Nonlinear Vibrations of a Structure Caused by Water Sloshing in a Cylindrical Tank*

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The nonlinear vibrations of a structure coupled with water sloshing in a circular cylindrical tank are theoretically and experimentally studied. For the purpose of vibration control the liquid tank is installed on the structure subjected horizontally to a sinusoidal excitation. In the theoretical analysis, the only nonlinearity of fluid force due to large sloshing amplitude is considered. The theoretical resonance curve for the harmonic oscillation is presented when the water surface is in a state of planar motion (i.e., when the excitation is small). When the excitation is comparatively large, the resonance curve is obtained by a swept sine test in a numerical simulation. As a result, it is found that the resonance curve changes from a soft spring type to a hard spring type as the water level decreases. In the case of a large excitation, a swirl motion of water surface and an almost periodic oscillation of the structure appear in certain regions of the excitation frequencies. Also, a super-summed-and differential harmonic oscillation can occur at the tuning frequency due to the nonlinearity of the fluid force. These theoretical results agreed quantitatively well with the experimental results.

Key Words: Vibration Coupled with Fluid Motion, Nonlinear Vibration, Sloshing, Cylindrical Tank, Vibration Control Device, Horizontal Excitation, Planar Motion, Swirl Motion, Resonance Curve

1. Introduction

The resonance phenomenon of the liquid surface, which is called sloshing, sometimes causes great damage to structures such as elevated water-tanks and oil tankers. Therefore, sloshing is major problem in the industry. As an aid to vibration control, a liquid container partially filled with liquid has been developed for utilizing the fluid force from the liquid sloshing^{(1),(2)}, and it has been put to practical use in some tower-like structures⁽³⁾⁻⁽⁶⁾. In the following, we abbreviate this damper to TSD (i.e., Tuned Sloshing Damper). However, these reports were limited to

experimental studies. In the case when the magnitude of the motion of the liquid surface becomes great, the effects of any nonlinearities due to the liquid motion should be considered in order to investigate the response of the system in a precise manner.

Up to now, many papers have reported on the nonlinear vibrations of only the liquid surface in rigid and circular cylinders under horizontal excitation. The response curves for the planar motion and the non-planar motion of a free surface have been presented theoretically, and confirmed experimentally by Hutton⁽⁷⁾. Experimental data by Abramson et al.⁽⁸⁾ have been compared with the theoretical results of Hutton⁽⁷⁾. Then, the theoretical response curves of the liquid surface in an axisymmetrical container have been determined and compared with the experimental results in a cylindrical tank by Kimura and Ohashi^{(9),(10)}. The response analysis from a planar motion to a chaotic motion has been discussed by Miles⁽¹¹⁾. As for the experimental studies, the nonlinear behaviors of liquid surfaces have been investigated in detail by Sudo and Hashimoto^{(12),(13)}. In addition, the statistical behaviors under earthquake excitation have been analyzed by Sakata et al.⁽¹⁴⁾ and

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Next, some papers on the vibrations of the structure coupled with sloshing have been published. For example, pioneering work has been done on the linearized theory by Senda and Nakagawa⁽¹⁶⁾. After that, a sloshing-structure interaction with due consideration of the nonlinearities has been studied. The effects of vibration control of the TSD have been investigated experimentally by Fujino et al.⁽¹⁷⁾. A coupled system satisfying the conditions of internal resonance by Ibrahim and Li⁽¹⁸⁾ and the torus shaped container by Welt and Modi⁽¹⁹⁾ have been treated. The interaction of fluid in a spacecraft by Peterson et al.⁽²⁰⁾ and that in a test rig by Barron and Chng⁽²¹⁾ have been investigated. These papers have already shown that not only the planar motion but also the swirl and the non-periodic motions of the liquid surface appear in the case of a circular cylinder under sinusoidal excitation^{(12),(20)}. However, as for the interactions of these complicated motions and the structure, a mathematical model with a quantitatively high accuracy has not been constructed yet. Also, there appear but few papers on the theoretical resonance curves.

In this paper, we treat a structure carrying a cylindrical tank partially filled with water. When the structure is horizontally subjected to a sinusoidal excitation, we theoretically and experimentally investigate the responses of the structure and the liquid surface. In a theoretical analysis, the modal equations are presented taking into account the effect of the only nonlinearity due to the fluid motion. These equations govern the coupled vibrations of the structure and sloshing. In the case when a planar motion of the water surface appears, the theoretical resonance curve for the harmonic oscillation is determined by the harmonic balance method. In the case of a large excitation, the resonance curve is obtained in a numerical simulation by using a swept sinusoidal excitation, whose frequency is slowly changed. Finally, the experimental data were compared with the theoretical results, and the validity of the theory discussed here was confirmed.

2. Nomenclature

- c : damping coefficient
- F_t : horizontal fluid force
- F_0 : amplitude of excitation
- g : acceleration of gravity
- h : water depth
- k : spring constant of cantilever
- M : equivalent mass of structure ($=m+m_t$)
- m : mass of structure
- m_t : mass of water
- P : fluid pressure

- p_0 : natural frequency of structure (or main system)
- p_{mn} : natural frequency of the (m, n) sloshing mode (m, n : integer)
- q_i : natural frequencies of system
- R : tank radius
- (r, θ) : polar coordinate system (see Fig. 1)
- t : time
- (x, y, z) : rectangular coordinate system (see Fig. 1)
- x_1 : horizontal displacement of structure
- α : angular acceleration of excitation
- β : angular position of nodal diameter
- η : displacement of fluid surface
- μ : fluid viscosity
- ν_1 : m/M
- ν_2 : $m_t R / (\pi M h)$
- ρ : fluid density
- ϕ : velocity potential
- ω : excitation frequency
- TSD: Tuned Sloshing Damper

3. Theoretical Analysis

3.1 Equations of motion

Figure 1 shows the theoretical model. A practical machine or building is modeled as a single-degree-of-freedom system, which is composed of a cantilever and a mass m . Here, we call this single DOF system a structure as a generic term. The structure is horizontally subjected to a sinusoidal excitation. A rigid circular tank of a radius R , partially filled with liquid to a depth h , is attached on the mass m .

We consider a moving rectangular coordinate system (x, y, z) and a polar coordinate system (r, θ, z) , which are fixed on the tank. The xy -plane coincides with the undisturbed water surface. Let the displacements of the structure and the water surface at the position (r, θ) be x_1 and η , respectively. In the case when the base motion of the tank is given, i.e.,

Fig. 1 A model for theoretical analysis

when x_1 is known, the equation of motion for the liquid has been derived by the references^{(14),(22)}. Let the velocity potential function of liquid governing the relative motion to the tank be ϕ . From the references mentioned above, Laplace's equation (or the continuity equation) is expressed by

$$\nabla^2 \phi(r, \theta, z, t) \equiv \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (1)$$

and the generalized Bernoulli's equation (or the pressure equation) is given by

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \left\{ \left(\frac{\partial \phi}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right\} + gz + \frac{P}{\rho} - \frac{1}{2} \dot{x}_1^2 + \mu \phi = -\dot{x}_1 r \cos \theta, \quad (2)$$

where P is the pressure of the liquid, ρ is the density, μ is the viscosity, g is the acceleration of the gravity, and t is the time. In Eq. (2) and the following, the symbol $\dot{}$ means the differentiation with respect to time. Equation (2) contains the viscosity of the liquid proportional to the fluid velocity taking account of all kinds of damping effects of the liquid. As long as this viscosity is sufficiently small, the fluid can be assumed to obey a potential flow with a high accuracy.

Next, let the spring constant of the structure be k , and the damping coefficient be c . Under the sinusoidal excitation of an amplitude F_0 and an excitation frequency ω , the equation of motion for the structure is given as follows:

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 = F_t + F_0 \cos \omega t, \quad (3)$$

where F_t is a horizontal fluid force acting from the tank wall to the structure in the direction of x_1 , and it is expressed by

$$F_t = \int_0^{2\pi} \int_{-h}^{\eta_1} P_{r=R} R \cos \theta dz d\theta, \quad (4)$$

where η_1 and $P_{r=R}$ represent the liquid elevation and the hydraulic pressure at the tank wall (i.e., at $r=R$), respectively. As stated in the reference⁽²²⁾, the longitudinal displacement of the mass m due to the lateral deflection is the order of x_1^2/L . L means a length of the cantilever. Actually, the terms, such as $x_1^2 \dot{x}_1/L^2$ and $x_1 \dot{x}_1^2/L^2$, exist in Eq. (3). However, since the orders of these terms are less than the order of x_1^3 when $2R/L < 1$ as treated here, these terms can be neglected in Eq. (3).

Now, we introduce the following dimensionless quantities:

$$\begin{aligned} x_1' &= x_1/R, \quad \eta' = \eta/R, \quad \eta_1' = \eta_1/R, \\ r' &= r/R, \quad z' = z/R, \quad h' = h/R, \\ \nu_1 &= m/M, \quad \nu_2 = m_i/(\pi M h'), \\ \phi' &= \phi/(R^2 p_{11}), \quad k' = k/(M p_{11}^2), \\ c' &= c/(M p_{11}), \quad \mu' = \mu/p_{11}, \\ F_t' &= F_t/(M R p_{11}^2), \quad P' = P/(\rho R^2 p_{11}^2), \\ F_0' &= F_0/(M R p_{11}^2), \quad \omega' = \omega/p_{11}, \\ t' &= p_{11} t, \quad \xi_{mn} = \lambda_{mn} R, \end{aligned}$$

$$[M = m + m_i, p_{11} = \sqrt{g \lambda_{11} \tanh(\lambda_{11} h)}], \quad (5)$$

where the quantities in the bracket have dimensions. m_i means the mass of the liquid. In Eq. (5), λ_{mn} is the n th root among the positive roots satisfying

$$\frac{dJ_m(\lambda r)}{dr} \Big|_{r=R} = 0. \quad (6)$$

In Eq. (6), J_m represents the Bessel function of the first kind and the m th order. Subscripts m and n , as seen in λ_{mn} , mean the numbers of nodal diameters and nodal circles, respectively. We designate the sloshing mode with such nodal lines as (m, n) mode. So p_{11} in Eq. (5) means the natural frequency corresponding to the (1, 1) mode. Using Eq. (5), we obtain the basic equations, which govern the coupled motion of the liquid and the structure, in the dimensionless form as follows:

$$\nabla^2 \phi(r, \theta, z, t) \equiv \phi_{rr} + \frac{1}{r} \phi_r + \frac{1}{r^2} \phi_{\theta\theta} + \phi_{zz} = 0, \quad (7)$$

$$\begin{aligned} \phi_t + \frac{1}{2} \left(\phi_r^2 + \frac{1}{r^2} \phi_\theta^2 + \phi_z^2 \right) + \frac{z}{\xi_{11} \tanh(\xi_{11} h)} \\ + P - \frac{1}{2} \dot{x}_1^2 + \mu \phi = -\dot{x}_1 r \cos \theta, \end{aligned} \quad (8)$$

$$\nu_1 \ddot{x}_1 + c \dot{x}_1 + k x_1 = F_t + F_0 \cos \omega t, \quad (9)$$

$$F_t = \nu_2 \int_0^{2\pi} \int_{-h}^{\eta_1} P_{r=1} \cos \theta dz d\theta. \quad (10)$$

In Eqs. (7)–(10) and the following, note that all of the symbol $\dot{}$ showing dimensionless quantities are omitted. The subscripts r, θ, z and t seen in ϕ mean the partial differentiation with respect to each variable.

The fluid velocity components normal to the wetted walls and bottom of the tank must vanish. Therefore, the boundary conditions are expressed by

$$\phi_r = 0 \quad \text{at } r=1, \quad (11.a)$$

$$\phi_z = 0 \quad \text{at } z=-h. \quad (11.b)$$

Next, the vertical velocity of a fluid particle on the free surface must be the same as the vertical velocity of the free surface. So, the boundary condition is obtained as follows^{(14),(22)}:

$$\frac{\partial \eta}{\partial t} = \phi_z - \phi_r \eta_r - \frac{1}{r^2} \phi_\theta \eta_\theta \Big|_{z=\eta}. \quad (12)$$

The boundary condition for the pressure Eq. (8) with $P=0$ at the free surface is as follows:

$$\begin{aligned} \phi_t + \frac{1}{2} \left(\phi_r^2 + \frac{1}{r^2} \phi_\theta^2 + \phi_z^2 \right) + \frac{z}{\xi_{11} \tanh(\xi_{11} h)} \\ - \frac{1}{2} \dot{x}_1^2 + \mu \phi \Big|_{z=\eta} = -\dot{x}_1 r \cos \theta. \end{aligned} \quad (13)$$

3.2 Modal equations

In order to determine the solutions of forced vibrations for ϕ and η by applying Galerkin's method, we assume these solutions in terms of the eigen functions obtained by a linear analysis as follows:

$$\begin{aligned} \phi(r, \theta, z, t) \\ = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \{ a_{mn}(t) \cos m\theta + b_{mn}(t) \sin m\theta \} \\ \times J_m(\xi_{mn} r) \cosh\{\xi_{mn}(z+h)\} / \cosh(\xi_{mn} h), \end{aligned} \quad (14.a)$$

$$\begin{aligned} \eta(r, \theta, t) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \{c_{mn}(t) \cos m\theta + d_{mn}(t) \sin m\theta\} \\ &\times J_m(\xi_{mn}r). \end{aligned} \quad (14.b)$$

A set of functions $\{J_m(\xi_{mn}r) \cos m\theta, J_m(\xi_{mn}r) \sin m\theta\}$ ($m=0, 1, 2, \dots; n=1, 2, \dots$), as known well, is orthogonal in the intervals of $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Therefore, an arbitrary function of r and θ can be expanded into a Fourier-Bessel series by using these eigen functions. The coefficients a_{mn} , b_{mn} , c_{mn} , and d_{mn} in Eq.(14) are functions of the time. a_{mn} and c_{mn} represent the components of the planar motion of the water surface, and b_{mn} and d_{mn} the components of the non-planar motion. In this paper, since we consider only the predominant components for the wavenumber $n=1$, the second subscripts of a_{mn} , b_{mn} , c_{mn} and d_{mn} will be omitted in the following.

Introducing the small parameter ε , and denoting the order of ε by $O(\varepsilon)$, we assume the orders of the quantities as follows⁽⁷⁾:

$$\begin{aligned} a_1, b_1, c_1, d_1, x_1 &\sim O(\varepsilon^{1/3}), \\ a_2, b_2, c_2, d_2, a_0, c_0; c, \mu &\sim O(\varepsilon^{2/3}), \\ a_3, b_3, c_3, d_3; F_0 &\sim O(\varepsilon^{3/3}). \end{aligned} \quad (15)$$

Here, we expand the free surface conditions (12) and (13) into a Taylor series about $\eta=0$. Then, substituting Eq.(14) into Eqs.(12) and (13), we expand these equations into a Fourier-Bessel series by applying Galerkin's method. Equating the coefficients of $J_0(\xi_{0n}r)$, $J_m(\xi_{mn}r) \cos m\theta$, and $J_m(\xi_{mn}r) \sin m\theta$ ($m=1, 2, \dots; n=1$) on both sides of the equations within the accuracy of $O(\varepsilon)$ by using Eq.(15), we obtain the ordinary differential equations. In addition, substituting the pressure P from Eq.(8) into Eq.(10), and integrating under the assumption (15) leads to the revised equation for Eq.(9) within the accuracy of $O(\varepsilon)$. Therefore, the modal equations for determining a_i , b_j , c_i , d_j ($i=0, 1, 2; j=1, 2$), and x_1 are written in the following abbreviated forms:

$$\begin{aligned} \dot{a}_1 + \mu a_1 + \frac{1}{\phi_{11}} c_1 + S_1 \ddot{x}_1 + G_1(\dot{a}_i, \dot{b}_j, a_i, b_j, c_i, d_j) \\ = 0, \end{aligned} \quad (16.a)$$

$$\begin{aligned} \dot{b}_1 + \mu b_1 + \frac{1}{\phi_{11}} d_1 + G_2(\dot{a}_i, \dot{b}_j, a_i, b_j, c_i, d_j) = 0, \end{aligned} \quad (16.b)$$

$$\dot{c}_1 - \phi_{11} a_1 + G_3(a_i, b_j, c_i, d_j) = 0, \quad (16.c)$$

$$\dot{d}_1 - \phi_{11} b_1 + G_4(a_i, b_j, c_i, d_j) = 0, \quad (16.d)$$

$$\begin{aligned} Q_{35} \ddot{x}_1 + c \dot{x}_1 + k x_1 + Q_{36} \dot{a}_1 + \mu Q_{37} a_1 \\ + G_5(\ddot{x}_1, \dot{x}_1, \dot{a}_i, \dot{b}_j, a_i, b_j, c_i, d_j) = F_0 \cos \omega t, \end{aligned} \quad (16.e)$$

$$\dot{a}_0 + \frac{1}{\phi_{11}} c_0 + G_6(\dot{a}_1, \dot{b}_1, a_1, b_1, c_1, d_1) = 0, \quad (16.f)$$

$$\dot{c}_0 - \phi_{01} a_0 + G_7(a_1, b_1, c_1, d_1) = 0, \quad (16.g)$$

$$\dot{a}_2 + \frac{1}{\phi_{11}} c_2 + G_8(\dot{a}_1, \dot{b}_1, a_1, b_1, c_1, d_1) = 0, \quad (16.h)$$

$$\dot{b}_2 + \frac{1}{\phi_{11}} d_2 + G_9(\dot{a}_1, \dot{b}_1, a_1, b_1, c_1, d_1) = 0, \quad (16.i)$$

$$\dot{c}_2 - \phi_{21} a_2 + G_{10}(a_1, b_1, c_1, d_1) = 0, \quad (16.j)$$

$$\dot{d}_2 - \phi_{21} b_2 + G_{11}(a_1, b_1, c_1, d_1) = 0, \quad (16.k)$$

where $S_1 = 2/\{(\xi_{11}^2 - 1)J_1(\xi_{11})\}$, $\phi_{mn} = \xi_{mn} \tanh(\xi_{mn}h)$, G_k ($k=1 \sim 11$) are nonlinear terms, and Q_l ($l=1 \sim 59$) are constants, as shown in the Appendix. The subscripts in G_k take the values of $i=0, 1, 2$, and $j=1, 2$ (e.g., i and j in a_i and b_j). The modal equations for a_3 , b_3 , c_3 , and d_3 are omitted here, because they are reduced to the equations for free vibrations. Equation (16) represents a set of ordinary differential equations, from which a time history for each variable can be obtained by a digital simulation.

From Eq.(16) without all of the nonlinear, damping and excitation terms, the natural frequencies q_1 and q_2 ($q_1 < q_2$) are given by

$$\begin{aligned} q_1^2, q_2^2 = \frac{1}{2(Q_{35} - S_1 Q_{36})} \left\{ k + Q_{35} \right. \\ \left. \mp \sqrt{(k + Q_{35})^2 - 4k(Q_{35} - S_1 Q_{36})} \right\}. \end{aligned} \quad (17)$$

3.3 Resonance curves

We consider the harmonic oscillation in the case of a planar motion of the water surface. The components of the non-planar motion b_j and d_j ($j=1, 2$) vanish, as shown in Fig. 2. Also, their FFT spectra are shown in Fig. 3. Taking account of the frequency components contained in x_1 , a_i and c_i , we assume the solutions of Eq.(16) for the harmonic oscillation in the same form as that seen in our previous paper⁽²³⁾:

$$\begin{aligned} x_1 &= u_1 \cos \omega t - v_1 \sin \omega t + w_1 \cos 3\omega t - z_1 \sin 3\omega t, \\ a_1 &= u_2 \cos \omega t - v_2 \sin \omega t + w_2 \cos 3\omega t - z_2 \sin 3\omega t, \\ c_1 &= u_3 \cos \omega t - v_3 \sin \omega t + w_3 \cos 3\omega t - z_3 \sin 3\omega t, \\ a_0 &= e_1 \cos 2\omega t - f_1 \sin 2\omega t, \\ a_2 &= e_2 \cos 2\omega t - f_2 \sin 2\omega t, \\ c_0 &= e_3 \cos 2\omega t - f_3 \sin 2\omega t + r_0, \\ c_2 &= e_4 \cos 2\omega t - f_4 \sin 2\omega t + r_2, \end{aligned} \quad (18)$$

where the amplitudes u_i , v_i , w_i , z_i , e_j and f_j ($i=1, 2, 3; j=1, 2, 3, 4$), and the constant terms r_0 and r_2 are the slowly varying functions of time. Their orders are assumed to have the same orders as those of a_i , c_i and x_1 , and to decrease by $O(\varepsilon^{2/3})$ for every additional differentiation with respect to time. Therefore, the order assumption is expressed as follows:

$$\begin{aligned} u_i, v_i &\sim O(\varepsilon^{1/3}), \\ w_i, z_i, e_j, f_j, r_0, r_2 &\sim O(\varepsilon^{2/3}), \\ \dot{u}_i, \dot{v}_i &\sim O(\varepsilon^{3/3}), \\ \dot{w}_i, \dot{z}_i, \dot{e}_j, \dot{f}_j, \dot{r}_0, \dot{r}_2 &\sim O(\varepsilon^{4/3}), \\ \ddot{u}_i, \ddot{v}_i &\sim O(\varepsilon^{5/3}), \\ \ddot{w}_i, \ddot{z}_i, \ddot{e}_j, \ddot{f}_j, \ddot{r}_0, \ddot{r}_2 &\sim O(\varepsilon^{6/3}). \end{aligned} \quad (19)$$

Substituting Eq.(18) into Eq.(16), equating the coefficients of the terms of ω , 2ω and 3ω and the constant terms on both sides of equations within the accuracy of $O(\varepsilon)$ by taking account of Eq.(19), we

obtain the following equations.

$$\begin{aligned}
 \dot{u}_1 &= g_1(u_i, v_i, e_j, f_j, r_k), \quad \dot{v}_1 = g_2(u_i, v_i, e_j, f_j, r_k), \\
 \dot{u}_2 &= g_3(u_i, v_i, e_j, f_j, r_k), \quad \dot{v}_2 = g_4(u_i, v_i, e_j, f_j, r_k), \\
 \dot{u}_3 &= g_5(u_i, v_i, e_j, f_j, r_k), \quad \dot{v}_3 = g_6(u_i, v_i, e_j, f_j, r_k), \\
 w_1 &= g_7(u_i, v_i, e_j, f_j), \quad z_1 = g_8(u_i, v_i, e_j, f_j), \\
 w_2 &= g_9(u_i, v_i, e_j, f_j), \quad z_2 = g_{10}(u_i, v_i, e_j, f_j), \\
 w_3 &= g_{11}(u_i, v_i, e_j, f_j), \quad z_3 = g_{12}(u_i, v_i, e_j, f_j), \\
 e_1 &= g_{13}(u_i, v_i, w_l, z_l), \quad f_1 = g_{14}(u_i, v_i, w_l, z_l), \\
 e_2 &= g_{15}(u_i, v_i, w_l, z_l), \quad f_2 = g_{16}(u_i, v_i, w_l, z_l), \\
 e_3 &= g_{17}(u_i, v_i, w_l, z_l), \quad f_3 = g_{18}(u_i, v_i, w_l, z_l), \\
 e_4 &= g_{19}(u_i, v_i, w_l, z_l), \quad f_4 = g_{20}(u_i, v_i, w_l, z_l), \\
 r_0 &= g_{21}(u_i, v_i), \quad r_2 = g_{22}(u_i, v_i),
 \end{aligned} \quad (20)$$

where $i=1, 2, 3$, $j=1, 2, 3, 4$, $k=0, 2$, and $l=2, 3$. The concrete form of Eq. (20) is omitted here. On the right side of Eq. (20), for example, u_i means u_1 , u_2 and u_3 . From Eq. (20), we can obtain the steady-state solutions of the harmonic oscillations by means of a numerical method, and can judge their stabilities in the same manner as that in the reference⁽²³⁾.

For comparison, we consider the main vibratory system without TSD (i.e., Tuned Sloshing Damper). When the fluid is assumed to be frozen, the amplitude of the structure is given by

$$R = \frac{F_0}{\sqrt{(k - Q_{35}\omega^2)^2 + c^2\omega^2}}, \quad (21)$$

where $Q_{35} = \nu_1 + \nu_2\pi h$, which represents the dimensionless quantity corresponding to the sum of masses $m + m_l$ of the structure and the liquid.

4. Numerical Results

In the following, the numerical calculations are carried out for a tuned condition such that the natural frequency of the structure p_0 is almost equal to the natural frequency p_{11} corresponding to the sloshing mode (1, 1), i.e., for the condition $p_0 (= \sqrt{k/Q_{35}}) = p_{11} (=1)$ expressed in the dimensionless form.

Figure 2 shows the steady-state time histories obtained by a digital simulation of Eq. (16). In the simulation, all the initial values at $t=0$ are selected as zeroes here, because non-planar components b_j and d_j keep vanishing. The values of the parameters are $h=1.2$, $\nu_1=0.92$, $\nu_2=0.02$, $k=1.0$, $c=0.01$, $\mu=0.021$, $F_0=0.0024$, $r=1$, $\theta=0$, and $\omega=1.065$. These values are determined on the basis of the values corresponding to the experimental apparatus as mentioned later. The waveform of η composes of c_0 , c_1 , and c_2 , as known in Eq. (14).

Figure 3 shows the FFT spectra for some waveforms shown in Fig. 2. For example, it can be seen that there are dominant components of frequencies, ω and 3ω in a_1 , 2ω and the constant component in c_0 , and ω in x_1 . These facts give the validity of the assumption of solutions (18).

Fig. 2 Steady-state time histories when $h=1.2$, $\nu_1=0.92$, $\nu_2=0.02$, $k=1.0$, $c=0.01$, $\mu=0.021$, $F_0=0.0024$, $r=1$, $\theta=0$, and $\omega=1.065$

Fig. 3 FFT spectra of the waveforms in Fig. 2

Figures 4(a) and 4(b) show the resonance curves for the component of ω (i.e., $\sqrt{u_1^2 + v_1^2}$) in x_1 , and the component of ω (i.e., $\sqrt{u_3^2 + v_3^2}$) in c_1 , respec-

tively. These curves can be obtained from Eq.(20) by numerical calculations. In this figure, the solid and broken lines represent the stable and unstable solu-

(a) Amplitude of the component of ω in x_1

(b) Amplitude of the component of ω in c_1

Fig. 4 Theoretical resonance curves when the values of the parameters are the same as in the case of Fig. 2 (i.e., for $h=1.2$)

Fig. 5 Theoretical resonance curves of x_1 when the values of parameters in Fig. 4 are changed to $h=0.5$, $\nu_1=0.97$, $\nu_2=0.02$, and $\mu=0.03$

Fig. 6 Theoretical resonance curves of x_1 when the values of parameters in Fig. 4 are changed to $h=0.24$, $\nu_1=0.96$, $\nu_2=0.06$, and $\mu=0.03$

tions, respectively. In addition, the dash-dotted line means the amplitude of the structure without TSD, which can be calculated by Eq.(21). In Fig. 4, the values of the parameters are the same as those in Fig. 2. The shape of this resonance curve is a soft spring type. The symbol $\bullet$ in Fig. 4 to Fig. 6 represents the amplitude of the component of ω obtained by a FFT analysis of the waveform from the digital simulation of Eq.(16). In these simulations, all the initial values are zeroes. The theoretical resonance curves show good agreement with the simulation results.

Figures 5 and 6 show the theoretical resonance curves for water depths different from Fig. 4. The values of the parameters in Fig. 4 are changed to $h=0.5$, $\nu_1=0.97$, $\nu_2=0.02$, and $\mu=0.03$ in Fig. 5, and to $h=0.24$, $\nu_1=0.96$, $\nu_2=0.06$, $\mu=0.03$ in Fig. 6. From these figures, it is found that the shape of the resonance curve changes from a soft spring type to a hard spring type as the water depth decreases. By the way, the peak near $\omega=0.95$ in Fig. 6 is due to the resonance phenomenon, in which the frequency component of 2ω of c_0 in Eq.(18) is almost equal to the natural frequency $p_{01}(=1.907)$ corresponding to the sloshing mode (0, 1).

Figure 7 shows the resonance curve in the swept sine test where the excitation frequency ω of in $F_0 \cos \omega t$ Eq.(16. e) is replaced by $\omega_s + \frac{1}{2}at$. ω_s is the initial value of ω and the angular acceleration $a = \pm 0.0001$ which is sufficiently small. In Fig. 7, ω_s is equal to 0.8, or 1.2. The fine and bold lines mean the envelopes of the maximum amplitudes in the cases of acceleration and deceleration, respectively. Only the value of F_0 is larger, changing from 0.0024 in Fig. 4 to 0.004. The symbol $\bullet$ in Fig. 7 represents the amplitude of the component of ω which is obtained by a numerical simulation of Eq.(16) under purely sinusoidal excitation for $a=0$. All the initial values are 0.01 except for point D (at $\omega=1.12$). It can be seen that, since the value of a is sufficiently small, the response

Fig. 7 Theoretical response curves of x_1 obtained from the simulation by swept sine test when the value F_0 of in Fig. 4 is changed to 0.004

curve agrees well with the simulation results for the sinusoidal excitation. New peaks P_1 and P_2 , which do not appear in the case of the small excitation, can be seen in Fig. 7.

Fig. 8 Steady-state time histories at point C in Fig. 7, i.e., at $\omega=1.08$

(a) Waveforms

(b) The angular position β of a nodal diameter

(c) The rotating speed $\dot{\beta}$ of a nodal diameter

Fig. 9 Waveforms and the direction of a nodal diameter during one cycle in Fig. 8

Figure 8 shows the steady-state time history at point C in Fig. 7, i.e., at $\omega=1.08$. The position for the measurement of η is $(r, \theta)=(1, 0)$. The waveforms appear even in b_1 and d_1 , and their dominant frequency is ω .

To make the phenomenon in Fig. 8 clear, we notice c_1 and d_1 . The liquid elevation η can be expressed approximately as follows:

$$\eta = (c_1 \cos \theta + d_1 \sin \theta) J_1(\xi_{11} r) = \sqrt{c_1^2 + d_1^2} \sin(\theta - \beta) J_1(\xi_{11} r), \quad (22)$$

where $\beta = -\tan^{-1}(c_1/d_1)$ and it represents the direction of the nodal diameter of the free surface. The derivative of β with respect to time is

$$\dot{\beta} = -\frac{\dot{c}_1 d_1 - c_1 \dot{d}_1}{c_1^2 + d_1^2}. \quad (23)$$

When $d_1=0$, i.e., when the liquid surface is in a planar motion, since $\beta=90^\circ$, the nodal diameter coincides with the y -axis.

Figures 9(a)-(c) show the waveforms during one cycle, and the corresponding changes of β and $\dot{\beta}$, respectively. From Fig. 9, it is found that the nodal diameter rotates at the average speed ω and its rotating speed changes twice during one cycle, and that the direction of this swirl motion is counterclockwise in the top view. This reason is that the phase of c_1 is shifted more forward by about 90° than that of d_1 . If the difference of these phases is equal to 90° , we can express that $c_1 = A \cos(\omega t + \pi/2)$ and $d_1 = B \cos \omega t$. Then, Eq. (23) leads to $\dot{\beta} = \omega AB / (A^2 \cos^2 \omega t + B^2 \sin^2 \omega t)$. Therefore, it can be concluded that the sign of $\dot{\beta}$ is positive. However, for example, if only the initial value of b_1 is changed to -0.01 , the clockwise swirl motion can be seen. Such kinds of swirl motions appear near intervals AB and CD in Fig. 7.

Next, we investigate the phenomena near peak P_1 in Fig. 7. Figure 10 is an expanded view of Fig. 7. The symbol $\bullet$ represents the total amplitude of the waveform obtained by numerical simulations. The solid line means the theoretical resonance curve, similar to that in Fig. 4. The data of $\bullet$ in the region AB of $\omega \geq 1.01$ are on the lower side of the resonance curve. The reason is that the amplitudes of the planar

Fig. 10 An expanded view near peak P_1 in Fig. 7

components become smaller due to the appearance of non-planar components. The peak appears near the tuning frequency of $\omega=1.0$.

Figures 11(a) and 11(b) show the steady-state time histories and its FFT spectrum at $\omega=0.99$ in Fig. 10, respectively. The values of the natural frequencies of this system with TSD are $q_1=0.927$ and $q_2=1.097$. From Fig. 11(b) it is seen that the vibratory components of $\omega_1(=0.891)$ and $\omega_2(=1.089)$ appear except for the component ω . These frequencies are close to the natural frequencies, and the relation $\omega_1+\omega_2=2\omega$ also holds. Therefore, it can be concluded that the oscillation shown in Fig. 11(a) is the super-summed-and-differential harmonic oscillation, which is a kind of a nonlinear forced oscillation⁽²⁴⁾. This kind of nonlinear, forced oscillation becomes a serious problem for vibration control of the structure.

Figure 12(a) shows the waveform near peak P_2 in Fig. 7, i.e., at $\omega=1.06$. This waveform is almost periodic in spite of sufficient time progress. The FFT spectrum of x_1 in Fig. 12(a) is shown in Fig. 12(b), where the time interval is from 3023 to 3527. A lot of various components of frequencies appear in both sides of ω . Figure 12(c) shows a Poincaré map on $(x_1, \dot{x}_1)$ plane. The curve consists of many points dotted every one cycle of frequency ω in the case of $t \geq 4267.8$. The representative points move on an identical curve, as shown in Fig. 12(c).

(a) Waveforms

(b) FFT spectrum of x_1

Fig. 11 Steady-state time histories of a super-summed-and-differential harmonic oscillation and its FFT spectrum at $\omega=0.99$ in Fig. 10

5. Experimental Apparatus and Experimental Results

Figure 13 shows a sketch of the experimental apparatus. This apparatus was similar to that detailed in our previous paper⁽²³⁾. A mass m with a

(a) Waveforms

(b) FFT spectrum of x_1

(c) Poincaré map on $(x_1, \dot{x}_1)$ plane

Fig. 12 Steady-state time histories of an almost periodic oscillation and its FFT spectrum near peak P_2 in Fig. 7, i.e., at $\omega=1.06$

Fig. 13 Experimental apparatus

Table 1 Dimensions of apparatuses A, B, C and D

Apparatus	A	B	C	D
Structure Mass m Kg	4.787	5.931	7.028	5.182
Spring Constant k N/m	1641	1606	687	1992
Damping Coefficient c Ns/m	0.983	0.507	0.485	0.911
Tank Radius R m	0.06	0.05	0.085	0.05
Water Depth h m	0.072	0.025	0.020	0.06
Water Mass m_l Kg	0.804	0.197	0.454	0.472
Viscosity μ 1/s	0.346	0.532	0.341	0.391
Force F_0 N	0.209	0.0968	0.162	0.424
Natural frequency p_0 Hz	2.727	2.576	1.525	2.987

(a) Amplitude of x_1 (b) Amplitude of η Fig. 14 Comparison of theoretical and experimental resonance curves in apparatus A when $h/R=1.2$

circular cylindrical tank was mounted at the end of a cantilever with a height of 50.0 mm and a thickness of 3.1 mm. The length of the cantilever, which was slightly different depending on the apparatuses, was about 350 mm. The mass m was horizontally subjected to sinusoidal excitation through a thin aluminium plate by a magnetic exciter. The displacement x_1 of m (i.e., the structure) and the elevation η of the water surface at the position of $r=15$ mm and $\theta=\pi$ were measured by the laser sensors S_1 and S_2 . In these experiments, apparatuses A, B, C and D were used, as shown in Table 1. In Table 1, the equivalent mass of the cantilever was included in the value of m , and the equivalent spring constant of the aluminium plate was contained in the value of k . The value of F_0 was determined from the experimental resonance curve for the main vibratory system by using the least

Fig. 15 Comparison of theoretical and experimental resonance curves of x_1 in apparatus B when $h/R=0.5$ Fig. 16 Comparison of theoretical and experimental resonance curves of x_1 in apparatus C when $h/R=0.235$

squares method and Eq.(21). In the experiments, a water solution with a white watercolor of about 0.28% concentration was used in order that the light of the laser could be reflected on the water surface. Each apparatus was set up in such tuned condition so that the natural frequency of the main system $p_0 = \frac{1}{2\pi} \sqrt{k/M}$ was equal to the natural frequency p_{11} corresponding to the sloshing mode (1,1).

Figures 14(a) and 14(b) show a comparison of the theoretical and experimental resonance curves in apparatus A, respectively. The water depth was 72 mm (i.e., $h/R=1.2$), that is, the highest level among apparatuses A, B, C, and D. The symbols $\circ$ and $\bullet$ represent the amplitudes obtained in the experiments with TSD and without TSD, respectively. The dash-dotted line means the theoretical resonance curve for the main system without TSD, which can be calculated by Eq.(21). The solid and broken lines mean the theoretical resonance curves for the system with TSD, which have the same meanings as explained in Figs. 4 ~6. Arrows $\uparrow$ and $\downarrow$ represent the appearance of jump phenomena. The shape of the resonance curve in apparatus A was a soft spring type.

Figures 15 and 16 show a comparison of the theoretical and experimental resonance curves of x_1 in apparatuses B (i.e., $h/R=0.5$) and C (i.e., $h/R=0.235$),

respectively. In apparatus B, the resonance curve did not incline. In apparatus C, as the radius of the tank was large and the water level was low, the shape of the resonance curve became a hard spring type as shown in Fig. 6. The peak near $f=1.44$ Hz was due to the resonance of the component of frequency $2f$.

By the way, the maximum values of symbols ● near the resonance frequency in Figs. 14, 15, and 16 were about 10.8 mm, 15.2 mm, and 22.9 mm, respectively. So, according to the device of TSD, the amplitudes of the structure became sufficiently small near the tuning frequency. From these figures it was clarified that the shape of the resonance curve tended to change from a soft spring type to a hard spring type as the water depth decreased. And the validity of the theoretical analysis was confirmed, because the theo-

Fig. 17 Experimental resonance curves of x_1 in apparatus D when the excitation was large

(a) Waveforms of x_1 and η at $f=2.961$ Hz

(b) FFT spectrum of x_1

Fig. 18 The waveforms of a super-summed-and-differential harmonic oscillation and its FFT spectrum obtained in apparatus D

retical resonance curves quantitatively coincide with the experimental results.

Figure 17 shows the experimental resonance curve of the structure in apparatus D. In this apparatus, the water level was high, and the magnitude of excitation was comparatively large. All data plotted in various symbols mean the total amplitudes. The symbols ○ and ▲ correspond to the cases of the planar motion and the swirl motion of the water surface, respectively. The symbol ● represents the super-summed-and-differential harmonic oscillation (i.e., SSD), and the symbol ○ shows the almost periodic oscillation (i.e., APO). The theoretical resonance curve in Fig. 7 is calculated by using the values close to those in Fig. 17. Therefore, the shape of the resonance curve in Fig. 17 was quite similar to that in Fig. 7. As shown in Fig. 17, when the excitation was comparatively large, the super-summed-and-differential harmonic oscillation occurred near the tuning frequency. Therefore, in such cases, TSD cannot show sufficient effects.

Figure 18(a) shows the experimental waveforms of x_1 and η at the excitation frequency $f=2.961$ Hz in Fig. 17. The FFT of x_1 is shown in Fig. 18(b). Two peaks of $f_1=2.657$ Hz and $f_2=3.250$ Hz besides f can be seen obviously. The measured values of the natural frequencies in apparatus D were $q_1=2.750$ Hz and $q_2=3.325$ Hz. The relations $f_1 \approx q_1$, $f_2 \approx q_2$ and $f_1+f_2=2f$ hold among these frequencies. Therefore, it was confirmed that this vibration was a super-summed-

(a) Waveforms of x_1 and η at $f=3.199$ Hz

(b) FFT spectrum of x_1

Fig. 19 The waveforms of an almost periodic oscillation and its FFT spectrum obtained in apparatus D

and-differential harmonic oscillation.

Figures 19(a), (b) show the waveforms of x_1 and η at $f=3.199$ Hz in Fig. 17, and its FFT spectrum, respectively. It can be observed that the waveform with a comparatively long period repeats itself, and that the dominant frequencies, such as 2.90, 3.05, 3.20, and 3.35 Hz, appear near the excitation frequency. These experimental results are very similar to those in Figs. 12(a) and 12(b). Therefore, it can be concluded that this vibration was an almost periodic oscillation.

6. Conclusions

The nonlinear vibrations of a structure coupled with sloshing in a circular cylindrical tank have been investigated theoretically and experimentally. The main results are summarized as follows:

(1) The modal equations of motion, which govern the nonlinear coupled vibrations of the structure and the sloshing, can be obtained.

(2) The shape of the resonance curve changes from a soft spring type to a hard spring type as the water depth decreases.

(3) In the case of a small magnitude of excitation, the amplitude of the structure is minimal near the tuning frequency.

(4) In the case of a large excitation, a super-summed-and-differential harmonic oscillation can occur near the tuning frequency. In addition, an almost periodic oscillation of the structure, and a swirl motion of water surface appear in certain regions of the excitation frequencies.

(5) The experimental data agreed well with the theoretical results. Therefore, from the experiments, the reliability of the mathematical model, i.e., the modal equations, and the validity of the theoretical analysis discussed here were confirmed quantitatively.

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Appendix

The nonlinear terms G_k in Eq. (16) are as follows:

$$\begin{aligned} G_1(\dot{a}_i, \dot{b}_j, a_i, b_j, c_i, d_j) &= Q_1(3\dot{a}_1c_1^2 + \dot{a}_1d_1^2 \\ &\quad + 2\dot{b}_1c_1d_1) + Q_2\dot{a}_0c_1 \\ &\quad + Q_3\dot{a}_1c_0 + Q_4(\dot{a}_1c_2 + \dot{b}_1d_2) + Q_5(\dot{a}_2c_1 + \dot{b}_2d_1) \\ &\quad + Q_6a_1^2c_1 + Q_7a_1b_1d_1 + Q_8b_1^2c_1 + Q_9a_0a_1 \\ &\quad + Q_{10}(a_1a_2 + b_1b_2), \\ G_2(\dot{a}_i, \dot{b}_j, a_i, b_j, c_i, d_j) &= Q_{11}(2\dot{a}_1c_1d_1 + \dot{b}_1c_1^2 + 3\dot{b}_1d_1^2) \\ &\quad + Q_{12}\dot{a}_0d_1 + Q_{13}\dot{b}_1c_0 \end{aligned}$$

$$\begin{aligned} &\quad + Q_{14}(\dot{a}_1d_2 - \dot{b}_1c_2) + Q_{15}(\dot{a}_2d_1 - \dot{b}_2c_1) + Q_{16}a_1^2d_1 \\ &\quad + Q_{17}a_1b_1c_1 + Q_{18}b_1^2d_1 + Q_{19}a_0b_1 + Q_{20}(a_1b_2 - a_2b_1), \\ G_3(a_i, b_j, c_i, d_j) &= Q_{21}a_1c_1^2 + Q_{22}b_1c_1d_1 + Q_{23}a_1d_1^2 \\ &\quad + Q_{24}a_0c_1 + Q_{25}a_1c_0 \\ &\quad + Q_{26}(a_1c_2 + b_1d_2) + Q_{27}(a_2c_1 + b_2d_1), \\ G_4(a_i, b_j, c_i, d_j) &= Q_{28}a_1c_1d_1 + Q_{29}b_1c_1^2 + Q_{30}b_1d_1^2 \\ &\quad + Q_{31}a_0d_1 + Q_{32}b_1c_0 \\ &\quad + Q_{33}(a_1d_2 - b_1c_2) + Q_{34}(a_2d_1 - b_2c_1), \\ G_5(\ddot{x}_1, \dot{x}_1, \dot{a}_i, \dot{b}_j, a_i, b_j, c_i, d_j) &= Q_{38}(3\dot{a}_1c_1^2 \\ &\quad + 2\dot{b}_1c_1d_1 + \dot{a}_1d_1^2) + Q_{39}(\dot{a}_0c_1 + \dot{a}_1c_0) \\ &\quad + Q_{40}(\dot{a}_1c_2 + \dot{b}_1d_2 + \dot{a}_2c_1 + \dot{b}_2d_1) + Q_{41}a_1^2c_1 \\ &\quad + Q_{42}a_1b_1d_1 \\ &\quad + Q_{43}b_1^2c_1 + Q_{44}(a_1a_2 + b_1b_2) + Q_{45}a_0a_1 + Q_{46}c_0c_1 \\ &\quad + Q_{47}(c_1c_2 + d_1d_2) + Q_{48}c_1\dot{x}_1^2 + Q_{49}c_0\dot{x}_1 + Q_{50}c_2\dot{x}_1, \\ G_6(\dot{a}_1, \dot{b}_1, a_1, b_1, c_1, d_1) &= Q_{51}(\dot{a}_1c_1 + \dot{b}_1d_1) + Q_{52}(a_1^2 \\ &\quad + b_1^2), \\ G_7(a_1, b_1, c_1, d_1) &= Q_{53}(a_1c_1 + b_1d_1), \\ G_8(\dot{a}_1, \dot{b}_1, a_1, b_1, c_1, d_1) &= Q_{54}(\dot{a}_1c_1 - \dot{b}_1d_1) + Q_{55}(a_1^2 \\ &\quad - b_1^2), \\ G_9(\dot{a}_1, \dot{b}_1, a_1, b_1, c_1, d_1) &= Q_{56}(\dot{a}_1d_1 + \dot{b}_1c_1) + Q_{57}a_1b_1, \\ G_{10}(a_1, b_1, c_1, d_1) &= Q_{58}(a_1c_1 - b_1d_1), \\ G_{11}(a_1, b_1, c_1, d_1) &= Q_{59}(a_1d_1 + b_1c_1). \end{aligned} \quad (A1)$$

where the coefficients Q_i ($i=1\sim 59$) are expressed as

$$\begin{aligned} Q_1 &= \frac{\xi_{11}^2}{8} K_1^{040}, \quad Q_2 = \phi_{01} K_1^{120}, \quad Q_3 = \phi_{11} K_1^{120}, \\ Q_4 &= \frac{\phi_{11}}{2} K_1^{021}, \quad Q_5 = \frac{\phi_{21}}{2} K_1^{021}, \\ Q_6 &= \frac{\phi_{11}}{4} (3\Gamma_1^{1111} + \chi_1^{040} + 3\xi_{11}^2 K_1^{040}), \\ Q_7 &= \frac{\phi_{11}}{2} (\Gamma_1^{1111} - \chi_1^{040} + \xi_{11}^2 K_1^{040}), \\ Q_8 &= \frac{\phi_{11}}{4} (\Gamma_1^{1111} + 3\chi_1^{040} + \xi_{11}^2 K_1^{040}), \\ Q_9 &= \gamma_1^{101} + \phi_{01}\phi_{11} K_1^{120}, \\ Q_{10} &= \frac{1}{2} (\gamma_1^{112} + 2\chi_1^{021} + \phi_{11}\phi_{21} K_1^{021}), \quad Q_{11} = Q_1, \quad Q_{12} = Q_2, \\ Q_{13} &= Q_3, \\ Q_{14} &= Q_4, \quad Q_{15} = -Q_5, \quad Q_{16} = Q_8, \quad Q_{17} = Q_7, \quad Q_{18} = Q_6, \\ Q_{19} &= Q_9, \quad Q_{20} = Q_{10}, \\ Q_{21} &= \frac{\phi_{11}}{8} (6\Gamma_1^{1111} + 2\chi_1^{040} - 3\xi_{11}^2 K_1^{040}), \\ Q_{22} &= \frac{\phi_{11}}{4} (2\Gamma_1^{1111} + 2\chi_1^{040} - \xi_{11}^2 K_1^{040}), \\ Q_{23} &= \frac{\phi_{11}}{8} (2\Gamma_1^{1111} - 2\chi_1^{040} - \xi_{11}^2 K_1^{040}), \\ Q_{24} &= \gamma_1^{101} - \xi_{01}^2 K_1^{120}, \quad Q_{25} = \gamma_1^{101} - \xi_{11}^2 K_1^{120}, \\ Q_{26} &= \frac{1}{2} (\gamma_1^{112} + 2\chi_1^{021} - \xi_{11}^2 K_1^{021}), \\ Q_{27} &= \frac{1}{2} (\gamma_1^{112} + 2\chi_1^{021} - \xi_{21}^2 K_1^{021}), \quad Q_{28} = Q_{22}, \\ Q_{29} &= Q_{23}, \quad Q_{30} = Q_{21}, \quad Q_{31} = Q_{24}, \quad Q_{32} = Q_{25}, \quad Q_{33} = Q_{26}, \\ Q_{34} &= -Q_{27}, \\ Q_{35} &= \nu_1 + \nu_2\pi h, \quad Q_{36} = \frac{\nu_2}{\xi_{11}^2} \pi \phi_{11} B_1, \quad Q_{37} = Q_{36}, \\ Q_{38} &= \frac{\nu_2}{8} \pi \phi_{11} B_1^3, \quad Q_{39} = \nu_2 \pi B_0 B_1, \end{aligned}$$

$$\begin{aligned}
Q_{40} &= \frac{\nu_2}{2} \pi B_1 B_2, \quad Q_{41} = \frac{\nu_2}{8} \pi B_1^3 (1 + 3\phi_{11}^2), \\
Q_{42} &= -\frac{\nu_2}{4} \pi B_1^3 (1 - \phi_{11}^2), \\
Q_{43} &= \frac{\nu_2}{8} \pi B_1^3 (3 + \phi_{11}^2), \\
Q_{44} &= \frac{\nu_2 \pi (\phi_{11} - \phi_{21}) B_1 B_2}{2(\xi_{11}^2 - \xi_{21}^2)} (2 + \xi_{11} \xi_{21}), \\
Q_{45} &= \frac{\nu_2 \pi \xi_{01} \xi_{11} (\phi_{01} - \phi_{11}) B_0 B_1}{\xi_{01}^2 - \xi_{11}^2}, \quad Q_{46} = \frac{Q_{39}}{\phi_{11}}, \quad Q_{47} = \frac{Q_{40}}{\phi_{11}}, \\
Q_{48} &= -\frac{\nu_2}{2} \pi B_1, \\
Q_{49} &= \nu_2 \pi B_0, \quad Q_{50} = \frac{\nu_2}{2} \pi B_2, \quad Q_{51} = \frac{\phi_{11}}{2} K_0^{120}, \\
Q_{52} &= \frac{1}{4} (\gamma_0^{011} + \kappa_0^{120} + \phi_{11}^2 K_0^{120}), \\
Q_{53} &= \frac{1}{2} (\gamma_0^{011} + \kappa_0^{120} - \xi_{11}^2 K_0^{120}), \quad Q_{54} = \frac{\phi_{11}}{2} K_2^{021}, \\
Q_{55} &= \frac{1}{4} (\gamma_2^{211} - \kappa_2^{021} + \phi_{11}^2 K_2^{021}), \\
Q_{56} &= Q_{54}, \quad Q_{57} = 2Q_{55}, \quad Q_{58} = \frac{1}{2} (\gamma_2^{211} - \kappa_2^{021} - \xi_{11}^2 K_2^{021}), \\
Q_{59} &= Q_{58}. \tag{A 2}
\end{aligned}$$

The symbols in Eq. (A 2) are as follows:

$$\begin{aligned}
G_{m1} &= \int_0^1 r J_m^2(\xi_{m1} r) dr, \quad B_m = J_m(\xi_{m1}), \\
S_1 &= \int_0^1 r^2 J_1(\xi_{11} r) dr / G_{11} = \frac{2}{(\xi_{11}^2 - 1) J_1(\xi_{11})}, \\
K_m^{ijk} &= \int_0^1 r J_0^i(\xi_{01} r) J_1^j(\xi_{11} r) J_2^k(\xi_{21} r) dr / G_{m1}, \\
\chi_m^{ijk} &= \int_0^1 \frac{1}{r} J_0^i(\xi_{01} r) J_1^j(\xi_{11} r) J_2^k(\xi_{21} r) dr / G_{m1}, \\
\Gamma_m^{ijkl} &= \int_0^1 r J_i(\xi_{i1} r) J_j(\xi_{j1} r) \frac{d}{dr} [J_k(\xi_{k1} r)] \\
&\quad \times \frac{d}{dr} [J_l(\xi_{l1} r)] dr / G_{m1}, \\
\gamma_m^{ijk} &= \int_0^1 r J_i(\xi_{i1} r) \frac{d}{dr} [J_j(\xi_{j1} r)] \frac{d}{dr} [J_k(\xi_{k1} r)] dr / \\
&\quad G_{m1}, \\
\psi_{mn} &= \xi_{mn} \tanh(\xi_{mn} h). \tag{A 3}
\end{aligned}$$

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