Internal Resonance of Rectangular Laminated Plates with Degenerate Modes

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Internal resonance of thin rectangular laminated plates with degenerate modes is investigated using the method of multiple scales. The plates are assumed to be simply supported along all edges, and both movable and immovable edge conditions are considered. In order to use the method of multiple scales properly, the definition of the detuning parameter σ , which expresses the relationship between the linear natural frequency ω and the driving frequency Ω , is changed from $\Omega = \omega + \varepsilon\sigma$ to $\Omega^2 = \omega^2 + \varepsilon\sigma$. The influence of the lamination sequence and the edge conditions on two-mode responses of the plates is shown. The accuracy of the analytical results is confirmed by comparing them with results of numerical integration of the equation of motion. Furthermore, it is shown that the two-mode responses that lose stability via a Hopf bifurcation lead to chaotic motion.

Key Words: Nonlinear Vibration, Vibration of Continuous System, Chaos, Internal Resonance, Laminated Plates, Method of Multiple Scales, Degenerate Modes, Bifurcation

1. Introduction

Fiber reinforced composite materials, e.g., fiber reinforced plastics (FRP), are widely used in industrial fields due to their excellent mechanical properties such as a high strength-to-weight ratio, high stiffness-to-weight ratio and anti-corrosiveness. It has therefore become important to understand not only their static properties but also their dynamic properties. On the other hand, the need for higher speed, lighter weight designs, or energy-cost savings has meant that structure members have become thinner, and hence they may encounter large amplitude vibrations.

Nonlinear vibrations of laminated plates, which are basic structure components, have been extensively studied^{(1),(2)}. There have been reports on forced vibrations of orthotropic plates^{(3),(4)}, two-mode responses of cross-ply laminated plates⁽⁵⁾, and so on. However, studies on multi-mode responses have been limited to isotropic materials⁽⁶⁾⁻⁽¹²⁾. In our previous paper⁽¹³⁾, we

studied the two-mode response of simply supported orthotropic, antisymmetric angle-ply laminated and symmetric cross-ply laminated plates when two natural frequencies, ω_1 and ω_2 , have the relationship $\omega_2 \approx 3\omega_1$. We proposed new detuning parameters for the method of multiple scales (MMS)⁽¹⁴⁾, and we reported the influence of lamination sequence on the two-mode responses.

Laminated plates may possess different degenerate modes from those of isotropic plates due to their anisotropy. In the present study, we examine the internal resonance of rectangular laminated plates with degenerate modes whose linear natural frequencies of two asymmetric vibration modes coincide with each other. The plate is simply supported along its four edges, and movable and immovable edge conditions are considered here for in-plane boundary conditions. By deriving the stress function satisfying each boundary condition and applying Galerkin's procedure to the equation of motion, ordinary differential equations in terms of time are obtained. We introduce a detuning parameter expressing the relationship between linear natural and driving frequencies⁽¹³⁾ and perform the MMS. Equations expressing the relationship among amplitudes, phases and a driving frequency in the steady-state are then derived. In numerical examples, it is shown that

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analytical results are also improved by modifying the detuning parameter on this kind of two-mode response and that frequency-response curves are dependent upon the lamination sequence and the boundary condition. Furthermore, we show that two-mode responses that lose stability via a Hopf bifurcation lead to chaotic vibrations.

2. Theoretical Analysis

Figure 1 shows a rectangular laminated plate consisting of N layers of an FRP lamina, with lengths a and b and thickness h . The distance from the midplane to the upper plane of the k th layer is h_k . The coordinate system (x, y, z) is taken in the midplane of the plate, as shown in the figure. The transverse displacement in the midplane is w . The principal axes of elasticity of k th layer are denoted by L and T , respectively, and θ_k is the angle between L and x axes.

The equation of motion and the compatibility condition for an antisymmetric angle-ply laminated plate, based on the von Kármán-type geometric non-linear theory, can be expressed as

$$\begin{aligned} \rho w_{,tt} + c w_{,t} + D_{11}^* w_{,xxxx} + 2(D_{12}^* + 2D_{66}^*) w_{,xxyy} \\ + D_{22}^* w_{,yyyy} + (2B_{26}^* - B_{61}^*) \Psi_{,xxxy} \\ + (2B_{16}^* - B_{62}^*) \Psi_{,xyyy} - w_{,yy} \Psi_{,xx} - w_{,xx} \Psi_{,yy} \\ + 2w_{,xy} \Psi_{,xy} - q(x, y) \cos \Omega' t = 0, \end{aligned} \quad (1)$$

$$\begin{aligned} A_{22}^* \Psi_{,xxxx} + (2A_{12}^* + A_{66}^*) \Psi_{,xxyy} + A_{11}^* \Psi_{,yyyy} \\ = (2B_{26}^* - B_{61}^*) w_{,xxxy} + (2B_{16}^* - B_{62}^*) w_{,xyyy} \\ - w_{,xx} w_{,yy} + w_{,xy}^2, \end{aligned} \quad (2)$$

where Ψ is the stress function, ρ is the mass per unit area of the plate, c is the viscous damping coefficient, and $q(x, y)$ and Ω' are the distribution and frequency of a harmonic excitation acting on the z direction, respectively. The subscripts following a comma stand for partial differentiation. Constants A_{ij}^* , B_{ij}^* and D_{ij}^* are the stiffness coefficients of the plate, which are derived from

Fig. 1 Geometry of a laminated plate and coordinate systems

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}, \quad (3)$$

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^N \int_{h_{k-1}}^{h_k} C_{ij}^{(k)} (1, z, z^2) dz, \quad (4)$$

$$A^* = A^{-1}, \quad B^* = -A^{-1}B, \quad D^* = D - BA^{-1}B, \quad (5)$$

in which $C_{ij}^{(k)}$ are the stiffness matrix elements expressing the stress-strain relation in the k th layer. The stress function Ψ is defined by

$$\Psi_{,yy} = N_x, \quad \Psi_{,xx} = N_y, \quad \Psi_{,xy} = -N_{xy}, \quad (6)$$

where N_x , N_y and N_{xy} are the membrane forces per unit length of the plate.

We assume that the plate is simply supported along its four edges, and both movable and immovable edges are adopted for in-plane boundary conditions. Each boundary condition can be given, respectively, by

$$\left. \begin{aligned} w = w_{,xx} = \Psi_{,xy} = \int_0^b \Psi_{,yy} dy = 0 \quad \text{at } x=0, a \\ w = w_{,yy} = \Psi_{,xy} = \int_0^a \Psi_{,xx} dx = 0 \quad \text{at } y=0, b \end{aligned} \right\} \quad (7)$$

$$\left. \begin{aligned} w = w_{,xx} = \Psi_{,xy} = 0 \quad \text{at } x=0, a \\ \int_0^b \int_0^a u^0_{,x} dx dy = 0 \\ w = w_{,yy} = \Psi_{,xy} = 0 \quad \text{at } y=0, b \\ \int_0^b \int_0^a v^0_{,y} dx dy = 0 \end{aligned} \right\} \quad (8)$$

where u^0 and v^0 are the displacements at the midplane in the x and y directions, respectively. The integrals in Eq. (7) indicate that the in-plane displacements in the direction perpendicular to the edge are constant at the boundaries. The integrals in Eq. (8), on average, satisfy $u^0=0$ and $v^0=0$.

In the present paper, we consider an internal resonance when the linear natural frequencies ω_1 and ω_2 of the vibration modes (1, 2) and (2, 1) have the relationship $\omega_1 = \omega_2$. It is assumed that the motion of the plate is dominated by two vibration modes and that displacements of other vibration modes are negligible. In vibration mode (m, n) , m and n are the number of half waves in the x and y directions, respectively. Thus, the transverse displacement w of the plate is approximated by using the eigenfunction of the linear vibration as

$$\begin{aligned} w = \frac{h^2}{a} \left\{ W_1(t) \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} \right. \\ \left. + W_2(t) \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} \right\}, \end{aligned} \quad (9)$$

in which $W_i(t)$ are non-dimensional displacements. Further, we assume that only vibration mode (1, 2) is directly excited by the harmonic excitation, and its distribution is defined as

$$q(x, y) = q_0 \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b}, \quad (10)$$

where q_0 is the amplitude of the load.

The stress function satisfying boundary conditions (7) and (8) can be written as

$$\Psi = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} B_{pq} \cos \frac{p\pi x}{a} \cos \frac{q\pi y}{b} + C_1 x^2 + C_2 y^2, \quad (11)$$

in which B_{pq} , C_1 and C_2 are unknown coefficients. By substituting Eqs. (9) and (11) into Eq. (2) and comparing the coefficients of trigonometric functions, B_{pq} can be determined as the quadratic form of W_i . By using the following equations

$$\begin{aligned} u_{,x}^0 &= A_{11}^* \Psi_{,yy} + A_{12}^* \Psi_{,xx} - 2 B_{16}^* w_{,xy} - w_{,x}^2/2 \\ v_{,y}^0 &= A_{12}^* \Psi_{,yy} + A_{22}^* \Psi_{,xx} - 2 B_{26}^* w_{,xy} - w_{,y}^2/2 \end{aligned} \quad (12)$$

and performing the integrations in Eq. (8), C_1 and C_2 can be determined. Although details of B_{pq} , C_1 and C_2 are omitted due to space restriction, C_1 and C_2 vanish in the case of boundary condition (7).

By substituting the obtained stress function, Eqs. (9) and (10) into the equation of motion (1) and applying Galerkin's procedure, simultaneous ordinary differential equations, which have cubic nonlinear terms, are obtained as

$$\begin{aligned} W_{1,\tau\tau} + 2\epsilon\mu W_{1,\tau} + \omega^2 W_1 \\ + \epsilon(G_1 W_1^3 + G_2 W_1 W_2^2) = 2\epsilon f \cos \Omega\tau \\ W_{2,\tau\tau} + 2\epsilon\mu W_{2,\tau} + \omega^2 W_2 \\ + \epsilon(G_3 W_1^2 W_2 + G_4 W_2^3) = 0 \end{aligned} \quad (13)$$

where μ , ω , G_i , f and Ω are the non-dimensional damping coefficient, the non-dimensional linear natural frequency, the non-dimensional coefficients of the nonlinear terms, and the non-dimensional amplitude and frequency of the excitation, respectively. Details of their coefficients are omitted due to space restriction, but G_2 and G_3 have the relationship $G_2 = G_3$. The non-dimensional time and small parameter are denoted by τ and ϵ , respectively, which are defined as

$$\tau = \sqrt{\frac{E_L h^3}{\rho a^4}} t, \quad \epsilon = \frac{h^2}{a^2}, \quad (14)$$

in which E_L is Young's modulus in the L direction.

Next, we apply the MMS to Eq. (13) to obtain steady-state responses near a primary resonance. According to the MMS, $W_i(\tau)$ are expanded as

$$W_i(\tau) = x_{i0}(T_0, T_1, \dots) + \epsilon x_{i1}(T_0, T_1, \dots) + \dots, \quad (15)$$

where $T_0 = \tau$, $T_1 = \epsilon\tau$, $\dots$ are different time scales. In order to properly use the MMS, a detuning parameter σ expressing the relationship between the frequency of the harmonic excitation and the linear natural frequency is redefined from $\Omega = \omega + \epsilon\sigma$ to

$$\Omega^2 = \omega^2 + \epsilon\sigma \quad (16)$$

in the same way as in our previous paper⁽¹³⁾. The substitution of Eqs. (15) and (16) into Eq. (13) yields the following equations in terms of ϵ^0 and ϵ^1 orders:

$$\begin{aligned} \epsilon^0: & \begin{cases} D_0^2 x_{10} + \Omega^2 x_{10} = 0 \\ D_0^2 x_{20} + \Omega^2 x_{20} = 0 \end{cases} \\ \epsilon^1: & \begin{cases} D_0^2 x_{11} + \Omega^2 x_{11} = -2 D_0 D_1 x_{10} - 2 \mu D_0 x_{10} + \sigma x_{10} \\ - G_1 x_{10}^3 - G_2 x_{10} x_{20}^2 + 2 f \cos \Omega T_0 \\ D_0^2 x_{21} + \Omega^2 x_{21} = -2 D_0 D_1 x_{20} - 2 \mu D_0 x_{20} + \sigma x_{20} \\ - G_3 x_{10}^2 x_{20} - G_4 x_{20}^3 \end{cases} \end{aligned} \quad (17)$$

in which $D_i = \partial/\partial T_i$. The first-order approximate solutions near the primary resonance, that is, the solutions of Eq. (17) for ϵ^0 order, are given in the form

$$x_{i0} = A_i(T_1) \exp(i\Omega T_0) + \bar{A}_i(T_1) \exp(-i\Omega T_0), \quad (18)$$

where A_i are unknown functions and $\bar{A}_i$ are complex conjugates of A_i . They are determined by solving the following equations

$$\begin{aligned} 2i\Omega(A_1' + \mu A_1) + 3G_1 A_1^2 \bar{A}_1 \\ + G_2(\bar{A}_1 A_2^2 + 2A_1 A_2 \bar{A}_2) - \sigma A_1 - f = 0 \\ 2i\Omega(A_2' + \mu A_2) + G_3(A_1^2 \bar{A}_2 + 2A_1 \bar{A}_1 A_2) \\ + 3G_4 A_2^2 \bar{A}_2 - \sigma A_2 = 0 \end{aligned} \quad (20)$$

which are derived from the condition that Eq. (18) for ϵ^1 order does not contain the secular terms. In Eq. (20), a prime denotes the derivative with respect to T_1 . To solve Eq. (20), we express A_i and $\bar{A}_i$ as

$$A_i = \frac{1}{2} a_i \exp(i\beta_i), \quad \bar{A}_i = \frac{1}{2} a_i \exp(-i\beta_i). \quad (21)$$

If we divide Eq. (20) into real and imaginary parts by means of the above equation, the following equations are obtained:

$$\begin{aligned} \Omega(a_1' + \mu a_1) + \frac{1}{8} G_2 a_1 a_2^2 \sin(2\beta_2 - 2\beta_1) = -f \sin \beta_1 \\ - \Omega a_1 \beta_1' - \frac{\sigma a_1}{2} + \frac{3}{8} G_1 a_1^3 \\ + \frac{1}{8} G_2 a_1 a_2^2 [\cos(2\beta_2 - 2\beta_1) + 2] = f \cos \beta_1 \\ \Omega(a_2' + \mu a_2) - \frac{1}{8} G_3 a_1^2 a_2 \sin(2\beta_2 - 2\beta_1) = 0 \\ - \Omega a_2 \beta_2' - \frac{\sigma a_2}{2} + \frac{3}{8} G_4 a_2^3 \\ + \frac{1}{8} G_3 a_1^2 a_2 [\cos(2\beta_2 - 2\beta_1) + 2] = 0 \end{aligned} \quad (22)$$

By substituting conditions $a_1' = a_2' = \beta_1' = \beta_2' = 0$ for the steady-state solutions into Eq. (22), algebraic equations in terms of unknown variables a_1 , a_2 , β_1 and β_2 are derived. The steady-state solutions are obtained from the algebraic equations by using the Newton-Raphson method.

It can be seen in Eq. (22) that the solutions can be classified into two types: $a_1 \neq 0$, $a_2 \neq 0$ and $a_1 \neq 0$, $a_2 = 0$, that is, two-mode and single-mode responses. In order to examine the stability of the two-mode responses, we derive characteristic equations in terms of small disturbances Δa_1 , Δa_2 , $\Delta \beta_1$ and $\Delta \beta_2$ on the steady-state solutions a_1 , a_2 , β_1 and β_2 from Eq. (22), and we calculate the eigenvalues of the coefficient

matrix of the equation. In the case of single-mode solutions, we derive characteristic equations in terms of small disturbances Δa_1 , $\Delta \beta_1$ and ΔA_2 on the steady-state solutions a_1 , β_1 and $A_2=0$ from Eqs. (22) and (20), respectively. If the real parts of all the eigenvalues are negative, the response is stable⁽⁶⁾. If stable steady-state solutions are obtained, the transverse deflection w is given by

$$\begin{aligned} \frac{wa}{h^2} &= W_1 \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} + W_2 \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} \\ &= a_1 \cos(\Omega \tau + \beta_1) \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} \\ &\quad + a_2 \cos(\Omega \tau + \beta_2) \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b}. \end{aligned} \quad (23)$$

3. Numerical Examples

In the following numerical examples, we treat an orthotropic plate ($\theta=0^\circ$), an antisymmetric angle-ply laminated plate ($\theta=45^\circ/-45^\circ$), and a symmetric cross-ply laminated plate ($\theta=90^\circ/0^\circ/90^\circ$). Each plate has the same thickness, and the thickness of each layer of the angle-ply and cross-ply laminations is the same. We use the following material properties of Graphite/Epoxy⁽¹⁵⁾:

$$E_L=138 \text{ GPa}, \quad E_T=8.96 \text{ GPa}, \quad G_{LT}=7.1 \text{ GPa}, \\ \nu_{LT}=0.3.$$

Table 1 shows aspect ratios a/b and non-dimensional natural frequencies for each lamination sequence satisfying the assumption of the analysis (i.e., vibration modes (1, 2) and (2, 1) are of equal natural frequency). Other parameters are taken as

$$\varepsilon=10^{-4}, \quad 2\varepsilon\mu=0.05,$$

and the value f is determined so as to induce the transverse deflection $a_1 \approx 70$ in the single-mode analysis.

3.1 Examination of the validity of the detuning parameter

First, we examine the validity of the detuning parameter proposed in this paper. Figure 2 shows only the stable responses of an orthotropic plate with immovable edges. Figures (a) and (b) present amplitudes a_1 and a_2 , respectively, and the abscissa is non-dimensional excitation frequency. The solid and broken lines denote the responses obtained by using the detuning parameters $\Omega^2 = \omega^2 + \varepsilon\sigma$ and $\Omega = \omega + \varepsilon\sigma$, respectively. As seen in Fig. 2, the responses obtained by using $\Omega = \omega + \varepsilon\sigma$ exist up to a higher frequency than those obtained by $\Omega^2 = \omega^2 + \varepsilon\sigma$. In order to confirm the validity of the analytical results, Eq. (13) is integrated numerically using the fourth-order Runge-Kutta method, and the results are plotted by the symbols $\bullet$ and $\circ$. The time step for the numerical integration is taken as $T/1024$, where T means one period (i.e., $T=2\pi/\Omega$). The results of numerical

Table 1 Aspect ratios and non-dimensional linear natural frequencies of plates

θ	0°	$45^\circ/-45^\circ$	$90^\circ/0^\circ/90^\circ$
a/b	1.981	1.000	0.567
ω	13.04	6.451	4.047

Fig. 2 Comparison of two detuning parameters on two-mode responses ($\theta=0^\circ$, immovable, $2\varepsilon f=60$), (a) amplitudes of (1, 2) mode, (b) amplitudes of (2, 1) mode

integration are in good agreement with those of $\Omega^2 = \omega^2 + \varepsilon\sigma$ and cannot be obtained in the frequency region over about 14.8. Therefore, it is shown that the detuning parameter proposed in this paper is more suitable for analysis than $\Omega = \omega + \varepsilon\sigma$.

3.2 Influences of the lamination sequence and the boundary condition on frequency-response curves

Frequency-response curves near a primary resonance are shown in Figs. 3 - 8. Figures (a) and (b) present the amplitudes of vibration modes (1, 2) and (2, 1), respectively, and Fig. (c) shows the phase difference ($\beta_2 - \beta_1$) of the two-mode response. In each figure, the solid and broken lines denote stable and unstable responses, respectively. The dash-dotted lines show unstable responses with complex conjugate eigenvalues (of the coefficient matrix) whose real

Fig. 3 Steady state solutions for the orthotropic plate (movable, $2\epsilon f=50$), (a) amplitudes of (1, 2) mode, (b) amplitudes of (2, 1) mode, (c) phase difference

Fig. 4 Steady state solutions for the angle-ply laminated plate (movable, $2\epsilon f=24$), (a) amplitudes of (1, 2) mode, (b) amplitudes of (2, 1) mode, (c) phase difference

parts are positive. They are depicted to make the Hopf bifurcation points clear. A boundary between solid (or broken) and dash-dotted lines is a Hopf bifurcation point. The figures show that the analytical results are valid because the numerical results agree with them.

Frequency-response curves for orthotropic, angle-ply laminated and cross-ply laminated plates with movable edges are shown in Figs. 3–5, respectively.

In the case of the orthotropic plate (Fig. 3), stable two-mode responses occur at $\sigma/\omega^2 \approx 100$ and 900 via pitchfork and Hopf bifurcations, respectively. Furthermore, there is a stable response, which achieves and loses its stability through Hopf bifurcation points, in a very small region at $\sigma/\omega^2 \approx 450$. The phase differences of the responses generated at $\sigma/\omega^2 \approx 100$ and 900 are nearly $\pi/2$ and 0 , respectively. We can categorize the former and latter into travelling

Fig. 5 Steady state solutions for the cross-ply laminated plate (movable, $2\epsilon f = 15.5$), (a) amplitudes of (1, 2) mode, (b) amplitudes of (2, 1) mode, (c) phase difference

waves in which the nodal line rotates at the same frequency as the driving frequency Ω and standing waves in which the nodal line is located at the diagonal of the plate, respectively^{(8),(10)}. It was found that two different types of two-mode response occur.

In the angle-ply laminated plate shown in Fig. 4, there is only a stable two-mode response generated at $\sigma/\omega^2 \approx 400$ via a saddle-node bifurcation, and a traveling wave by a pitchfork bifurcation does not occur.

Fig. 6 Steady state solutions for the orthotropic plate (immovable, $2\epsilon f = 60$), (a) amplitudes of (1, 2) mode, (b) amplitudes of (2, 1) mode, (c) phase difference

As seen in Fig. 5(c), only a standing wave occurs in the cross-ply laminated plate, for which the results are contrary to those for the angle-ply laminated plate. Further, a Hopf bifurcation, which does not change the stability of the solution, occurs at $\sigma/\omega^2 \approx 720$ in the case of the cross-ply laminated plate. As mentioned above, the frequency-response curves are quite different for each lamination sequence. Especially, in the case of the angle-ply laminated plate, the

Fig. 7 Steady state solutions for the angle-ply laminated plate (immovable, $2\epsilon f=32$), (a) amplitudes of (1, 2) mode, (b) amplitudes of (2, 1) mode, (c) phase difference

range of the two-mode resonance is longer than that of the single-mode resonance, and we can see the necessity of two-mode analysis. The nonlinearity (i.e., hard spring behavior) is stronger in the order of orthotropic plate (Fig. 3) > cross-ply laminated plate (Fig. 5) > angle-ply laminated plate (Fig. 4), and the range of the two-mode response becomes longer in contract to this order. Therefore, it can be assumed that the range depends on the magnitude of the non-

Fig. 8 Steady state solutions for the cross-ply laminated plate (immovable, $2\epsilon f=18$), (a) amplitudes of (1, 2) mode, (b) amplitudes of (2, 1) mode, (c) phase difference

linearity.

Frequency-response curves for the orthotropic, angle-ply laminated and cross-ply laminated plates with immovable edges are shown in Figs. 6 - 8, respectively. As can be seen in each figure, the nonlinearity is much larger for the immovable than movable edge conditions because the in-plane displacements are restrained. Although the frequency-response curves for the cross-ply laminated plate with immovable

edges are similar to those for the cross-ply laminated plate with movable edges, those for the orthotropic and angle-ply laminated plates are different from those of the movable boundary condition. In the case of the orthotropic plate in Fig. 6, the single-mode response, which loses its stability at $\sigma/\omega^2 \approx 0$ via a pitchfork bifurcation, recovers its stability at $\sigma/\omega^2 \approx 500$ via a pitchfork bifurcation, and such a bifurcation does not occur under the movable edge condition. The frequency-response curves for the angle-ply laminated plate shown in Fig. 7 are similar to those for the orthotropic plate with movable edges, except for a stable response, which occurs and vanishes through Hopf bifurcation points, around $\sigma/\omega^2 \approx 1670$.

In the immovable edge condition, the range of the two-mode response becomes longer in the order of angle-ply laminated > cross-ply laminated > orthotropic plates. This tendency is the same as that under movable edge conditions. However, since the nonlinearity becomes stronger in the order of angle-ply laminated > orthotropic > cross-ply laminated plates, it cannot be said that the range depends only on the magnitude of nonlinearity. Backbone curves expressing the magnitude of nonlinearity are given by

$$\frac{\sigma}{\omega^2} = \frac{3}{4} \frac{G_1}{\omega^2} a_1^2. \quad (24)$$

It can be seen in the above equation that G_1/ω^2 is a parameter representing the magnitude of nonlinearity. Hence, the nonlinear term G_2 , which denotes the two-mode interaction, divided by ω^2 is listed in Table 2. It can be seen from Table 2 that if the in-plane boundary condition is the same, the range of the two-mode response becomes longer with an increase in G_2/ω^2 . It is thought that the range also depends on the nonlinear term denoting the two-mode interaction.

3.3 Quasi-periodic and chaotic vibrations

In this section, we show that stable two-mode responses that lose their stability through a Hopf bifurcation point lead to quasi-periodic vibrations, eventually giving rise to chaotic vibrations. As an example, we treat the cross-ply laminated plate with immovable edges shown in Fig. 8. Unstable responses after a Hopf bifurcation in $\sigma/\omega^2 = 1126 \sim 1129$ are simulated by numerical integration, and Poincaré sections on the a_1 - β_1 plane, which are constructed by sampling the motion at the period of the driving frequency, are shown in Fig. 9. We use the results obtained by the MMS as the initial conditions, and we present the results calculated from 10 000 to 40 000 periods. Further, maximum Lyapunov exponents $\lambda_{\max}^{(16)}$ are shown in the figure to determine whether the motions are chaotic vibrations.

It can be seen in Figs. (a), (b) and (c) that the closed curves become double and fourfold with

Table 2 Values of coefficient G_2/ω^2

θ	0°	45°/-45°	90°/0°/90°
Movable	0.789	2.024	0.838
Immovable	1.459	3.918	2.256

Fig. 9 Poincaré sections for the cross-ply laminated plate (immovable, $2\epsilon f=18$), (a) $\sigma/\omega^2=1127.0$, (b) $\sigma/\omega^2=1127.6$, (c) $\sigma/\omega^2=1127.9$, (d) $\sigma/\omega^2=1128.8$

increases in the driving frequency. As the frequency is increased further, the motion leads to vibrations such as the chaos shown in Fig. (d). However, eightfold closed curves cannot be observed. Since the maximum Lyapunov exponents, except in Fig. (d), are very small and can be taken to be zero, these motions are quasi-periodic. In the case of Fig. (d), the maximum Lyapunov exponent is a positive value. Therefore, we conclude that the motion is chaotic (other Lyapunov exponents are not positive).

Figure 10 shows the maximum Lyapunov exponents for the quasi-periodic ($\sigma/\omega^2=1127.0$) and chaotic ($\sigma/\omega^2=1128.8$) vibrations. The abscissa is a period. The maximum Lyapunov exponent for $\sigma/\omega^2=1127.0$ converges to zero. In the case of $\sigma/\omega^2=1128.8$, it is almost a constant positive value after 10 000 periods. The above results confirm the validity of the maximum Lyapunov exponent, and it is concluded that the vibration at $\sigma/\omega^2=1128.8$ is chaos.

4. Conclusions

In the present study, we investigated responses near a primary resonance of rectangular laminated plates with degenerate modes. The boundary conditions were assumed for simply supported plates, and in-plane movable and immovable edge conditions

Fig. 10 Convergence characteristics of maximum Lyapunov exponents

were considered. In the analysis of the MMS, a detuning parameter σ , which expresses the relationship between the frequency Ω of a harmonic excitation and linear natural frequency ω of the plate, was redefined from $\Omega = \omega + \varepsilon\sigma$ to $\Omega^2 = \omega^2 + \varepsilon\sigma$. An orthotropic plate and angle-ply and cross-ply laminated plates were treated as numerical examples. The results can be summarized as follows:

(1) We confirmed the validity of the present results in a comparison with those of numerical integration, and we confirmed that analysis using the proposed detuning parameter is more accurate than analysis using the conventional detuning parameter.

(2) The two-mode responses were classified into standing waves whose phase difference is nearly zero and travelling waves whose phase difference is nearly $\pi/2$. The frequency-response curves are dependent upon the lamination sequence and the in-plane boundary condition, and in some cases two types of stable response occur, while in other cases only one type occurs.

(3) The range of the two-mode response depends on not only the magnitude of nonlinearity but also on the coefficient G_2 of the nonlinear term denoting the two-mode interaction. In the case of the same in-plane boundary condition, the range becomes longer with increases in the value of G_2/ω^2 .

(4) Stable two-mode responses that lose their stability via a Hopf bifurcation lead to quasi-periodic vibrations, which eventually give rise to chaotic vibrations.

References

(1) Chia, C.Y., *Nonlinear Analysis of Plates*, (1980),

McGraw-hill.

- (2) Sathyamoorthy, M., *Nonlinear Vibration Analysis of Plates: A Review and Survey of Current Developments*, *Appl. Mech. Rev.*, Vol. 40, No. 11 (1987), p. 1553.
- (3) Eslami, H. and Kandil O.A., *Nonlinear Forced Vibration of Orthotropic Plates Using the Method of Multiple Scales*, *AIAA J.*, Vol. 27, No. 7 (1989), p. 955.
- (4) Eslami, H. and Kandil O.A., *Two-Mode Nonlinear Vibration of Orthotropic Plates Using the Method of Multiple Scales*, *AIAA J.*, Vol. 27, No. 7 (1989), p. 961.
- (5) Hadian, J., Nayfeh, A.H. and Nayfeh, J.F., *Modal Interaction in the Response of Antisymmetric Cross-Ply Laminated Rectangular Plates*, *Journal of Vibration and Control*, Vol. 1 (1995), p. 159.
- (6) Yasuda, K. and Uno, H., *Multimode Response of a Circular Membrane*, *Bull. JSME*, Vol. 26, No. 216 (1983), p. 1050.
- (7) Yasuda, K. and Asano, T., *Nonlinear Forced Oscillations of a Rectangular Membrane with Degenerate Modes*, *Bull. JSME*, Vol. 29, No. 255 (1986), p. 3090.
- (8) Yasuda, K. and Torii, T., *Multi-Mode Response of a Square Membrane*, *JSME Int. J.*, Vol. 30, No. 264 (1987), p. 963.
- (9) Hadian, J. and Nayfeh, A.H., *Modal Interaction in Circular Plates*, *J. Sound Vib.*, Vol. 142, No. 2 (1990), p. 279.
- (10) Yang, X.L. and Sethna, P.R., *Non-Linear Phenomena in Forced Vibrations of a Nearly Square Plate: Antisymmetric Case*, *J. Sound Vib.*, Vol. 155, No. 3 (1992), p. 413.
- (11) Nayfeh, T.A. and Vakakis, A.F., *Subharmonic Travelling Waves in a Geometrically Non-Linear Circular Plate*, *Int. J. Non-Linear Mech.*, Vol. 29, No. 2 (1994), p. 233.
- (12) Ye, J. and Yasuda, K., *Nonlinear Oscillations of Three-Degree-of-Freedom Vibratory Systems with 1:1:1 Internal Resonance*, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 63, No. 613, C (1997), p. 2967.
- (13) Yamada, G., Kobayashi, Y. and Abe, A., *Multimode response of Rectangular Laminated Plates*, *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 62, No. 600, C (1996), p. 2976.
- (14) Nayfeh, A.H. and Mook, D.T., *Nonlinear Oscillations*, (1979), John Wiley and Sons.
- (15) Vinson, J.R. and Sierakowski, R.L., *The Behavior of Structures Composed of Composite Materials*, (1986), p. 303, Martinus Nijhoff Publishers, Dordrecht.
- (16) Wolf, A., Swift, H.L. and Vastano, J.A., *Determining Lyapunov Exponents from a Time Series*, *Physica*, 16D, (1985), p. 285.