center at $(\xi, \eta) = (0, 0)$. Near the center there exist periodic orbits, whose frequencies are approximately given by $\omega_0 = \sqrt{\cos a_0 + \alpha}$. Hence, if the ratio of the input frequency ω to ω_0 is close to a rational number, then resonance behavior may occur near the unperturbed center in Eq. (22). It was shown⁽¹²⁾ that the second-order averaging method⁽⁴⁾ is useful to describe such behavior in strongly nonlinear systems like Eq. (22), although it is originally applicable only to weakly nonlinear systems. See also Ref. (3). In the following, we consider the primary and second-order subharmonic resonance cases, $\omega/\omega_0 \approx 1$ and 2, and use the second-order averaging method to analyze the resonance behavior in Eq. (22) and hence Eq. (11). From the results we can describe when and how the prescribed motion becomes unstable and what motions the pendulum

exhibits then. The reader may also think that our scaling of parameters used below is very special. We remark that the scaling is adopted to adequately take into account the effects of the damping, restoring and external forcing terms after carrying out averaging up to second-order.

4.1 Primary resonance

We first discuss the primary resonance. Introduce a small parameter ε such that $0 < \varepsilon \ll 1$, and replace $\bar{\delta}$ and a_1 by $\varepsilon\delta$ and $\sqrt{\varepsilon}\gamma$, respectively. We assume that $\omega^2 - \omega_0^2 = O(\varepsilon)$, and set $\varepsilon\nu = \omega^2 - \omega_0^2$. Let $c_1 = \cos a_0 + \alpha$, $c_2 = -\sin a_0/2$, $c_3 = -\cos a_0/6$ and $\xi = \sqrt{\varepsilon}\zeta$. Then Eq. (22) is rewritten as

$$\begin{aligned} \ddot{\zeta} + \omega_0^2 \zeta + \sqrt{\varepsilon} c_2 (\zeta^2 + 2\gamma\zeta \cos \omega t) \\ + \varepsilon [\delta \dot{\zeta} + c_3 (\zeta^3 + 3\gamma\zeta^2 \cos \omega t \\ + 3\gamma^2 \zeta \cos^2 \omega t)] + O(\varepsilon^{3/2}) = 0. \end{aligned} \quad (23)$$

We use the van der Pol transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \frac{\omega}{k} t & -\frac{k}{\omega} \sin \frac{\omega}{k} t \\ -\sin \frac{\omega}{k} t & -\frac{k}{\omega} \cos \frac{\omega}{k} t \end{pmatrix} \begin{pmatrix} \zeta \\ \dot{\zeta} \end{pmatrix} \quad (24)$$

with $k=1$ and carry out averaging up to second order, so that Eq. (23) becomes

$$\begin{aligned} \dot{u} &= \frac{\varepsilon}{\omega_0} \{ -\delta \omega_0 u + (\nu - A\gamma^2)v \\ &\quad - B[2\gamma uv + (u^2 + v^2)v] \}, \\ \dot{v} &= \frac{\varepsilon}{\omega_0} \{ -(\nu - C\gamma^2)u - \delta \omega_0 v \\ &\quad + B[\gamma(3u^2 + v^2) + (u^2 + v^2)u] \}, \end{aligned} \quad (25)$$

where

$$\begin{aligned} A &= \frac{9c_1 c_3 + 4c_2^2}{12c_1}, \quad B = \frac{9c_1 c_3 - 10c_2^2}{12c_1}, \\ C &= \frac{27c_1 c_3 - 20c_2^2}{12c_1}. \end{aligned} \quad (26)$$

The trivial equilibrium $(u, v) = (0, 0)$ of Eq. (25) corresponds to the prescribed motion.

We now analyze Eq. (25) near the trivial equilibrium using the center manifold theory^{(5),(6)}. We choose γ as a control parameter for bifurcation.

We first linearize Eq. (25) about $(u, v) = (0, 0)$ as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -\delta \omega_0 & \nu - A\gamma^2 \\ -(\nu - C\gamma^2) & -\delta \omega_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \quad (27)$$

If the Jacobian matrix has a zero eigenvalue, i.e.,

$$(\delta \omega_0)^2 + (\nu - C\gamma^2)(\nu - A\gamma^2) = 0, \quad (28)$$

then a saddle-node, transcritical or pitchfork bifurcation^{(5),(6)} occurs in Eq. (25). The center manifold is an invariant set tangent to the eigenspace for the zero eigenvalue at the fixed point. Denote by γ_0 the value of γ such that condition (28) holds. We have

$$\gamma_0 = \sqrt{\frac{(A+C)\nu \pm \sqrt{(A-C)^2\nu^2 - 4AC\delta^2\omega_0^2}}{2AC}}. \quad (29)$$

The other eigenvalue of the Jacobian matrix is $-2\delta\omega_0$.

Under a linear transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \nu - A\gamma_0^2 & \nu - A\gamma_0^2 \\ \delta\omega_0 & -\delta\omega_0 \end{pmatrix} \begin{pmatrix} \rho \\ \sigma \end{pmatrix}, \quad (30)$$

Eq.(25) becomes

$$\dot{\rho} = f(\rho, \sigma, \mu), \quad \dot{\sigma} = -2\delta\omega_0\sigma + g(\rho, \sigma, \mu), \quad (\dot{\mu}=0), \quad (31)$$

where $\mu = \gamma - \gamma_0$ and

$$\begin{aligned} f(\rho, \sigma, \mu) &= f_{200}\rho^2 + f_{110}\rho\sigma + f_{101}\mu\rho + f_{300}\rho^3 + \dots, \\ g(\rho, \sigma, \mu) &= g_{200}\rho^2 + g_{101}\mu\rho + \dots \end{aligned} \quad (32)$$

with

$$\begin{aligned} f_{200} &= \frac{B\gamma_0(3A^2\gamma_0^4 - 6A\gamma_0^2\nu + 3\nu^2 - \delta^2\omega_0^2)}{\delta\omega_0}, \quad f_{110} = 2f_{200}, \\ f_{101} &= \left[\frac{C(\nu - A\gamma_0^2)}{\delta\omega_0} - \frac{A\delta\omega_0}{\nu - A\gamma_0^2} \right] \gamma_0, \\ f_{300} &= \frac{B(A^4\gamma_0^8 - 4A^3\gamma_0^6\nu + 6A^2\gamma_0^4\nu^2 - 4A\gamma_0^2\nu^3 + \nu^4 - \delta^4\omega_0^4)}{2\delta\omega_0(\nu - A\gamma_0^2)}, \\ g_{200} &= -\frac{3B\gamma_0(A^2\gamma_0^4 - 2A\gamma_0^2\nu + \nu^2 + \delta^2\omega_0^2)}{2\delta\omega_0}, \\ g_{101} &= -\frac{A^2C\gamma_0^4 - 2AC\gamma_0^2\nu + C\nu^2 - A\delta^2\omega_0^2}{2\delta\omega_0(\nu - A\gamma_0^2)}. \end{aligned} \quad (33)$$

Here $\rho=0$ and $\sigma=0$ indicate the eigenspaces for the nonzero and zero eigenvalues, respectively. Note that Eq.(31) always has a trivial equilibrium $(\rho, \sigma)=(0, 0)$ since Eq.(25) does.

We can represent the center manifold as the graph of a function of ρ and μ :

$$\sigma = h(\rho, \mu) = h_{20}\rho^2 + h_{11}\rho\mu + h_{02}\mu^2 + \dots, \quad (34)$$

where h_{ij} , $i, j=0, 1, 2, \dots$, are some constants. Substitution of Eq.(34) and the first equation of (31) into the second equation of (31) yields

$$\begin{aligned} \frac{\partial h}{\partial \rho}(\rho, \mu)f(\rho, h(\rho, \mu), \mu) + 2\delta\omega_0h(\rho, \mu) \\ - g(\rho, h(\rho, \mu), \mu) = 0, \end{aligned} \quad (35)$$

from which we obtain

$$h_{20} = -\frac{g_{200}}{2\delta\omega_0}, \quad h_{11} = \frac{g_{101}}{2\delta\omega_0}, \quad h_{02} = 0. \quad (36)$$

Here we also used Eqs.(32) and (34). Finally we substitute Eq.(34) into the first equation of (31) to have

$$\dot{\rho} = \rho[f_{101}\mu + f_{200}\rho + (f_{300} + f_{110}h_{20})\rho^2], \quad (37)$$

by which the behavior of solutions on the center manifold is approximately described. Here we ignored the higher-order terms such as $O(\mu^2)$, $O(\rho\mu)$ and $O(\rho^4)$.

Eq.(37) always has a trivial equilibrium at $\rho=0$, which is stable for $f_{101}\mu < 0$ and unstable for $f_{101}\mu > 0$. Moreover, if

$$f_{200} - 4f_{101}(f_{300} + f_{110}h_{20})\mu > 0, \quad (38)$$

then there are other two fixed points, one of which collides with the trivial equilibrium when $\mu=0$. Thus, we see that if $f_{200}, f_{101}, f_{300} + f_{110}h_{20} \neq 0$, then a saddle-node bifurcation occurs at

$$\mu = \frac{f_{200}}{4f_{101}(f_{300} + f_{110}h_{20})} \quad (39)$$

and a transcritical bifurcation occurs at $\mu=0$. At the

Fig. 4 The behavior of solutions on the center manifold for $f_{101} > 0$ and $f_{300} + f_{110}h_{20} < 0$. (a) $f_{200} < 0$; (b) $f_{200} = 0$; (c) $f_{200} > 0$. The labels sn, tr and pf represent saddle-node, transcritical and pitchfork bifurcations, respectively. Stable and unstable fixed points are shown solid and dotted, respectively

transcritical bifurcation the stability of the trivial equilibrium changes. The saddle-node bifurcation is supercritical or subcritical depending on whether $f_{101}(f_{300} + f_{110}h_{20}) < 0$ or > 0 . If $f_{200} = 0$ but $f_{101}, f_{300} + f_{110}h_{20} \neq 0$, then a pitchfork bifurcation occurs at $\mu = 0$. See Fig. 4.

The above observations mean, via the center manifolds theory^{(5),(6)}, for Eq.(25) that

(i) if $f_{200}, f_{101}, f_{300} + f_{110}h_{20} \neq 0$, then a saddle-node bifurcation occurs near

$$\gamma = \frac{f_{200}}{4f_{101}(f_{300} + f_{110}h_{20})} + \gamma_0, \quad (40)$$

and a transcritical bifurcation occurs near $\gamma = \gamma_0$;

(ii) if another parameter, say ν , can also be varied and $f_{200} = 0$ for $(\gamma, \nu) = (\gamma_0, \nu_0)$, then a pitchfork bifurcation occurs near $\gamma = \gamma_0$ for a value of ν near ν_0 . Moreover, it follows from the averaging theorem^{(4),(5)} that a fixed point (u_0, v_0) of the averaged system (25) corresponds to a harmonic orbit of the original system (11),

$$\begin{aligned} x &= \sqrt{\varepsilon}(u_0 \cos \omega t - v_0 \sin \omega t) + a_1 \cos \omega t + a_0 \\ &+ O(\varepsilon). \end{aligned} \quad (41)$$

In addition, saddle-node, transcritical and pitchfork bifurcations of fixed points in the averaged system correspond to those of harmonic orbits in the original system.

Figure 5 shows the approximate bifurcation sets in the (ω, a_1) -space for the original system (11) when $a_0 = 2$, $\bar{\delta} = 0.12$ and $\alpha = 1.2$ or 0.7 . The curves labeled sn and tr represent saddle-node and transcritical bifurcation sets, respectively, and the point labeled pf represents a pitchfork bifurcation set. These bifurcations are all supercritical if γ is a control parameter. The saddle-node bifurcation curve obtained by using a computer software called AUTO⁽¹³⁾ for the averaged system (25) is also plotted as a thin curve. We see that both results are in good agreement for the saddle-node bifurcations near the primary resonances.

Thus, when the amplitude a_1 is continued to increase, the prescribed motion loses its stability at the transcritical or pitchfork bifurcation. Moreover, even if the prescribed motion is stable, there may

Fig. 5 Approximate bifurcation sets of Eq.(11) near the primary resonance for $\bar{\delta}=0.12$ and $a_0=2$. (a) $\alpha=1.2$; (b) $\alpha=0.7$. See also the caption of Fig. 4 for the labels

exist a stable orbit born at the saddle-node bifurcation.

4.2 Second-order subharmonic resonance

We next discuss the second-order subharmonic resonance, $\omega/\omega_0 \approx 2$. Let $\bar{\delta} = \varepsilon \delta$ and $a_1 = \varepsilon \gamma$, where $0 < \varepsilon \ll 1$. We assume that $\omega^2 - 4\omega_0^2 = O(\varepsilon)$, and set $\varepsilon \nu = \omega^2 - 4\omega_0^2$ and $\xi = \sqrt{\varepsilon} \zeta$. Then Eq.(22) is rewritten as

$$\ddot{\zeta} + \omega_0^2 \zeta = -\sqrt{\varepsilon} c_2 \zeta^2 + \varepsilon [-2c_2 \gamma \zeta \cos \omega t - c_3 \zeta^3 - \delta \zeta] + O(\varepsilon^{3/2}). \quad (42)$$

We use the van der Pol transformation (24) with $k=2$ and carry out averaging up to second order, so that Eq.(42) becomes

$$\begin{aligned} \dot{u} &= \frac{\varepsilon}{2\omega_0} [-\delta \omega_0 u + (\nu + c_2 \gamma) v - B(u^2 + v^2)v], \\ \dot{v} &= \frac{\varepsilon}{2\omega_0} [(-\nu + c_2 \gamma) u - \delta \omega_0 v + B(u^2 + v^2)u], \end{aligned} \quad (43)$$

Dynamics of averaged equations of type (43) and their implications for the associated original systems were described⁽¹²⁾ (see also Ref.(3)). Henceforth we briefly state the results in the context of Eq.(11).

Choose a_1 as a control parameter. For a_1 (i.e., the absolute value of γ) sufficiently small, the prescribed motion is globally asymptotically stable. This fact precisely follows from structural stability⁽⁵⁾ of Eq.(11) at $a_1=0$. As a_1 (i.e., $|\gamma|$) increases, the prescribed motion becomes unstable near

Fig. 6 Bifurcation diagrams predicted by the averaging analysis for Eq.(11). (a) $B\nu < 0$; (b) $B\nu > 0$. The labels pd and pd' represent supercritical and subcritical period-doubling bifurcations of the desired orbit, respectively, and sn represents supercritical saddle-node bifurcations of second-order subharmonics. Stable and unstable orbits are shown solid and dotted, respectively

$$\gamma^2 = \frac{1}{c_2^2} (\nu^2 + \delta^2 \omega_0^2). \quad (44)$$

If $B\nu < 0$, then a stable second-order subharmonic orbit is born near the value of $|\gamma|$ given by Eq.(44): a supercritical period-doubling bifurcation occurs (see Fig.6(a)). On the other hand, if $B\nu > 0$, then a pair of second-order subharmonic orbits, one of which is stable and the other unstable, are created near

$$\gamma^2 = \frac{1}{c_2^2} \delta^2 \omega_0^2, \quad (45)$$

and the unstable subharmonic orbit disappears near equation (44): a supercritical saddle-node bifurcation of second-order subharmonics occurs near equation (45), and a subcritical period-doubling bifurcation occurs near equation (44) (see Fig.6(b)). Eqs. (44) and (45) thus give approximate period-doubling and saddle-node bifurcation sets.

Figure 7 shows the approximate bifurcation curves in the (ω, a_1) -space for $a_0=2$, $\bar{\delta}=0.12$ and $\alpha=1.2$ or 0.7 . The label sn represents supercritical saddle-node bifurcations of second-order subharmonics, and the labels pd and pd', respectively, represent supercritical and subcritical period-doubling bifurcations of the desired orbits. Thus, when the amplitude a_1 is continued to increase, the prescribed motion loses its stability at the period-doubling bifurcation. Moreover, even if the prescribed motion is stable, there may exist a stable second-order subharmonic orbit born at the saddle-node bifurcation.

5. The Existence of Chaos: The Melnikov Analysis

When $\bar{\delta} = a_1 = 0$, if the gain α is not large enough for the fixed point $(a_0, 0)$ to be globally asymptotically stable, then Eq.(11) has multiple equilibria including a (hyperbolic) saddle point. Moreover, the saddle has homoclinic and/or heteroclinic orbits, i.e., its stable and unstable manifolds coincide (see Fig.8(a)). For $\bar{\delta}, a_1 \neq 0$, this structure is probably broken up: Eq.(11) has a hyperbolic periodic orbit near the saddle

Fig. 7 Approximate bifurcation sets of Eq. (11) near the second-order subharmonic resonance for $\bar{\delta}=0.12$ and $a_0=2$. (a) $\alpha=1.2$; (b) $\alpha=0.7$. See also the caption of Fig. 6 for the labels

Fig. 8 Stable and unstable manifolds. (a) $\bar{\delta}=a_1=0$; (b) $\bar{\delta}=a_1 \neq 0$

but its stable and unstable manifolds split and may intersect transversely (see Fig. 8(b)). By the Smale-Birkhoff homoclinic theorem^{(5),(6)}, the existence of such orbits results in chaotic dynamics. Melnikov's

Fig. 9 Approximate homoclinic bifurcation sets of Eq. (11) for $\alpha=0.7$, $\bar{\delta}=0.12$ and $a_0=2$. R^+ and R^- , respectively, represent homoclinic tangencies for the right and left branches of the manifolds

method^{(5),(6)} is an analytical tool to detect the occurrence of this type of chaotic dynamics. We consider the case in which Eq. (11) with $\bar{\delta}=a_1=0$ has a hyperbolic saddle p_0 possessing a pair of homoclinic orbits $q^+(t)$ and $q^-(t)$ and apply Melnikov's method to obtain criteria for the existence of chaos. Again, we set $\bar{\delta}=\varepsilon\delta$ and $a_1=\varepsilon\gamma$.

The Melnikov functions $M^\pm(t)$ for homoclinic orbits $q^\pm(t)=(x^\pm(t), y^\pm(t))$ are given by

$$M^\pm(t_0) = \int_{-\infty}^{\infty} y^\pm(t) [\gamma(\cos a_0 + \alpha - \omega^2) \cos \omega(t+t_0) - \delta y^\pm(t)] dt. \quad (46)$$

Since $y^\pm(t)$ are odd functions of t , Eq. (46) becomes

$$M^\pm(t_0) = -\gamma(\cos a_0 + \alpha - \omega^2) \sin \omega t_0 \times \int_{-\infty}^{\infty} y^\pm(t) \sin \omega t dt - \delta \int_{-\infty}^{\infty} [y^\pm(t)]^2 dt = -\gamma A^\pm(\alpha, \omega, a_0) \sin \omega t_0 - \delta B^\pm(\alpha, a_0), \quad (47)$$

where

$$A^\pm(\alpha, \omega, a_0) = (\cos a_0 + \alpha - \omega^2) \int_{-\infty}^{\infty} y^\pm(t) \sin \omega t dt, \\ B^\pm(\alpha, a_0) = \int_{-\infty}^{\infty} [y^\pm(t)]^2 dt. \quad (48)$$

Note that $y^\pm(t)$ also depend on α and a_0 . Hence, if

$$\frac{\gamma}{\delta} > \left| \frac{B^\pm(\alpha, a_0)}{A^\pm(\alpha, \omega, a_0)} \right| \equiv R^\pm(\alpha, \omega, a_0),$$

then $M^\pm(t)$ has a simple zero and chaotic dynamics may occur in Eq. (11). Moreover, near $\gamma/\delta = R^\pm(\alpha, \omega, a_0)$, the stable and unstable manifolds have a quadratic tangency: a homoclinic bifurcation occurs.

Figure 9 shows the approximate homoclinic bifurcation curves in the (ω, a_1) -space for $\alpha=0.7$, $\bar{\delta}=0.12$ and $a_0=2$. Here the integrals $A^\pm$ and $B^\pm$ were numerically computed using Simpson's rule as in Ref. (2). Chaotic motions may occur in regions above one of these curves. In particular, we see that chaotic motions are hard to occur near the primary resonance of the desired motion, $\omega \approx 0.5328$ (see Eq. (49)).

6. Numerical Simulation

Numerical simulations of Eq.(11) were performed using the computer softwares Dynamics⁽¹¹⁾ and AUTO⁽¹³⁾. The damping coefficient and constant component were fixed as $\bar{\delta}=0.12$ and $a_0=2$ while the amplitude and frequency of the desired motion, a_1 and ω , were varied. For the feedback gain, two cases were considered for $\alpha=1.2$ and 0.7. Note that the prescribed position is globally and locally asymptotically stable for $\alpha=1.2$ and 0.7, respectively, if $a_1=0$.

We first give the results for $\alpha=1.2$. Figure 10 shows bifurcation sets obtained by AUTO. The primary and second-order subharmonic resonance results are displayed in Figs. 10(a) and (b), respectively. We see that the theoretical results of Figs. 5(a) and 7(a) are at least qualitatively in good agreement with the numerical simulation results of Fig. 10 near the resonance points.

Figure 11 shows several examples of "stable" phase plane motions obtained by Dynamics. The unstable prescribed orbit is also plotted as a thin curve in Figs. 11(b) and (d). In Fig. 11(a), besides the prescribed orbit, a stable harmonic orbit, which is born at a saddle-node bifurcation, appears. In Fig. 11(b) the prescribed motion becomes unstable at a transcritical bifurcation and two harmonic orbits born

at the saddle-node bifurcation are stable. In Fig. 11(c), besides the prescribed orbit, a stable second-order subharmonic orbit, which is born at a saddle-

Fig. 11 Numerically computed stable phase plane motions for $\alpha=1.2$, $\bar{\delta}=0.12$ and $a_0=2$. (a) $a_1=0.8$ and $\omega=0.85$; (b) $a_1=1$ and $\omega=0.85$; (c) $a_1=0.27$ and $\omega=1.7$; (d) $a_1=0.3$ and $\omega=1.8$. The unstable prescribed motions are also plotted as thin curves

Fig. 10 Numerically computed bifurcation sets of Eq. (11) for $\alpha=1.2$, $\bar{\delta}=0.12$ and $a_0=2$. (a) Near the primary resonance; (b) near the second-order subharmonic resonance. See also the captions of Figs. 4 and 6 for the labels

Fig. 12 Numerically computed bifurcation sets of Eq. (11) for $\alpha=0.7$, $\bar{\delta}=0.12$ and $a_0=2$. (a) Near the primary resonance; (b) near the second-order subharmonic resonance. See also the captions of Figs. 4 and 6 for the labels

Fig. 13 Numerically computed stable phase plane motions for $\alpha=0.7$, $\bar{\delta}=0.12$ and $a_0=2$. (a) $a_1=0.3$ and $\omega=0.5$; (b) $a_1=0.4$ and $\omega=0.5$; (c) $a_1=0.2$ and $\omega=0.9$; (d) $a_1=0.3$ and $\omega=1.1$; (e) $a_1=0.6$ and $\omega=0.9$. The unstable prescribed motions are also plotted as thin curves

node bifurcation, appears. In Fig. 11(d) the prescribed orbit becomes unstable and a stable second-order subharmonic orbit is born at a supercritical period-doubling bifurcation.

We next give the results for $\alpha=0.7$. Figure 12 shows bifurcation sets obtained by AUTO. Again, the theoretical results of Figs. 5(b) and 7(b) are in good agreement with the numerical simulation results of Fig. 12 near the resonance points. Figure 13 shows several examples of “stable” phase plane motions obtained by Dynamics. The unstable prescribed orbit is also plotted as a thin curve in Figs. 13(b) and (d). In Figs. 13(a)–(d) we observe the same behavior as in Fig. 11(a)–(d). In Fig. 13(e) a chaotic motion occurs as expected by the Melnikov analysis of section 5.

7. Conclusions

In this paper we have studied the nonlinear

dynamics of a pendulum subjected to feedforward and feedback control with periodic desired motions. We concentrated our attention on the case in which the feedback gain is sufficient large so that the desired position is globally or locally stable when its amplitude is zero.

First, the constant desired motion case was discussed, and equilibrium states and their stability, especially the stability type of the desired position, were described. Second, the stability of the desired motion and bifurcations of periodic orbits were analyzed by using the second-order averaging method and center manifold theory when the primary or second-order subharmonic resonance occurs near the desired motion. It was shown that there occur transcritical, pitchfork and period-doubling bifurcations, at which the desired motion loses its stability, and saddle-node bifurcations, at which different stable periodic orbits are born. Third, criteria for the existence of chaos were obtained by using Melnikov’s method, and it was shown that chaotic motions occur in some parameter regions. Finally, the above theoretical results were demonstrated by the corresponding numerical simulation results.

Thus, even if both feedforward and feedback control is applied and the feedback gain is sufficient large, the desired motion can become unstable and different periodic or chaotic motions occur when the amplitude of the desired motion is increased. Note that if it is unstable, the desired motion cannot be performed by the pendulum, to which small disturbance necessarily applies.

It is plausible to think that such situations generally occur in many control systems. So control engineers will also have to pay attention to this fact.

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