Nonlinear Lateral Vibration of a Vertical Fluid-Conveying Pipe with End Mass*

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Flow-induced vibration of a vertical cantilevered pipe with an end mass, is examined theoretically and experimentally under the supercritical condition that the fluid velocity of an axial flow in the pipe is slightly over its critical value. As the fluid velocity is increased above the critical value of the fluid velocity, i.e., the critical velocity, the planar or nonplanar pipe vibrations are self-excited due to the internal flow. In this paper, the effect of the end mass is discussed with the nonlinear coupled equations governing the nonplanar pipe vibration. Moreover the experiments were conducted with a silicon rubber pipe conveying water, and the spatial behaviors of the pipe were observed by means of the image-processing system.

Key Words: Nonlinear Dynamics, Cantilevered Pipe, Lumped Mass, Flow-Induced Vibration, Nonplanar, Nonlinear Analysis, Experiment

1. Introduction

Flow-induced vibration of a cantilevered fluid-conveying pipe is an easily realizable and physically interesting problem in mechanical systems⁽¹⁾. In particular, nonplanar vibration of a fluid-conveying pipe, which is caused by the nonlinear terms with respect to the lateral deflections of the pipe and has been studied theoretically by Bajaj and Sethna⁽²⁾, Troger and Steindl⁽³⁾ among others, is a most interesting phenomenon from the viewpoint of nonlinear dynamics. Bajaj and Sethna⁽⁴⁾ have also conducted a theoretical study of three-dimensional oscillatory motions of a cantilevered pipe, where small different bending stiffnesses in two mutually perpendicular directions were imposed to break the rotational symmetry, and have confirmed the theoretical results with visual inspection.

Furthermore, it was observed with visual inspection that the addition of an end mass to a cantilevered pipe yielded a rotational symmetric system with many types of nonplanar vibrations⁽⁵⁾. Those nonplanar flow-

induced pipe vibrations are very complicated and include the chaotic vibration with increasing the fluid velocity of an internal flow. Such stability of the nonplanar pipe vibrations is analyzed according to the nonlinear stability theory, and depends sensitively on the mass ratio of an end mass to the total mass. However, the effects of the end mass on the fluid-conveying pipe vibration were investigated by the linear stability theory alone⁽⁶⁾.

In this paper, the nonlinear lateral vibrations of a cantilevered pipe, which is hung vertically with an end mass and conveys fluid, are examined theoretically and experimentally under the supercritical condition that the fluid velocity of the axial flow in the pipe is slightly over its critical value, i.e., the planar and nonplanar pipe vibrations are self-excited due to an internal flow above the critical velocity.

First, the effect of an end mass on the pipe vibration is discussed theoretically with the nonlinear coupled equations of the lateral deflections v and w of the pipe. Second, the experiments were conducted with a silicon rubber pipe conveying water. The spatial behaviors of the fluid-conveying pipe were observed by means of the image-processing system based on the images from two CCD cameras.

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2. Basic Equations

2.1 Equations of motion

The system under consideration (see Fig. 1), consists of a flexible pipe with an end mass M , conveying an incompressible fluid which is discharged into the atmosphere at the free end of the pipe.

The pipe of length l , flexural rigidity EI , mass per unit length m and cross-sectional flow area S , is hung vertically under the influence of gravity g in its equilibrium state. The pipe is long compared to its radius, and its centerline is inextensible. The internal fluid of density ρ is incompressible. The axial fluid velocity V relative to the pipe motion is assumed to be maintained at a constant, as the relaxation time of the unsteady flow is small compared to the period of the flow-induced vibration, under the condition that the frictional force between the fluid and the pipe wall is large⁽⁷⁾.

We use a fixed system of co-ordinates xyz to describe the nonplanar motion of a pipe. The origin of the moving system is taken to coincide with the upper clamped end of the pipe. Let $v(s, t)$ and $w(s, t)$ be the deflections of the pipe centerline in the y and z directions respectively, which are expressed as functions of co-ordinate s along the pipe axis and time t .

The equations governing the spatial behavior of the pipe are then derived under the assumptions that v and w are small but finite, and the pipe has no torsion about its centerline⁽⁸⁾.

Introducing the dimensionless variables which carry an asterisk, i.e., $v = lv^*$, $w = lw^*$, $s = ls^*$, $t = \sqrt{(m + \rho S)l^4/(EI)}t^*$, and retaining terms up to the

Fig. 1 Analytical model

third order of v^* and w^* , the equation governing the pipe motion in the xy plane is expressed in the dimensionless form as follows:

$$\begin{aligned}
 & \ddot{v} + v'''' - \gamma[(\alpha + 1 - s)v']' + V_s^2 v'' + 2\sqrt{\beta} V_s \dot{v}' \\
 &= -v'' \int_s^1 \left[-\frac{1}{2} \frac{\partial^2}{\partial t^2} \left\{ \int_0^s (v'^2 + w'^2) ds \right\} \right. \\
 &+ v' \dot{v} + w' \dot{w} \Big] ds \\
 &- \frac{1}{2} \left[v' \frac{\partial^2}{\partial t^2} \left\{ \int_0^s (v'^2 + w'^2) ds \right\} - v'^2 \dot{v} - \dot{w} \{ v' w' \right. \\
 &- \left. \int_0^s (v' w'' - v'' w') ds \} \right] \\
 &+ v'' \int_s^1 (v'' v''' + w'' w''') ds - \frac{1}{2} \left[v'(v' v'' + w' w'') \right. \\
 &+ w'' \int_0^s (v' w'' - v'' w') ds \Big]'' \\
 &- \frac{1}{2} \gamma v'' \int_s^1 (v'^2 + w'^2) ds - \frac{1}{2} \gamma w' \int_0^s (v' w'' \\
 &- v'' w') ds \\
 &+ \frac{1}{2} (1 - s) \gamma \left[v'(v' v'' + w' w'') + w'' \int_0^s (v' w'' \right. \\
 &- v'' w') ds \Big] \\
 &+ \alpha v'' \left[\frac{1}{2} \frac{\partial^2}{\partial t^2} \left\{ \int_0^1 (v_1'^2 + w_1'^2) ds \right\} - v_1' \dot{v}_1 - w_1' \dot{w}_1 \right] \\
 &+ \frac{1}{2} \alpha \gamma \left[v'(v' v'' + w' w'') + w'' \int_0^s (v' w'' - v'' w') ds \right. \\
 &- \left. v'' (v_1'^2 + w_1'^2) \right] \\
 &- \frac{1}{2} V_s^2 \left[v'(v' v'' + w' w'') + w'' \int_0^s (v' w'' \right. \\
 &- v'' w') ds \Big] \\
 &- \sqrt{\beta} V_s \left[v'(v' \dot{v}' + w' \dot{w}') + \dot{w}' \int_0^s (v' w'' \right. \\
 &- v'' w') ds \Big] \quad (1)
 \end{aligned}$$

where $(\dot{})$ and $()'$ denote the derivatives with respect to t and s , respectively. The asterisks indicating the dimensionless variables are omitted in Eq. (1) and thereafter.

The boundary conditions for both ends of the pipe in xy plane are expressed as follows:

$$\begin{aligned}
 & s = 0 : v = v' = 0 \\
 & s = 1 : v'' = 0 \\
 & v''' - \alpha(\gamma v' + \dot{v}) \\
 &= \frac{1}{2} \alpha \left[v' \frac{\partial^2}{\partial t^2} \left\{ \int_0^1 (v'^2 + w'^2) ds \right\} - \dot{v} v'^2 \right. \\
 &- \dot{w} \{ v' w' - \int_0^1 (v' w'' - v'' w') ds \} \Big] \\
 &- \frac{1}{2} \left[\{ v'(v' v'' + w' w'') \}' \right. \\
 &+ w''' \int_0^1 (v' w'' - v'' w') ds + w'' (v' w'' - v'' w') \Big] \quad (2)
 \end{aligned}$$

The equation governing the pipe motion in the xz plane and its boundary conditions, are expressed by exchanging v for w and w for v in Eqs. (1) and (2).

As a result, the spatial behavior of the pipe is described by two equations and eight boundary conditions with respect to v and w . When α is equal to 0, those equations and boundary conditions are reduced to the equations and boundary conditions governing flow-induced vibration of a cantilevered pipe with no end mass⁽²⁾.

There are four dimensionless parameters involved in Eqs. (1), (2) and their symmetric equations, i.e. the dimensionless velocity $V_s = V/\sqrt{EI/(\rho S l^2)}$, the ratio of the lumped mass to the total mass $\alpha = M/(m + \rho S)l$, the ratio of the fluid mass to the total mass $\beta = \rho S/(m + \rho S)$, the ratio of the gravity force to the elastic force of the pipe $\gamma = (m + \rho S)gl^3/EI$.

2.2 Equations in vector form

By defining

$$\mathbf{v} = {}^t[v \partial v / \partial t \ w \ \partial w / \partial t] \quad (3)$$

The governing equations of $\mathbf{v}$ are expressed in the vector form as follows:

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{L}\mathbf{v} + \mathbf{N}, \quad (4)$$

$$\begin{cases} s=0 : B_1 \mathbf{v} = \mathbf{0} \\ s=1 : B_2 \dot{\mathbf{v}} = B_3 \mathbf{v} - \mathbf{N}_b \end{cases} \quad (5)$$

from two equations and eight boundary conditions with respect to v and w , where

$$\mathbf{L} = \text{diag}(\hat{\mathbf{L}}, \hat{\mathbf{L}}),$$

$$\hat{\mathbf{L}} = \begin{bmatrix} 0 \\ -(\cdot)'''' + \gamma((\alpha + 1 - s)(\cdot)')' - V_s^2(\cdot)'' \\ 1 \\ -2\sqrt{\beta} V_s(\cdot)' \end{bmatrix}$$

$$B_1 = \text{diag}(\hat{B}_1, \hat{B}_1), \quad \hat{B}_1 = \begin{bmatrix} 1 & 0 \\ (\cdot)' & 0 \end{bmatrix}$$

$$B_2 = \text{diag}(\hat{B}_2, \hat{B}_2), \quad \hat{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}$$

$$B_3 = \text{diag}(\hat{B}_3, \hat{B}_3), \quad \hat{B}_3 = \begin{bmatrix} (\cdot)'' & 0 \\ (\cdot)''' - \alpha\gamma(\cdot)' & 0 \end{bmatrix}$$

$$\mathbf{N}_b = \begin{bmatrix} 0 \\ b(v, w) \\ 0 \\ b(w, v) \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 \\ n(v, w) \\ 0 \\ n(w, v) \end{bmatrix}$$

Furthermore $n(v, w)$, $n(w, v)$, $b(v, w)$ and $b(w, v)$ in Eqs. (4) and (5) are expressed as the third order nonlinear polynomials with respect to v and w .

3. Method of Solution

In this section, first, the linear stability of the lateral vibration of the vertical fluid-conveying pipe with an end mass is briefly described in order to determine the flow-induced vibration mode. Second, the nonlinear first-order ordinary differential equations governing the amplitudes and phases of the lateral displacements v and w are derived for the case

of a nonplanar flow-induced vibration of the pipe.

3.1 Linear stability

Neglecting the nonlinear terms with respect to v and w in Eqs. (4) and (5),

$$\bar{\mathbf{v}}_v = {}^t[v \ \partial v / \partial t], \quad \bar{\mathbf{v}}_w = {}^t[w \ \partial w / \partial t] \quad (6)$$

are independent each other, and described by the same equations and the same boundary conditions as follows:

$$\frac{\partial \bar{\mathbf{v}}_i}{\partial t} = \hat{\mathbf{L}} \bar{\mathbf{v}}_i, \quad (i = v, w) \quad (7)$$

$$\begin{cases} s=0 : \hat{B}_1 \bar{\mathbf{v}}_i = \mathbf{0} \\ s=1 : \hat{B}_2 \dot{\bar{\mathbf{v}}}_i = \hat{B}_3 \bar{\mathbf{v}}_i \end{cases} \quad (8)$$

Therefore $\bar{\mathbf{v}}_v$ and $\bar{\mathbf{v}}_w$ have the identical linear eigenvalues and eigenfunctions. We solve the boundary value problem for $\bar{\mathbf{v}}_v$ in this subsection.

Letting $\bar{\mathbf{v}}_v = \hat{\mathbf{q}}_v e^{\lambda_v t}$, $\hat{\mathbf{q}}_v = {}^t[\Phi_{v1}(s), \Phi_{v2}(s)]$ and substituting them into Eqs. (4) and (5), we can cast into the eigenvalue problem

$$\lambda_v \hat{\mathbf{q}}_v = \mathbf{L} \hat{\mathbf{q}}_v \quad (9)$$

$$s=0 : \hat{B}_1 \hat{\mathbf{q}}_v = \mathbf{0}, \quad s=1 : \hat{B}_4 \hat{\mathbf{q}}_v = \mathbf{0} \quad (10)$$

where

$$\hat{B}_4 = \begin{bmatrix} (\cdot)'' & 0 \\ (\cdot)''' - \alpha\gamma & -\alpha\lambda_v \end{bmatrix}$$

The complex eigenvalue λ_v , being the root of the characteristic equation which is symbolically described by

$$f(\lambda_v : V_s, \alpha, \beta, \gamma) = 0 \quad (11)$$

can be found numerically from the condition that Eqs. (9) and (10) has a non-trivial solution $\hat{\mathbf{q}}_v$. The eigenvalue λ_v is equal to $i(\omega_r + i\omega_i)$, where ω_r is the linear eigenfrequency and ω_i corresponds to the damping coefficient.

The component $\Phi_{v1}(s)$ of an eigenvector $\hat{\mathbf{q}}_v$, which is used in the following section, can be expressed in the form of a power series of s , and the component $\Phi_{v2}(s)$ is equal to $\lambda_v \Phi_{v1}(s)$. $\hat{\mathbf{q}}_v$ satisfies the condition $\langle \hat{\mathbf{q}}_v, \hat{\mathbf{q}}_v \rangle = 1$ where brackets denote the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \int_0^1 {}^t \mathbf{x}(s) \overline{{}^t \mathbf{y}(s)} ds$.

Figure 2 shows the critical velocity V_{cr} , above which the lateral pipe vibration is excited by the internal flow, as a function of α . That is, ω_i is equal

Fig. 2 Critical velocity V_{cr} for oscillatory instability

to zero when $V_s = V_{cr}$. The values of β and γ are 0.26 and 20.6 respectively. Those values correspond to the values of the experimental example in Section 5 and are used in the numerical example henceforward. It is found from Fig. 2 that the third mode of the lateral pipe vibration becomes unstable in the region $0 < \alpha < 0.11$, and the second mode becomes unstable in the region $0.11 < \alpha$, slightly above the critical velocity V_{cr} .

Figures 3 shows the evolutions of the complex frequencies $\omega = \omega_r + i\omega_i$ with increasing the V_s for two typical cases, i.e. $\alpha = 0.06$ and 0.12 . The numbers along the curves in Fig. 3 indicate the value of the fluid velocity V_s . In the next subsection, we derive the nonlinear first-order ordinary differential equations governing the amplitudes and phases of v and w under the supercritical condition that V_s is slightly above V_{cr} .

Figure 4 shows the complex eigenfunction Φ_{v1} , which is expressed as follows:

$$\Phi_{v1} = \Phi_r + i\Phi_i$$

where Φ_r and Φ_i are the real and imaginary parts of Φ_{v1} , respectively, and i is the imaginary unit. Φ_r and Φ_i in Fig. 4(a) correspond to the 3rd mode of the lateral pipe vibration for the case of $\alpha = 0.06$ and $V_s = 6.37$. Φ_r and Φ_i in Fig. 4(b) correspond to the 2nd mode of the lateral pipe vibration for the case of $\alpha = 0.12$ and $V_s = 8.35$.

Moreover from the condition:

$$\langle L\hat{q}_v, \hat{q}_v^* \rangle = \langle \hat{q}_v, \hat{L}^* \hat{q}_v^* \rangle \quad (12)$$

(a) $\alpha = 0.06$

(b) $\alpha = 0.12$

Fig. 3 Dimensionless complex frequencies ω as a function of the flow velocity V_s

we get the equations of the adjoint vector $\hat{q}_v^* = {}^t[\Psi_{v1}(s), \Psi_{v2}(s)]$ of $\hat{q}_v$ as follows:

$$\bar{\lambda}_v \hat{q}_v^* = \hat{L}^* \hat{q}_v^* \quad (13)$$

$$s=0: \hat{C}_1 \hat{q}_v^* = \mathbf{0}, \quad s=1: \hat{C}_2 \hat{q}_v^* = \mathbf{0} \quad (14)$$

where

$$\hat{L}^* = \begin{bmatrix} 0 & L_{12}^* \\ 1 & 2\sqrt{\beta} V_s(\cdot)' \end{bmatrix}$$

$$\hat{C}_1 = \begin{bmatrix} 0 & 1 \\ 0 & (\cdot)'' \end{bmatrix}, \quad \hat{C}_2 = \begin{bmatrix} 0 & -(\cdot)'' - V_s^2 \\ 0 & C_{22} \end{bmatrix}$$

and

$$L_{12}^* = -(\cdot)''' + \gamma\{(\alpha+1-s)(\cdot)'\}' - V_s^2(\cdot)''$$

$$C_{22} = (\cdot)''' + (V_s^2 - \alpha\gamma)(\cdot)' - 2\lambda_v\sqrt{\beta} V_s - \alpha\bar{\lambda}_v^2$$

The adjoint vector $\hat{q}_v^*$, which is expressed in the form of a power series of s , satisfies the condition $\langle \hat{q}_v, \hat{q}_v^* \rangle = 1$ and is also used in the following section.

3.2 Nonlinear stability

In this subsection, the equations governing the amplitudes and phases of v and w are derived for the case when the flow velocity V_s is near the critical velocity V_{cr} .

The Banach space, which includes v , is expressed as $Z = X \oplus M^{(1),(2),(9)}$. X is the eigenspace spanned by the eigenvectors

$$q_v = {}^t[\Psi_{v1}(s), \Psi_{v2}(s), 0, 0],$$

$$q_w = {}^t[0, 0, \Psi_{w1}(s), \Psi_{w2}(s)]$$

and their complex conjugate eigenvectors. q_v and q_w correspond to the linear unstable vibration modes of v and w , respectively. M is the subspace of X .

Therefore v are expressed as follows:

$$v(s, t) = a_v(t)q_v(s) + a_w(t)q_w(s) + y + c.c. \quad (15)$$

where y is the element of M and $c.c.$ means the complex conjugate.

Using the projection P_i onto X , Eqs.(4) with boundary conditions (5) are decomposed as follows:

$$P_i \frac{\partial v_i}{\partial t} = P_i L v_i + P_i N_i \quad (i=v, w) \quad (16)$$

where $P_i x = \langle x, q_i^* \rangle q_i$.

From Eqs.(16), the equations of a_i are derived as follows:

(a) $\alpha = 0.06$

(b) $\alpha = 0.12$

Fig. 4 Complex eigenfunction $\Phi_{v1} (= \Phi_r + i\Phi_i)$

$$\dot{a}_i = \lambda_i a_i + (\xi_1 a_i^3 + \xi_2 a_i a_j^2 + \xi_3 |a_i|^2 a_i + \xi_4 |a_j|^2 a_i + \xi_5 \bar{a}_i a_j^2) \quad (17)$$

where $i=v$, $j=w$ and $i=w$, $j=v$. The constant coefficients $\xi_1, \xi_2, \dots, \xi_5$, are depend on α, β, γ and V_s .

Letting $a_v = h_v e^{i\varphi}/2$ and $a_w = h_w e^{i\vartheta}/2$, separating the real and imaginary parts of Eq.(17), and averaging them by the period $2\pi/\omega_r$, we obtain the nonlinear first-order ordinary differential equations which govern the amplitude $h_v(t)$ and the phase $\varphi(t)$ of v , and the amplitude $h_w(t)$ and the phase $\vartheta(t)$ of w .

Furthermore we define the phase difference $\Omega = \vartheta - \varphi$. The equation of Ω is obtained from the equations of φ and ϑ . Finally the autonomous equations governing h_v, h_w and Ω are obtained as follows:

$$\dot{h}_v = -\omega_i h_v + \frac{1}{4}(\xi_{4re} + \xi_{5re})h_v^3 + \frac{1}{4}(\xi_{4re} + \xi_{5re} \cos 2\Omega - \xi_{5im} \sin 2\Omega)h_v h_w^2 \quad (18)$$

$$\dot{h}_w = -\omega_i h_w + \frac{1}{4}(\xi_{4re} + \xi_{5re})h_w^3 + \frac{1}{4}(\xi_{4re} + \xi_{5re} \cos 2\Omega + \xi_{5im} \sin 2\Omega)h_w h_v^2 \quad (19)$$

$$\dot{\Omega} = \frac{1}{4}\{-\xi_{5re} \sin 2\Omega + \xi_{5im}(\cos 2\Omega - 1)\}h_v^2 - \frac{1}{4}\{\xi_{5re} \sin 2\Omega + \xi_{5im}(\cos 2\Omega - 1)\}h_w^2 \quad (20)$$

where $\xi_j = \xi_{jre} + i\xi_{jim}$ ($j=4, 5$).

The equation of $\mathbf{y}$ is obtained by the projection $Q_i = I - P_i$ where I is an unit matrix. However $\mathbf{y}$, which includes the components of the linear stable vibration modes and the high frequency components of $v(s, t)$ and $w(s, t)$, is omitted in this paper. Therefore the pipe deflections v and w are expressed as follows:

$$v = h_v |\Phi_{v1}| \cos(\varphi + \angle \Phi_{v1}) + o(v^3) \quad (21)$$

$$w = h_w |\Phi_{w1}| \cos(\vartheta + \angle \Phi_{w1}) + o(w^3) \quad (22)$$

where $|\Phi_{i1}|$ and $\angle \Phi_{i1}$ in Eqs.(21) and (22) are the magnitudes and arguments of the complex eigenfunctions Φ_{i1} ($i=v, w$), respectively.

4. Theoretical Results

Equations (18) through (20) lead to the following two types of the steady-state solutions ($d/dt=0$).

(i) Planar Flow-induced Vibration of the Pipe:

$$\Omega = 0(\text{or } \pi), h_v = h_w \equiv h_p, \dot{\varphi} = \dot{\vartheta} \equiv \omega_p \quad (23)$$

where

$$h_p = \sqrt{\frac{2\omega_i}{\xi_{4re} + \xi_{5re}}} \quad (24)$$

$$\omega_p = \omega_r + \frac{(\xi_{4im} + \xi_{5im})h_p^2}{2} \quad (25)$$

The frequency ω_p of the planar self-excited vibration of the pipe is obtained from the nonlinear coupled first-order differential equations describing h_v, h_w, φ and ϑ . Moreover the eigenvalues of the equations governing the small disturbances near the steady-state solution (23) are obtained as follows:

$$[\lambda_{11}, \lambda_{12}, \lambda_{13}] = \left[0, 4\omega_i, -\frac{2\xi_{5re}\omega_i}{\xi_{4re} + \xi_{5re}} \right] \quad (26)$$

from Eqs.(18) through (20).

(ii) Nonplanar Flow-induced Vibration of the Pipe:

$$\Omega = \pi/2 (\text{or } 3\pi/2), h_v = h_w \equiv h_{np}, \dot{\varphi} = \dot{\vartheta} \equiv \omega_{np} \quad (27)$$

where

$$h_{np} = \sqrt{\frac{2\omega_i}{\xi_{4re}}} \quad (28)$$

$$\omega_{np} = \omega_r + \frac{\xi_{4im}h_{np}^2}{2} \quad (29)$$

The frequency ω_{np} of the nonplanar self-excited vibration of the pipe is obtained from the nonlinear coupled first-order differential equations describing h_v, h_w, φ and ϑ . Moreover the eigenvalues of the equations governing the small disturbances near the steady-state solution (27) are obtained in the same way as Eq.(26) as follows:

$$[\lambda_{21}, \lambda_{22}, \lambda_{23}] = \left[2\omega_i, \frac{2\omega_i(\xi_{5re} + i\xi_{5im})}{\xi_{4re}}, \frac{2\omega_i(\xi_{5re} - i\xi_{5im})}{\xi_{4re}} \right] \quad (30)$$

where i is the imaginary unit.

These results correspond to those already obtained by Bajaj and Sethna⁽²⁾, and Troger and Steindl⁽³⁾ for the case of the fluid-conveying pipe with no end mass.

The equations (18) through (20) facilitate the study of many kinds of the nonlinear lateral vibrations of a vertical fluid-conveying pipe with the end mass. This paper focuses on the effect of the end mass on the steady-state flow-induced vibration of the pipe, under the supercritical condition that the flow velocity V_s is hereafter slightly over the critical value V_{cr} , i.e. $\omega_i < 0$.

Figure 5(a) shows the transient time histories of h_v, h_w and Ω , for the case of $\alpha=0.06$, $V_s=8.35$. This example is obtained numerically from Eqs.(18) through (20), where the initial values of h_v, h_w and Ω are $1 \times 10^{-3}, 1 \times 10^{-3}$ and $\pi \times 10^{-5}$, respectively. h_v and h_w become constant after a while, and their value is 0.022 which is equal to the analytical values obtained from Eqs.(24). Ω also becomes constant and its value is 1.7×10^{-120} , which is almost zero as indicated analytically in Eq.(23). Figure 5(b) shows the steady-state planar pipe motion in a horizontal plane at $s=0.3$. The direction in the horizontal plane depends on the initial conditions of h_v, h_w and Ω .

The amplitude of the lateral deflection $w(0.3, t)$ and the frequency ω_p of the stable steady-state planar vibration, which are shown as a function of V_s in Fig. 6, are calculated numerically from Eqs.(22), (24) and (25) for the case of $\alpha=0.06$. The stability of the steady-state planar pipe vibration is examined by

means of Eqs.(26). Those results show that the amplitude of $w(0.3, t)$ increases with increasing V_s , and the effect of α on $w(0.3, t)$ does not depend on the value of the fluid velocity V_s as shown in Fig. 6(a), and ω_p is almost the same as ω_r because h_p is very

(a) Transient time histories of h_v , h_w and Ω

(b) Pipe motion in a horizontal plane at $s=0.3$

Fig. 5 Transient time histories of h_v , h_w and Ω and the steady-state pipe motion in a horizontal plane ($\alpha=0.06$, $V_s=8.35$, Theory)

(a) Amplitude of the lateral deflection $w(0.3, t)$

(b) Frequencies ω_p and ω_r

Fig. 6 Amplitude of the lateral deflection $w(0.3, t)$, frequencies ω_p and ω_r of the stable steady-state planar vibration ($\alpha=0.06$, Theory)

small compared with unity in Eq.(25).

Figure 7(a) shows the transient time histories of h_v , h_w and Ω , for the case of $\alpha=0.12$, $V_s=6.37$. This example is obtained numerically from Eqs.(18) through (20), where the initial values of h_v , h_w and Ω are 1×10^{-3} , 1×10^{-3} and $\pi \times 10^{-5}$, respectively, i.e., the same initial values for the former case. h_v and h_w become constant after a while, and their value is 0.0174, which is equal to the analytical values obtained from Eqs. (28). Ω also becomes constant and its value is 1.57, which is almost equal to $\pi/2$ as indicated analytically in Eq.(27). Figure 7(b) shows the steady-state nonplanar pipe motion in a horizontal plane at $s=0.3$.

The amplitude of the lateral deflection $w(0.3, t)$ and the frequency ω_{np} of the stable steady-state nonplanar vibration, which are shown as a function of V_s in Fig. 8, are calculated numerically from Eqs. (22), (28) and (29) for the case of $\alpha=0.12$. The stability of the steady-state nonplanar pipe vibration is examined using Eqs.(30). Those results show that the amplitude of $w(0.3, t)$ increases with increasing V_s , and the effect of α on $w(0.3, t)$ does not depend on the value of the fluid velocity V_s as shown in Fig. 8(a), and ω_{np} is almost the same as ω_r because h_{np} is very small compared with unity in Eq.(29).

Therefore, we examined the effect of α on the steady-state amplitudes h_p and h_{np} for the constant damping ratio $\omega_i=-0.1$. Figure 16(a), shows that

(a) Transient time histories of h_v , h_w and Ω

(b) Pipe motion in a horizontal plane at $s=0.3$

Fig. 7 Transient time histories of h_v , h_w and Ω and the steady-state pipe motion in a horizontal plane ($\alpha=0.12$, $V_s=6.37$, Theory)

(a) Amplitude of the lateral deflection $w(0.3, t)$ (b) Frequencies ω_{np} and ω_r

Fig. 8 Amplitude of the lateral deflection $w(0.3, t)$, frequencies ω_{np} and ω_r of the stable steady-state nonplanar vibration ($\alpha=0.12$, Theory)

the nonplanar pipe vibration is stable for $0.11 < \alpha < 0.14$, and the planar pipe vibration is stable for $0 < \alpha < 0.11$ and $0.14 < \alpha < 0.25$. In Fig. 16(a), the solid and broken lines correspond to the stable and unstable steady-state amplitudes, respectively.

5. Experiments

5.1 Experimental apparatus

The experimental setup is shown in Fig. 9. The experiments were conducted with the silicone rubber pipe of 12 (mm) external diameter, 7 (mm) internal diameter and 518 (mm) length. The total mass of the pipe is 58.5 (g), and the natural frequency of the first vibration mode is 1.02 (Hz) at $V=0$. The conveyed fluid is water. As a result, the values of β and γ were determined experimentally as 0.26 and 20.6, respectively. The values of α are 0, 0.06, 0.12, 0.16 and 0.25.

The spatial displacements of the flexible pipe were measured by the image-processing system that could perform measurements of the marker in three-dimensional space, based on images from two CCD cameras (OKK Inc., Quick MAG System).

5.2 Experimental results

Figure 10 shows the stroboscopic photographs of the planar pipe vibration, which is taken twice a period, for the case of $\alpha=0.06$ and $V_s=7.45$ [7, 2 (m/sec)]. Figures 10(a) and 10(b) correspond to the real and imaginary parts of the 3rd eigenfunction of the pipe vibration as shown in Fig. 4(a), respectively.

Fig. 9 Experimental setup

Fig. 10 Stroboscopic photographs of the planar pipe vibration ($\alpha=0.06$, $V_s=7.45$)

Figure 11 shows the time histories of v and w , their spectrum analyses and the pipe motions in a horizontal yz plane at $s=0.3$ [156 (mm)]. The planar pipe vibration, which remained almost stationary and had

Fig. 11 Time histories of v and w , their spectrum analyses and the pipe motion in a horizontal plane at $s=0.3$ ($\alpha=0.06$, $V_s=7.45$, Experiment)

the single mode, was observed in a yz plane as predicted in the theory.

The dimensionless amplitude of the pipe vibration at $s=0.3$ is shown as a function of the fluid velocity V_s in Fig. 12(a) for the same case. The experimental critical velocity $V_{cr}=7.25$ is small compared with the theoretical one, i.e. 8.15. We suppose that this is mainly caused by neglecting the damping effects of the pipe vibration in the theory. As the fluid velocity V_s increases, the experimental value of the pipe deflection becomes large qualitatively in the same manner as the theoretical result as shown in Fig. 6(a). Figure 12(b) indicates that the experimental frequency of the flow-induced pipe vibration is almost constant with increasing V_s , in the same manner as the theoretical result as shown in Fig. 6(b). However, the experimental frequency is small compared with the theoretical ones as shown in Fig. 6(b). We suppose that this is mainly caused by the difference between the experimental and theoretical values of the critical fluid velocity.

Figure 13 shows the stroboscopic photographs of the nonplanar pipe vibration, which is taken once a period, for the case of $\alpha=0.12$ and $V_s=6.22$. Figures 13(a) and 13(b) correspond to the real and imaginary parts of the 2nd eigenfunction of the pipe vibration as

Fig. 12 Amplitude of the lateral deflection $w(0.3, t)$, frequencies ω_p of the stable steady-state planar vibration ($\alpha=0.06$, Experiment)

Fig. 13 Stroboscopic photographs of the nonplanar pipe vibration ($\alpha=0.12$, $V_s=6.22$)

shown in Fig. 4(b), respectively. Figures 14 shows the time histories of v and w , their spectrum analyses and the pipe motions in a horizontal yz plane at $s=0.3$. The steady-state nonplanar pipe vibration with the single mode, was observed in the $y-z$ plane as predicted in the theory.

The dimensionless amplitude of the pipe vibration at $s=0.3$ as a function of the fluid velocity V_s is shown

Fig. 14 Time histories of v and w , their spectrum analyses and the pipe motion in a horizontal plane at $s=0.3$ ($\alpha=0.12$, $V_s=6.22$, Experiment)

in Fig. 15(a) for the same case. The experimental critical velocity is $V_{cr}=6.00$ is slightly small compared with the theoretical one, i.e. 6.15. As the fluid velocity V_s increases, the experimental value of the pipe deflection becomes qualitatively large in the same manner as the theoretical result as shown in Fig. 8 (a). Figure 15(b) indicates that the experimental frequency of the flow-induced pipe vibration is almost constant with increasing V_s , in the same manner as the theoretical result as shown in Fig. 8(b).

Figure 16 shows the effect of the end mass on the lateral pipe vibration theoretically and experimentally. The planar pipe vibrations were observed for $\alpha=0, 0.06$ and 0.09 . The nonplanar pipe vibrations were observed for $\alpha=0.12$ and 0.16 . The planar pipe vibration was observed again for $\alpha=0.24$. Those experimental results are in qualitative agreement with the theoretical ones.

6. Conclusions

The effect of the mass ratio of the end mass to the total mass of the pipe α on a lateral flow-induced vibration of a cantilevered pipe, has been studied from the viewpoint of nonlinear dynamics.

First, the nonlinear first-order ordinary differential equations governing the amplitudes and

(a) Amplitude of the lateral deflection $w(0.3, t)$

(b) Frequency ω_{np}

Fig. 15 Amplitude of the lateral deflection $w(0.3, t)$, frequencies ω_{np} of the stable steady-state nonplanar vibration ($\alpha=0.12$, Experiment)

(a) Theoretical result

(b) Experimental result

Fig. 16 Stabilities of nonlinear lateral vibrations of a fluid-conveying pipe with an end mass

phases of v and w , which are the lateral deflections of the pipe perpendicular to each other, have been derived from the nonlinear coupled integro-partial differential equations for v and w . By means of the derived equations, the effect of the mass ratio α on the stability of the flow-induced pipe vibration has been

examined theoretically for the typical example of the fluid-conveying pipe.

Second, the spacial behaviors of a silicon rubber pipe conveying water were observed quantitatively by means of the image-processing system based on images from two CCD cameras. As predicted in the theory, the nonplanar pipe vibration has been observed for some finite range of the mass ratio α , for the case that the planar flow-induced vibration of the pipe without the end mass, i.e., $\alpha=0$.

Furthermore, the flow-induced lateral vibration of a cantilevered pipe with an end mass becomes very complicated with increasing the fluid velocity V_s , i.e. the bifurcation phenomena and the chaotic vibration. At present, it is very difficult to analyze such flow-induced vibrations. In this paper, we have analyzed the flow-induced vibration under the supercritical condition that the fluid velocity in the pipe is slightly above its critical value, i.e., the same condition as Bajaj and Sethna (1984).

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