Nonlinear Lateral Vibration of a Vertical Fluid-Conveying Pipe with End Mass*

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Flow-induced vibration of a vertical cantilevered pipe with an end mass, is examined theoretically and experimentally under the supercritical condition that the fluid velocity of an axial flow in the pipe is slightly over its critical value. As the fluid velocity is increased above the critical value of the fluid velocity, i.e., the critical velocity, the planar or nonplanar pipe vibrations are self-excited due to the internal flow. In this paper, the effect of the end mass is discussed with the nonlinear coupled equations governing the nonplanar pipe vibration. Moreover the experiments were conducted with a silicon rubber pipe conveying water, and the spatial behaviors of the pipe were observed by means of the image-processing system.

**Key Words**: Nonlinear Dynamics, Cantilevered Pipe, Lumped Mass, Flow-Induced Vibration, Nonplanar, Nonlinear Analysis, Experiment

1. Introduction

Flow-induced vibration of a cantilevered fluid-conveying pipe is an easily realizable and physically interesting problem in mechanical systems\(^{(1)}\). In particular, nonplanar vibration of a fluid-conveying pipe, which is caused by the nonlinear terms with respect to the lateral deflections of the pipe and has been studied theoretically by Bajaj and Sethna\(^{(2)}\), Troger and Steindlid\(^{(3)}\) among others, is a most interesting phenomenon from the viewpoint of nonlinear dynamics. Bajaj and Sethna\(^{(4)}\) have also conducted a theoretical study of three-dimensional oscillatory motions of a cantilevered pipe, where small different bending stiffnesses in two mutually perpendicular directions were imposed to break the rotational symmetry, and have confirmed the theoretical results with visual inspection.

Furthermore, it was observed with visual inspection that the addition of an end mass to a cantilevered pipe yielded a rotational symmetric system with many types of nonplanar vibrations\(^{(5)}\). Those nonplanar flow-induced pipe vibrations are very complicated and include the chaotic vibration with increasing the fluid velocity of an internal flow. Such stability of the nonplanar pipe vibrations is analyzed according to the nonlinear stability theory, and depends sensitively on the mass ratio of an end mass to the total mass. However, the effects of the end mass on the fluid-conveying pipe vibration were investigated by the linear stability theory alone\(^{(6)}\).

In this paper, the nonlinear lateral vibrations of a cantilevered pipe, which is hung vertically with an end mass and conveys fluid, are examined theoretically and experimentally under the supercritical condition that the fluid velocity of the axial flow in the pipe is slightly over its critical value, i.e., the planar and nonplanar pipe vibrations are self-excited due to an internal flow above the critical velocity.

First, the effect of an end mass on the pipe vibration is discussed theoretically with the nonlinear coupled equations of the lateral deflections \(v\) and \(w\) of the pipe. Second, the experiments were conducted with a silicon rubber pipe conveying water. The spatial behaviors of the fluid-conveying pipe were observed by means of the image-processing system based on the images from two CCD cameras.
2. Basic Equations

2.1 Equations of motion

The system under consideration (see Fig. 1), consists of a flexible pipe with an end mass \( M \), conveying an incompressible fluid which is discharged into the atmosphere at the free end of the pipe.

The pipe of length \( l \), flexural rigidity \( EI \), mass per unit length \( m \) and cross-sectional flow area \( S \), is hung vertically under the influence of gravity \( g \) in its equilibrium state. The pipe is long compared to its radius, and its centerline is inextensible. The internal fluid of density \( \rho \) is incompressible. The axial fluid velocity \( V \) relative to the pipe motion is assumed to be maintained at a constant, as the relaxation time of the unsteady flow is small compared to the period of the flow-induced vibration, under the condition that the frictional force between the fluid and the pipe wall is large.

We use a fixed system of co-ordinates \( xyz \) to describe the nonplanar motion of a pipe. The origin of the moving system is taken to coincide with the upper clamped end of the pipe. Let \( v(s, t) \) and \( w(s, t) \) be the deflections of the pipe centerline in the \( y \) and \( z \) directions respectively, which are expressed as functions of co-ordinate \( s \) along the pipe axis and time \( t \).

The equations governing the spatial behavior of the pipe are then derived under the assumptions that \( v \) and \( w \) are small but finite, and the pipe has no torsion about its centerline.

Introducing the dimensionless variables which carry an asterisk, i.e., \( v = lv^* \), \( w = lw^* \), \( s = ls^* \), \( t = \sqrt{(m+\rho S)} t^*/(E) t^* \), and retaining terms up to the third order of \( v^* \) and \( w^* \), the equation governing the pipe motion in the \( xy \) plane is expressed in the dimensionless form as follows:

\[
\begin{align*}
\frac{\partial v^*}{\partial t} + v^* w^* - \gamma (lv^*) \frac{\partial v^*}{\partial s} + V^2 v^* + 2\sqrt{\beta} V^2 v^* + \frac{1}{2} \left[ v^* \frac{\partial^2 v^*}{\partial t^2} \right] \\
+ \frac{1}{2} \left[ v^* \frac{\partial^2 v^*}{\partial s^2} \right] - \frac{1}{2} \left[ v^* \frac{\partial^2 v^*}{\partial s \partial t} \right] + \frac{1}{2} \left[ v^* \frac{\partial^2 v^*}{\partial s^2} \right] = 0
\end{align*}
\]

where \( \gamma \) and \( \gamma' \) denote the derivatives with respect to \( t \) and \( s \), respectively. The asterisks indicating the dimensionless variables are omitted in Eq. (1) and thereafter.

The boundary conditions for both ends of the pipe in \( xy \) plane are expressed as follows:

\[
\begin{align*}
\text{at } s = 0: & \quad v = v' = 0 \\
\text{at } s = 1: & \quad v'' = 0 \\
\gamma' & = \frac{1}{2} \left[ v^* \frac{\partial^2 v^*}{\partial s^2} \right] - \frac{1}{2} \left[ v^* \frac{\partial^2 v^*}{\partial s \partial t} \right] \\
\gamma'' & = \frac{1}{2} \left[ v^* \frac{\partial^2 v^*}{\partial s^2} \right] - \frac{1}{2} \left[ v^* \frac{\partial^2 v^*}{\partial s \partial t} \right] \\
\end{align*}
\]

The equation governing the pipe motion in the \( zx \) plane and its boundary conditions, are expressed by exchanging \( v \) for \( w \) and \( w \) for \( v \) in Eqs. (1) and (2).
As a result, the spatial behavior of the pipe is described by two equations and eight boundary conditions with respect to \( v \) and \( w \). When \( a \) is equal to 0, those equations and boundary conditions are reduced to the equations and boundary conditions governing flow-induced vibration of a cantilevered pipe with no end mass.

There are four dimensionless parameters involved in Eqs. (1), (2) and their symmetric equations, i.e. the dimensionless velocity \( V_s = \nu/\sqrt{EI/(\rho S^2)} \), the ratio of the lumped mass to the total mass \( \alpha = M/(m + \rho S) \), the ratio of the fluid mass to the total mass \( \beta = \rho S/(m + \rho S) \), the ratio of the gravity force to the elastic force of the pipe \( \gamma = (m + \rho S)g/\nu EI \).

2.2 Equations in vector form

By defining

\[
\mathbf{v} = [v \ \partial v/\partial t \ \ w \ \partial w/\partial t]
\]

(3)

The governing equations of \( \mathbf{v} \) are expressed in the vector form as follows:

\[
\frac{\partial \mathbf{v}}{\partial t} = L \mathbf{v} + N_s
\]

(4)

\[
s = 0 : B_s \mathbf{v} = \mathbf{m}, \quad s = 1 : B_s \mathbf{v} = \mathbf{B}_s \mathbf{v} - \mathbf{N}_s
\]

(5)

from two equations and eight boundary conditions with respect to \( v \) and \( w \), where

\[
L = \text{diag} (L, L),
\]

\[
\begin{bmatrix}
\lambda (s+1) & \mathbf{V} (s) \\
-2\sqrt{\beta} \mathbf{V} (s) & 1
\end{bmatrix}
\]

\[
B_s = \text{diag} (\bar{B}_s, \bar{B}_s), \quad \bar{B}_s = \begin{bmatrix}
1 & 0 \\
(\lambda)^{s+1} & 0
\end{bmatrix}
\]

\[
B_\infty = \text{diag} (\bar{B}_\infty, \bar{B}_\infty), \quad \bar{B}_\infty = \begin{bmatrix}
0 & \alpha \\
0 & 0
\end{bmatrix}
\]

\[
B_\infty = \text{diag} (\bar{B}_\infty, \bar{B}_\infty), \quad \bar{B}_\infty = \begin{bmatrix}
(\lambda)^{s+1} & 0 \\
0 & 0
\end{bmatrix}
\]

\[
N_s = \begin{bmatrix}
b(v, w) \\
b(w, v)
\end{bmatrix}, \quad N = \begin{bmatrix}
n(v, w) \\
n(w, v)
\end{bmatrix}
\]

Furthermore \( n(v, w) \), \( n(w, v) \), \( b(v, w) \) and \( b(w, v) \) in Eqs. (4) and (5) are expressed as the third order nonlinear polynomials with respect to \( v \) and \( w \).

3. Method of Solution

In this section, first, the linear stability of the lateral vibration of the vertical fluid-conveying pipe with an end mass is briefly described in order to determine the flow-induced vibration mode. Second, the nonlinear first-order ordinary differential equations governing the amplitudes and phases of the lateral displacements \( v \) and \( w \) are derived for the case of a nonplanar flow-induced vibration of the pipe.

3.1 Linear stability

Neglecting the nonlinear terms with respect to \( v \) and \( w \) in Eqs. (4) and (5),

\[
\mathbf{v}_s = \mathbf{v}, \quad \frac{\partial \mathbf{v}}{\partial t} = \mathbf{v}, \quad \frac{\partial w}{\partial t} = \mathbf{w}
\]

(6)

are independent each other, and described by the same equations and the same boundary conditions as follows:

\[
\frac{\partial \mathbf{v}_s}{\partial t} = \mathbf{L} \mathbf{v}_s, \quad (i = v, w)
\]

(7)

\[
s = 0 : \bar{B}_s \mathbf{v}_s = \mathbf{m}, \quad s = 1 : \bar{B}_s \mathbf{v}_s = \mathbf{B}_s \mathbf{v}_s
\]

(8)

Therefore \( \mathbf{v}_v \) and \( \mathbf{v}_w \) have the identical eigenvalues and eigenvectors. We solve the boundary value problem for \( \mathbf{v}_s \) in this subsection.

Letting \( \mathbf{v}_s = \mathbf{q}_s \mathbf{e}^{\omega t}, \quad \mathbf{q}_s = [\mathbf{q}_s (s), \mathbf{q}_s (s)] \) and substituting them into Eqs. (4) and (5), we can cast into the eigenvector problem

\[
\lambda \mathbf{q}_s = \mathbf{L} \mathbf{q}_s
\]

(9)

\[
s = 0 : \bar{B}_s \mathbf{q}_s = \mathbf{m}, \quad s = 1 : \bar{B}_s \mathbf{q}_s = \mathbf{B}_s \mathbf{q}_s
\]

(10)

where

\[
\mathbf{B}_s = \begin{bmatrix}
(\cdot)^s & 0 \\
(\cdot)^{s-1} & \cdot^s & -a \cdot^s, & -w \cdot^s
\end{bmatrix}
\]

The complex eigenvalue \( \lambda_\omega \), being the root of the characteristic equation which is symbolically described by

\[
f(\lambda_\omega : V_s, \alpha, \beta, \gamma) = 0
\]

(11)

can be found numerically from the condition that Eqs. (9) and (10) has a non-trivial solution \( \mathbf{q}_s \). The eigenvalue \( \omega_\cdot \) is equal to \( \iota (\omega_\cdot + i \omega_\cdot) \), where \( \omega_\cdot \) is the linear eigenfrequency and \( \omega_\cdot \) corresponds to the damping coefficient.

The component \( \Phi (s) \) of an eigenvector \( \mathbf{q}_s \), which is used in the following section, can be expressed in the form of a power series of \( s \), and the component \( \Phi (s) \) is equal to \( \lambda_\omega \mathbf{q}_s (s) \). \( \mathbf{q}_s \) satisfies the condition \( \mathbf{q}_s (s) = \mathbf{B}_s \mathbf{q}_s (s) = 0 \) where brackets denote the inner product \( \langle x, y \rangle = \int_0^1 x (s) y (s) ds \).

Figure 2 shows the critical velocity \( V_c \), above which the lateral pipe vibration is excited by the internal flow, as a function of \( a \). That is, \( \omega_\cdot \) is equal
to zero when $V_e = V_c$. The values of $\beta$ and $\gamma$ are 0.26 and 20.6 respectively. Those values correspond to the values of the experimental example in Section 5 and are used in the numerical example henceforward. It is found from Fig. 2 that the third mode of the lateral pipe vibration becomes unstable in the region $0 < \alpha < 0.11$, and the second mode becomes unstable in the region $0.11 < \alpha$, slightly above the critical velocity $V_c$.

Figures 3 shows the evolutions of the complex frequencies $\omega = \omega_r + i\omega_i$ with increasing the $V_e$ for two typical cases, i.e., $\alpha = 0.06$ and 0.12. The numbers along the curves in Fig. 3 indicate the value of the fluid velocity $V_e$. In the next subsection, we derive the nonlinear first-order ordinary differential equations governing the amplitudes and phases of $v$ and $w$ under the supercritical condition that $V_e$ is slightly above $V_c$.

Figure 4 shows the complex eigenfunction $\Phi_e$, which is expressed as follows:

$$\Phi_e^{(3)} = \Phi_r^{(3)} + i\Phi_i^{(3)}$$

where $\Phi_r$ and $\Phi_i$ are the real and imaginary parts of $\Phi_e$, respectively, and $i$ is the imaginary unit. $\Phi_r$ and $\Phi_i$ in Fig. 4(a) correspond to the 3rd mode of the lateral pipe vibration for the case of $\alpha = 0.06$ and $V_e = 6.37$. $\Phi_r$ and $\Phi_i$ in Fig. 4(b) correspond to the 2nd mode of the lateral pipe vibration for the case of $\alpha = 0.12$ and $V_e = 8.35$.

Moreover from the condition:

$$\langle L^{*} \tilde{\Phi}, \tilde{\Phi} \rangle = \langle \tilde{\Phi}, L^{*} \tilde{\Phi} \rangle$$

we get the equations of the adjoint vector $\tilde{\Phi}^{*}$ of $\Phi_e$ as follows:

$$\frac{d}{ds} \tilde{\Phi}^{*} = L^{*} \tilde{\Phi}^{*}$$

where

$$s = 0 : \tilde{\Phi}_v^{*} = 0, \quad s = 1 : \tilde{\Phi}_w^{*} = 0$$

and

$$\bar{L} = \begin{bmatrix} 0 & L^*_1 \\ 1 & 2\sqrt{\beta} V_e(s) \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 0 & 1 \\ 0 & (-\gamma) \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & (\gamma + 1) \end{bmatrix}$$

and

$$L^*_1 = (\gamma + 1)(V_e (s) - \alpha) - \beta \gamma V_e (s) - \alpha \beta$$

$$C = (\gamma + 1)(V_e (s) - \alpha) - \beta \gamma V_e (s) - \alpha \beta$$

The adjoint vector $\tilde{\Phi}^{*}$, which is expressed in the form of a power series of $s$, satisfies the condition $\langle \tilde{\Phi}, \tilde{\Phi}^{*} \rangle = 1$ and is also used in the following section.

3.2 Nonlinear stability

In this subsection, the equations governing the amplitudes and phases of $v$ and $w$ are derived for the case when the flow velocity $V_e$ is near the critical velocity $V_c$.

The Banach space, which includes $v$, is expressed as $Z = \mathbf{X} \oplus \mathbf{M}^{(3,2,1)}$. $\mathbf{X}$ is the eigenspace spanned by the eigenvectors:

$$q_e = [\mathcal{W}_e(s), \mathcal{W}_w(s), 0, 0]$$

and their complex conjugate eigenvectors. $q_e$ and $q_w$ correspond to the linear unstable vibration modes of $v$ and $w$, respectively. $\mathbf{M}$ is the subspace of $\mathbf{X}$.

Therefore $v$ and $w$ are expressed as follows:

$$v(s,t) = a_v(s)q_v(s) + a_w(s)q_w(s) + y + c.c.$$  (15)

where $y$ is the element of $\mathbf{M}$ and $c.c.$ means the complex conjugate.

Using the projection $P_v$ onto $\mathbf{X}$, Eqs. (4) with boundary conditions (5) are decomposed as follows:

$$P_v \frac{\partial \mathbf{x}}{\partial s} = P_v L_v \mathbf{v} + P_v N_v \quad (i = v, w)$$  (16)

From Eqs. (16), the equations of $a_i$ are derived as follows:

$$a_i = \langle x, q_i^* \rangle q_i$$

Fig. 3  Dimensionless complex frequencies $\omega$ as a function of the flow velocity $V_e$

Fig. 4  Complex eigenfunction $\Phi_{ei}$ (= $\Phi_r + i\Phi_i$)

\[ a_i = \lambda a_i + (\xi a_i + \eta a_i a_i + \zeta a_i) a_i \]
\[ + \zeta_i a_i a_i a_i - \zeta_i a_i a_i \]  
(17)  
where \( i = v, j = w \) and \( i = w, j = v \). The constant coefficients \( \xi, \eta, \zeta, \zeta_2, \ldots, \zeta_n \) depend on \( a, \beta, \gamma \) and \( V_s \).  

Letting \( a_0 = h_0 e^{i\omega t/2} \) and \( a_w = h_w e^{i\omega t/2} \), separating the real and imaginary parts of Eq.(17), and averaging them by the period \( 2\pi/\omega \), we obtain the nonlinear first-order ordinary differential equations which govern the amplitude \( h_i(t) \) and the phase \( \varphi(t) \) of \( v \), and the amplitude \( h_v(t) \) and the phase \( \delta(t) \) of \( w \).  

Furthermore, we define the phase difference \( Q = \delta - \varphi \). The equation of \( Q \) is obtained from the equations of \( \varphi \) and \( \delta \). Finally, the autonomous equations governing \( h_v, h_w \) and \( Q \) are obtained as follows:

\[ \dot{h}_v = -\omega h_v + \frac{1}{4} (\xi_{21} + \xi_{22}) h_w^2 \]

\[ + \frac{1}{4} (\xi_{21} + \xi_{22}) \cos 2\Omega - \xi_{21} \sin 2\Omega) h_v h_w \]

\[ \dot{h}_w = -\omega h_w + \frac{1}{4} (\xi_{21} + \xi_{22}) h_v^2 \]

\[ + \frac{1}{4} (\xi_{21} + \xi_{22}) \cos 2\Omega - \xi_{21} \sin 2\Omega) h_v h_w \]

\[ \dot{Q} = \frac{1}{4} (\xi_{21} \sin 2\Omega + \xi_{22} \cos 2\Omega - 1) h_v \]

\[ - \frac{1}{4} (\xi_{21} \sin 2\Omega + \xi_{22} \cos 2\Omega - 1) h_w \]

where \( \xi_{21} = \xi_{12} = \xi_{21} \) (j=4,5).  

The equation of \( y \) is obtained by the projection \( Q_i = I - \hat{P} \), where \( I \) is an unit matrix. However \( y \), which includes the components of the linear stable vibration modes and the high frequency components of \( v(s, t) \) and \( w(s, t) \), is omitted in this paper. Therefore the pipe deflections \( v \) and \( w \) are expressed as follows:

\[ v = h_v \Phi(v) \cos(\varphi + \Phi(v)) + o(v^3) \]

\[ w = h_w \Phi(w) \cos(\gamma + \Phi(w)) + o(w^3) \]

where \( \Phi(v) \) and \( \Phi(w) \) in Eqs.(21) and (22) are the magnitudes and arguments of the complex eigenfunctions \( \Phi_{i1}(i = v, w) \), respectively.  

4. Theoretical Results  

Equations (18) through (20) lead to the following two types of the steady-state solutions \( (d/dt = 0) \).  

(i) Planar Flow-induced Vibration of the Pipe:  

\[ Q = 0 (or \pi) , h_v = h_w = h_p, \phi = \delta = \omega_p \]

where

\[ h_p = \sqrt{\frac{2\omega_p}{\xi_{21} + \xi_{22}}} \]

\[ \omega_p = \omega_r + \frac{\xi_{22} + \xi_{22}}{2} \]

The frequency \( \omega_p \) of the planar self-excited vibration of the pipe is obtained from the nonlinear coupled first-order differential equations describing \( h_v, h_w, \varphi \) and \( \delta \). Moreover the eigenvalues of the equations governing the small disturbances near the steady-state solution (23) are obtained as follows:

\[ [\lambda_{11}, \lambda_{12}, \lambda_{13}] = \left[ 0, 4 \omega_r - \frac{2 \xi_{21} \omega_r}{\xi_{21} + \xi_{22}}, \xi_{22} + \xi_{22} \right] \]

from Eqs.(18) through (20).  

(ii) Nonplanar Flow-induced Vibration of the Pipe:  

\[ Q = \pi/2 (or 3\pi/2) , h_v = h_w = h_p, \phi = \delta = \omega_p \]

where

\[ h_p = \sqrt{\frac{2\omega_r}{\xi_{21}}} \]

\[ \omega_p = \omega_r + \frac{\xi_{22}}{2} \]

The frequency \( \omega_p \) of the nonplanar self-excited vibration of the pipe is obtained from the nonlinear coupled first-order differential equations describing \( h_v, h_w, \varphi \) and \( \delta \). Moreover the eigenvalues of the equations governing the small disturbances near the steady-state solution (27) are obtained in the same way as Eq.(26) as follows:

\[ [\lambda_{11}, \lambda_{12}, \lambda_{13}] = \left[ 2 \omega_r, 2 \omega_r (\xi_{21} + \xi_{22}), \xi_{22} + \xi_{22} \right] \]

where \( i = 4,5 \).  

These results correspond to those already obtained by Bajaj and Sethna(19), and Troger and Steindl(20) for the case of the fluid-conveying pipe with no end mass.  

The equations (18) through (20) facilitate the study of many kinds of the nonlinear lateral vibrations of a vertical fluid-conveying pipe with the end mass. This paper focuses on the effect of the end mass on the steady-state flow-induced vibration of the pipe, under the supercritical condition that the flow velocity \( V_s \) is hereafter slightly over the critical value \( V_{rt} \), i.e. \( \omega_r < 0 \).  

Figure 5 (a) shows the transient time histories of \( h_v, h_w \) and \( Q \), for the case of \( \alpha = 0.06, V_s = 8.35 \). This example is obtained numerically from Eqs.(18) through (20), where the initial values of \( h_v, h_w \) and \( Q \) are \( 1 \times 10^{-3}, 1 \times 10^{-3} \) and \( 2 \times 10^{-3} \), respectively. \( h_v \) and \( h_w \) become constant after a while, and their value is 0.022 which is equal to the analytical values obtained from Eqs.(24). \( Q \) also becomes constant and its value is \( 1.7 \times 10^{-3} \), which is almost zero as indicated analytically in Eq.(23). Figure 5 (b) shows the steady-state planar pipe motion in a horizontal plane at \( s = 0.3 \). The direction in the horizontal plane depends on the initial conditions of \( h_v, h_w \) and \( Q \).  

The amplitude of the lateral deflection \( w(0.3, t) \) and the frequency \( \omega_p \) of the stable steady-state planar vibration, which are shown as a function of \( V_s \) in Fig. 6, are calculated numerically from Eqs.(22), (24) and (25) for the case of \( \alpha = 0.06 \). The stability of the steady-state planar pipe vibration is examined by
means of Eqs.(26). Those results show that the amplitude of $w(0.3,t)$ increases with increasing $V_s$, and the effect of $\alpha$ on $w(0.3,t)$ does not depend on the value of the fluid velocity $V_s$ as shown in Fig. 6(a), and $\omega_p$ is almost the same as $\omega_r$ because $h_p$ is very small compared with unity in Eq.(25).

Figure 7(a) shows the transient time histories of $h_v$, $h_w$ and $\Omega$, for the case of $\alpha=0.12$, $V_i=6.37$. This example is obtained numerically from Eqs.(18) through (20), where the initial values of $h_v$, $h_w$ and $\Omega$ are $1 \times 10^{-3}$, $1 \times 10^{-3}$ and $\pi \times 10^{-5}$, respectively, i.e., the same initial values for the former case. $h_v$ and $h_w$ become constant after a while, and their value is 0.0174, which is equal to the analytical values obtained from Eqs. (28). $\Omega$ also becomes constant and its value is 1.57, which is almost equal to $\pi/2$ as indicated analytically in Eq.(27). Figure 7(b) shows the steady-state nonplanar pipe motion in a horizontal plane at $s=0.3$.

The amplitude of the lateral deflection $w(0.3,t)$ and the frequency $\omega_{np}$ of the stable steady-state nonplanar vibration, which are shown as a function of $V_s$ in Fig.8, are calculated numerically from Eqs. (22), (28) and (29) for the case of $\alpha=0.12$. The stability of the steady-state nonplanar pipe vibration is examined using Eqs.(30). Those results show that the amplitude of $w(0.3,t)$ increases with increasing $V_s$, and the effect of $\alpha$ on $w(0.3,t)$ does not depend on the value of the fluid velocity $V_s$ as shown in Fig.8 (a), and $\omega_{np}$ is almost the same as $\omega_r$ because $h_{np}$ is very small compared with unity in Eq.(29).

Therefore, we examined the effect of $\alpha$ on the steady-state amplitudes $h_p$ and $h_{np}$ for the constant damping ratio $\omega_r=-0.1$. Figure 16(a), shows that...
the nonplanar pipe vibration is stable for \(0.11 < a < 0.14\), and the planar pipe vibration is stable for \(0 < a < 0.11\) and \(0.14 < a < 0.25\). In Fig. 16(a), the solid and broken lines correspond to the stable and unstable steady-state amplitudes, respectively.

5. Experiments

5.1 Experimental apparatus

The experimental setup is shown in Fig. 9. The experiments were conducted with the silicone rubber pipe of 12 (mm) external diameter, 7 (mm) internal diameter and 518 (mm) length. The total mass of the pipe is 58.5 (g), and the natural frequency of the first vibration mode is 1.02 (Hz) at \(V = 0\). The conveyed fluid is water. As a result, the values of \(\beta\) and \(\gamma\) were determined experimentally as 0.26 and 20.6, respectively. The values of \(a\) are 0.06, 0.12, 0.16 and 0.25.

The spatial displacements of the flexible pipe were measured by the image processing system that could perform measurements of the marker in three-dimensional space, based on images from two CCD cameras (OKK Inc., Quick MAG System).

5.2 Experimental results

Figure 10 shows the stroboscopic photographs of the planar pipe vibration, which is taken twice a period, for the case of \(a = 0.06\) and \(V_z = 7.45\) [7, 2 (m/ sec)]. Figures 10(a) and 10(b) correspond to the real and imaginary parts of the 3rd eigenfunction of the pipe vibration as shown in Fig. 4(a), respectively.

Fig. 10 Stroboscopic photographs of the planar pipe vibration \((a = 0.06, V_z = 7.45)\)

Figure 11 shows the time histories of \(v\) and \(w\), their spectrum analyses and the pipe motions in a horizontal \(yz\) plane at \(s = 0.3\) [156 (mm)]. The planar pipe vibration, which remained almost stationary and had
Fig. 11 Time histories of \( v \) and \( w \), their spectrum analyses and the pipe motion in a horizontal plane at \( s=0.3 \) (\( \alpha=0.06 \), \( V_s=7.45 \), Experiment)

The single mode, was observed in a \( yz \) plane as predicted in the theory.

The dimensionless amplitude of the pipe vibration at \( s=0.3 \) is shown as a function of the fluid velocity \( V_s \) in Fig. 12(a) for the same case. The experimental critical velocity \( V_c=7.25 \) is small compared with the theoretical one, i.e. 8.15. We suppose that this is mainly caused by neglecting the damping effects of the pipe vibration in the theory. As the fluid velocity \( V_s \) increases, the experimental value of the pipe deflection becomes large qualitatively in the same manner as the theoretical result as shown in Fig. 6(a). Figure 12(b) indicates that the experimental frequency of the flow-induced pipe vibration is almost constant with increasing \( V_s \), in the same manner as the theoretical result as shown in Fig. 6(b). However, the experimental frequency is small compared with the theoretical ones as shown in Fig. 6(b). We suppose that this is mainly caused by the difference between the experimental and theoretical values of the critical fluid velocity.

Figure 13 shows the stroboscopic photographs of the nonplanar pipe vibration, which is taken once a period, for the case of \( \alpha=0.12 \) and \( V_s=6.22 \). Figures 13(a) and 13(b) correspond to the real and imaginary parts of the 2nd eigenfunction of the pipe vibration as shown in Fig. 4(b), respectively. Figures 14 shows the time histories of \( v \) and \( w \), their spectrum analyses and the pipe motions in a horizontal \( yz \) plane at \( s=0.3 \). The steady-state nonplanar pipe vibration with the single mode, was observed in the \( y-z \) plane as predicted in the theory.

The dimensionless amplitude of the pipe vibration at \( s=0.3 \) as a function of the fluid velocity \( V_s \) is shown.
Fig. 14 Time histories of $v$ and $w$, their spectrum analyses and the pipe motion in a horizontal plane at $s=0.3$ ($\alpha=0.12$, $V_s=6.22$, Experiment)

in Fig. 15(a) for the same case. The experimental critical velocity is $V_c=6.00$ is slightly small compared with the theoretical one, i.e. 6.15. As the fluid velocity $V_s$ increases, the experimental value of the pipe deflection becomes qualitatively large in the same manner as the theoretical result as shown in Fig. 8(a). Figure 15(b) indicates that the experimental frequency of the flow-induced pipe vibration is almost constant with increasing $V_s$, in the same manner as the theoretical result as shown in Fig. 8(b).

Figure 16 shows the effect of the end mass on the lateral pipe vibration theoretically and experimentally. The planar pipe vibrations were observed for $\alpha=0$, 0.06 and 0.09. The nonplanar pipe vibrations were observed for $\alpha=0.12$ and 0.16. The planar pipe vibration was observed again for $\alpha=0.24$. Those experimental results are in qualitative agreement with the theoretical ones.

6. Conclusions

The effect of the mass ratio of the end mass to the total mass of the pipe $\alpha$ on a lateral flow-induced vibration of a cantilevered pipe, has been studied from the viewpoint of nonlinear dynamics.

First, the nonlinear first-order ordinary differential equations governing the amplitudes and phases of $v$ and $w$, which are the lateral deflections of the pipe perpendicular to each other, have been derived from the nonlinear coupled integro-partial differential equations for $v$ and $w$. By means of the derived equations, the effect of the mass ratio $\alpha$ on the stability of the flow-induced pipe vibration has been
examined theoretically for the typical example of the fluid-conveying pipe.

Second, the spacial behaviors of a silicon rubber pipe conveying water were observed quantitatively by means of the image-processing system based on images from two CCD cameras. As predicted in the theory, the nonplanar pipe vibration has been observed for some finite range of the mass ratio \( a \), for the case that the planar flow-induced vibration of the pipe without the end mass, i.e., \( a=0 \).

Furthermore, the flow-induced lateral vibration of a cantilevered pipe with an end mass becomes very complicated with increasing the fluid velocity \( V_f \), i.e., the bifurcation phenomena and the chaotic vibration. At present, it is very difficult to analyze such flow-induced vibrations. In this paper, we have analyzed the flow-induced vibration under the supercritical condition that the fluid velocity in the pipe is slightly above its critical value, i.e., the same condition as Bajaj and Sethna (1984).

Acknowledgments

We would like to thank Messrs M. Watanabe, M. Uchiyama and G. Kitadai for their assistance in the experiments. This work was supported by Grant-in-Aid for Science Research of Japanese Ministry of Education, Science and Culture.

References