The Ideal Tooth Profiles of Conical-External and -Internal Gears Meshing with Cylindrical-Involute Gears over the Entire Tooth Width*

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A new synthesis method for tooth profiles using vector equations is proposed to precisely derive the curve of tooth profile of conical-external and -internal gears which engage a cylindrical-involute gear over the entire tooth width. As an example, the ideal tooth profiles of conical-external and -internal gears which engage with an inclined cylindrical-involute gear over the entire tooth width have been derived, and their characteristics with respect to a plane perpendicular to the axis of the inclined cylindrical gear have been investigated. The results obtained are as follows. (1) The tooth profiles of conical-external and -internal gears favorably coincide with plane involute curves. (2) The radius of the base circle is given by using the equivalent tooth number of the conical-external or -internal gear. (3) The profile-shift coefficient is given by the center-to-center distance between the inclined involute gear and the conical-external gear, or between the cylindrical gear and the conical-internal gear. (4) The profile-shift coefficient of conical-external or -internal gear changes nonlinearly along the axis of the cylindrical-involute gear.

Key Words: Machine Element, Gear, Design, Involute Conical Gear, Tooth Profile Synthesis, Planetary Gear Drive, Mechanical Paradox Gear, Profile-Shift Coefficient

1. Introduction

Traditional 3K-type mechanical paradox planetary gear drives, which are composed of cylindrical gears only, have a high reduction ratio and high efficiency⁽¹⁾⁻⁽³⁾. In the field of industrial robotics, however, gear drives with a low reduction ratio of 1/30 to 1/60 and with high efficiency are now required to drive industrial robots at high speed. The authors

have hence proposed a new mechanical paradox planetary gear drive in which the cylindrical planet gears are inclined at an angle and are engaged with a conical sun gear and two conical internal gears so that the difference between the tooth numbers of the two internal gears can be increased, thereby obtaining a low reduction ratio⁽⁴⁾. This new mechanical paradox planetary gear drive has many advantages, such as high efficiency⁽⁵⁾, adjustable backlash, and low noise since the tooth engagement is similar to that of helical gears⁽⁶⁾.

This gear drive, however, has the disadvantage of a small load capacity because the tooth engagements between the cylindrical planet gears and the conical-internal gears are point contact⁽⁷⁾. It is therefore an important issue to realize line contact tooth engagement. The authors hence attempted to cut and generate a conical gear by moving the cutting hob not along a straight line, as in the traditional method, but along a circle of large radius so that the conical gear could

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engage with a cylindrical gear over the entire tooth width⁽⁸⁾. However the proposed method could not be evaluated adequately, since it was not theoretically derived by taking line contact conditions into consideration, and the experimental inspection of line contact was also very difficult.

On the other hand, a synthesis method for the three-dimensional curved surface of tooth profile has already been proposed to derive the tooth profiles of special skew gears, taking line contact conditions into consideration⁽⁹⁾. But this method has not been applied to other cases.

In this paper, the authors propose a new synthesis method for tooth profile using vector equations by which the curved surface of tooth profile of one gear can easily be derived from the given curved surface of tooth profile of the other mating gear, derive the precise curved surfaces of tooth profile of conical-external and -internal gears which engage with a cylindrical-involute external gear over the entire tooth width, and discuss the basic characteristics of the synthesized curves with respect to a section perpendicular to the axis of the cylindrical involute external gear.

2. Mechanical Paradox Planetary Gear Drive with Inclined Planet Gears

2.1 Construction of gear drive

Figure 1 shows the construction of a mechanical paradox planetary gear drive which is composed of three inclined cylindrical-involute planet gears, a conical sun gear fixed on the input axle, a fixed conical-internal gear, and a rotatory conical-internal gear fixed on the output axle. The sun and the two internal gears simultaneously engage with the three planet gears. The planet gear axis crosses the center axis of the reduction gear drive; the center-to-center distances between the planet and sun gears, between the planet and fixed internal gears, and between the planet and rotatory internal gears hence vary linearly along the center axis of the reduction gear drive.

2.2 Characteristics of the reduction gear drives

This reduction gear drive can provide a low

Fig. 1 The configuration of 3K-type mechanical paradox planetary gear drive with inclined planet gears

reduction ratio because the diameters of the two internal gears are different due to the inclination of planet gears, and the difference between their tooth numbers can then be increased by more than three, which is the difference in tooth number for traditional mechanical paradox planetary gear drives composed of cylindrical-involute gears only. For instance, suppose that the number of planet gears is three, the difference between the tooth numbers of two internal gears is two times larger than three, namely six, and the tooth numbers of sun and two internal gears are equal to an integral value times the number of planet gears. The assembly conditions of the gear drive can thereby be satisfied since the tooth phases and tooth spaces coincide with each other for the engagement of a pair of planet-sun gears, a pair of planet-fixed internal gears, and a pair of planet-rotary internal gears, respectively. The reduction ratio, in this case, is one half the value of a traditional mechanical paradox planetary gear drive composed of cylindrical-involute gears only, and also can be lower than one sixtieth. Other advantages such as high efficiency, adjustable backlash, and low noise can be expected.

For the practical application of reduction gear drives which have the above-mentioned advantages, it is necessary to make the teeth of the gears contact on a line on the tooth surfaces, so that the gears can engage with each other over the entire tooth width and thus can have a sufficiently large load capacity for practical use. In chapter 3, the authors propose a new synthesis method for tooth profiles to synthesize the precise tooth profile curves of conical-external and -internal gears, and, in chapter 4, synthesize them and investigate their characteristics.

3. Synthesis Method for Tooth Profiles Using Vector Equations

3.1 Coordinate systems and definitions of vectors for describing points on tooth surfaces

Figure 2 shows the three right-hand rectangular coordinate systems defined as follows. (1) Static coordinate system $O-x_0y_0z_0$: z_0 axis coincides with the axis of cylindrical-external gear; an arbitrary point on z_0 axis is origin O . A plane which passes through origin O and is perpendicular to z_0 axis is designated as the axis-perpendicular reference plane S_0 . x_0 axis is OO_2 axis, where O_2 is the intersection between the axis-perpendicular reference plane and the axis of conical gear. (2) Moving coordinate system $O_1-x_1y_1z_1$: the coordinate system fixed on the cylindrical gear. Origin O_1 is the same as origin O defined above, and z_1 axis is also the same as z_0 axis defined above. In the initial state, x_1 and y_1 axes are superimposed on

Fig. 2 The coordinate system for the tooth profile synthesis of cylindrical and conical-external and -internal gears

x_0 and y_0 axes, respectively. (3) Moving coordinate system $O_2-x_2y_2z_2$: the coordinate system fixed on the conical gear. The origin is the intersection, O_2 , defined above. z_2 axis is the axis of conical gear which is inclined by an angle γ to the axis of the cylindrical gear. In the initial state, x_2 axis is located in plane x_0z_0 .

The position vectors of a point on a tooth surface are here defined. Let K_1 denote a point on the tooth surface of cylindrical-external gear, also let $\mathbf{r}_{1,1}$ denote the position vector of point K_1 , and let $\mathbf{r}_{1,1} = \{x_1, y_1, z_1\}^T$, which is described on moving coordinate system $O_1-x_1y_1z_1$ from origin O_1 . The first subscript of position vector $\mathbf{r}$ indicates the gear (1: cylindrical gear, 2: conical gear) and the second subscript indicates the coordinate on which the position vector is described (1: moving coordinate system $O_1-x_1y_1z_1$, 2: moving coordinate system $O_2-x_2y_2z_2$). The superscript T in the component expression of a vector indicates the transpose of a vector. Let $\mathbf{r}_{1,0}$ denote the position vector of the same point K_1 and let $\mathbf{r}_{1,0} = \{x_0, y_0, z_0\}^T$, which is described on static coordinate system $O-x_0y_0z_0$ from origin O .

Next, let K_2 denote a point on the tooth surface of conical gear; let $\mathbf{r}_{2,2}$ denote the position vector of point K_2 ; let $\mathbf{r}_{2,2} = \{x_2, y_2, z_2\}^T$, which is described on moving coordinate system $O_2-x_2y_2z_2$ from origin O_2 ; and also let $\mathbf{r}_{2,1}$ denote the position vector of this point K_2 , which is described on moving coordinate system $O_1-x_1y_1z_1$ from origin O_1 .

3.2 Equation of tooth engagement and its solution

Suppose that the cylindrical gear (angular velocity ω_1) and the conical gear (angular velocity ω_2) engage with each other and rotate with a reduction ratio $u (= \omega_2/\omega_1)$, and also that point K_1 contacts point K_2 at point K when the cylindrical gear rotates by angle ϕ_1 and the conical gear rotates by the corresponding angle ϕ_1 . In this case, the normal-to-tooth surface component of the relative velocity of the

mating cylindrical and conical gears should be zero, since their tooth surfaces contact and rotate without separation. Consideration of this condition on moving coordinate $O_1-x_1y_1z_1$ gives the following equation of tooth engagement.

$$(\mathbf{v}_{2,1} - \mathbf{v}_{1,1}) \cdot \mathbf{n}_{1,1} = 0 \quad (1)$$

where $\mathbf{v}_{1,1}$ and $\mathbf{v}_{2,1}$ are the velocity vectors of points K_1 and K_2 , respectively, which are described on moving coordinate $O_1-x_1y_1z_1$; $\mathbf{n}_{1,1}$ is the normal vector on the tooth surface of cylindrical gear at point K_1 (refer to Appendix A1); and indicates the inner product of vectors. $\mathbf{v}_{1,1}$ and $\mathbf{v}_{2,1}$ are given as follows.

$$\left. \begin{aligned} \mathbf{v}_{1,1} &= \boldsymbol{\omega}_{1,1} \times \mathbf{r}_{1,1} \\ \mathbf{v}_{2,1} &= [T]_{21} \mathbf{v}_{2,2}, \quad \mathbf{v}_{2,2} = \boldsymbol{\omega}_{2,2} \times \mathbf{r}_{2,2} \end{aligned} \right\} \quad (2)$$

where, $[T]_{21}$ is the transformation matrix from moving coordinate system $O_2-x_2y_2z_2$ to moving coordinate system $O_1-x_1y_1z_1$ (refer to Appendix A2); $\mathbf{v}_{2,2}$ is the velocity vector of point K_2 described on moving coordinate system $O_2-x_2y_2z_2$, and $\boldsymbol{\omega}_{1,1} = \{0, 0, \omega_1\}^T$, $\boldsymbol{\omega}_{2,2} = \{0, 0, \omega_2\}^T$; and $\times$ indicates the outer product of vectors.

Next, let $\mathbf{r}_{0,0}$ denote the position vector of point O_2 , and, $\mathbf{r}_{0,0} = \{a_w, 0, 0\}^T$, which is described on static coordinate system $O-x_0y_0z_0$ from origin O , where a_w is the center-to-center distance of the two mating gears on axis-perpendicular reference plane S_0 . The position vector, $\mathbf{r}_{2,2}$, of point K_2 is given as follows.

$$\mathbf{r}_{2,2} = [T]_{12}(\mathbf{r}_{2,1} - [T]_{01}\mathbf{r}_{0,0}) \quad (3)$$

where, $[T]_{12}$ and $[T]_{01}$ are the transformation matrices from moving coordinate system $O_1-x_1y_1z_1$ to moving coordinate system $O_2-x_2y_2z_2$ and from static coordinate system $O-x_0y_0z_0$ to moving coordinate system $O_1-x_1y_1z_1$, respectively (refer to Appendix A2). Since points K_1 and K_2 contact each other, the following relation is applied.

$$\mathbf{r}_{1,1} = \mathbf{r}_{2,1} \quad (4)$$

Substituting this relation into Eq. (3) gives position vector $\mathbf{r}_{2,2}$ in the following form.

$$\mathbf{r}_{2,2} = [T]_{12}(\mathbf{r}_{1,1} - [T]_{01}\mathbf{r}_{0,0}) \quad (5)$$

Substituting Eq. (5) into the second equation of Eq. (2), substituting Eq. (2) into Eq. (1), and rearranging the results by using the normal vectors and also the component expressions of transformation matrices shown in Appendices A1 and A2 gives the following equations which have ϕ_1 as an unknown variable.

$$\begin{aligned} & [-\omega_1 y_1 + \omega_2 z_1 \sin \gamma \sin \phi_1 \\ & + \omega_2 (y_1 + a_w \sin \phi_1) \cos \gamma] \frac{dy_1}{da} \\ & + [\omega_1 x_1 + \omega_2 z_1 \sin \gamma \cos \phi_1 \\ & - \omega_2 (x_1 - a_w \cos \phi_1) \cos \gamma] \left(-\frac{dx_1}{da} \right) = 0 \end{aligned} \quad (6)$$

Here, however, it is assumed that reduction ratio u is given previously, and that the components, x_1 , y_1 , and z_1 , of the position vector of point K_1 which is a point

curved surface of tooth profile of cylindrical-involute gear, and the coordinates, (x_2, y_2, z_2) , of the corresponding point on the curved surface of tooth profile of conical gear are thus determined. The precise curved surface of tooth profile of conical gear can then be obtained by changing the value of radius r_1 from the radius of base circle, r_{g1} , to the desired length, and also by changing the value of coordinate, z_1 , from one end to the other of tooth width successively.

(2) Simultaneous contact line

The simultaneous contact line on the tooth surface of cylindrical gear is obtained by connecting the coordinates, (x_1, y_1, z_1) , which have the same rotational angle, ϕ_1 . The corresponding simultaneous contact line on the tooth surface of conical gear is also obtained by connecting the coordinates, (x_2, y_2, z_2) , which have the same rotational angle, $\phi_2 (=u\phi_1)$.

(3) Curved surface of contact

Let $\mathbf{r}_{1,1}$ denote the position vector of contact point K_1 which corresponds to a constant coordinate value, z_1 , and also let $\mathbf{r}_{1,0}$ denote the position vector obtained by representing $\mathbf{r}_{1,1}$ on the static coordinate system (refer to Eqs. (11) and (12)). Position vector $\mathbf{r}_{1,0}$ then represents the line of contact of the engaging pair of conical- and cylindrical-gears, since the line of contact is the trajectory of contact point, which appears on the static coordinate system due to the rotation of the pair of gears. The curved surface of contact of the engaging pair of conical- and cylindrical-gears is thus obtained by calculating the line of contact which corresponds to each coordinate value z_1 from one end to the other end of tooth width successively and then connecting the obtained lines of contact.

(4) Definition of pitch point

Here, it is defined that the point which is distant r_{p10} by from origin O on x_0 axis is the pitch point, P_0 , of the engaging pair of conical and cylindrical gears. r_{p10} and r_{p20} , where r_{p20} is the length of the perpendicular segment from pitch point P_0 on z_2 axis, are given by the following equations.

$$r_{p10} = a_w \frac{\pm z_b}{z_{ae} \pm z_b}, \quad r_{p20} = a_w \frac{z_{ae}}{z_{ae} \pm z_b} \cos \gamma \quad (17)$$

In the initial state where $\phi_1 = \phi_2 = 0$, pitch point P_1 , which was used when x_1 axis was defined, coincides with this pitch point P_0 .

The method described in this chapter is the proposed synthesis method for tooth profile of a conical-external or -internal gear which engages with an involute cylindrical gear over the entire tooth width. If plus is chosen for the double signatures in the given equations, the curve of tooth profile is obtained for a pair of external gears; and if minus is chosen, the curve of tooth profile is obtained for a pair of internal

gears.

4. Ideal Curved Surface of Tooth Profile of a Conical Gear

As an example, the precise curved surfaces of tooth profile of conical-external and internal gears, which engage with a cylindrical-involute gear with module $m=1.0$, the pressure angle of basic rack $\alpha_c=20^\circ$, tooth number $z_b=17$, and tooth profile-shift coefficient $x_b=0.3$, are synthesized, and the characteristics of the obtained curved surfaces are investigated in the following sections.

4.1 Conical-external gear

4.1.1 Ideal curved surface of tooth profile

The ideal curved surface of tooth profile was synthesized for a conical-external gear with the following data: tooth number $z_a=12$, the tooth profile-shift coefficient on axis-perpendicular reference plane S_0 , $x_a=0$, and inclined angle $\gamma=20^\circ$. Figures 4(a), 4(b), and 4(c) show an example of the obtained curved surface of tooth profile. Figure 4(a) shows the involute tooth profile curve of cylindrical-external gear given on axis-perpendicular reference plane S_0 , the projection of the synthesized three-dimensional curved surface of conical-external gear on axis-perpendicular reference plane S_0 , and the line of contact. The bottom left of the tooth profile curve of conical-external gear corresponds to the tooth tip, the top right of the curve corresponds to the tooth root, and the synthesized curve touches the given involute curve at pitch point P_0 , which is indicated with a black filled circle. The line of contact is a slightly curved line, although it appears to be a straight line in the figure. Figure 4(b) shows the projections of the three synthesized tooth profile curves of conical-external gear on plane x_2z_2 : the curves were calculated for $z_1=0, \pm 0.020$ mm. They are all slightly curved and downward convex lines; the center curve, which corresponds to $z_1=0$, touches axis-perpendicular reference plane S_0 at pitch point P_0 . Figure 4(c) shows an oblique projection of the curved surface of tooth profile of conical-external gear which was synthesized by changing z_1 from -1.0 mm to 1.0 mm with an interval of 0.25 mm. Plane ABCD is the cylindrical-involute surface of the cylindrical-external gear; plane A'B'C'D' is the synthesized, slightly skewed and curved surface of tooth profile of conical external gear. As is shown in the figure, the simultaneous contact points are not the same point on the involute curve, but shift slightly according to the value of z_1 , and the positions and angles of the lines of contact also shift slightly accordingly, since each of the lines of contact intersects the involute curve at the simultaneous contact point at a right angle. Plane A''B''C''D'',

(a) Projection on x_0y_0 -plane (b) Projection on x_2y_2 -plane

(c) Oblique projection

Fig. 4 The calculated tooth profile of conical-external gear

which was obtained by connecting the lines of contact, hence became a slightly curved and skewed surface as shown in the figure.

4.1.2 The curve of tooth profile on a section perpendicular to the axis of cylindrical-external gear

(a) Calculation of the curve of tooth profile

Let section S denote an axis-perpendicular plane separated by z_{1s} from axis-perpendicular reference plane S_0 as shown in Fig. 5. The equation for calculat-

Fig. 5 Calculation of the tooth profile of conical gear on plane S perpendicular to the axis of cylindrical gear

ing the curve of tooth profile on section S is derived here. The equation for section S on coordinate system $O_2-x_2y_2z_2$ is given below.

$$z_{2s} = x_{2s} \tan \gamma + z_{1s} / \cos \gamma \quad (18)$$

Next, put $z_1 = z_{1s} - (i-1)\Delta z_1$, where, Δz_1 is the increase in coordinate z_1 , and $i=1, 2, \dots, n$. The precise curve of tooth profile, l_i , of conical-external gear is then derived for a given z_1 by the synthesis method for tooth profiles described in chapter 3 giving the coordinates of the involute tooth profile curve of the cylindrical-external gear. Searching the points which satisfy Eq. (18) on the obtained curve of tooth profile, l_i , gives Q_{i1} and Q_{i2} , which are the two intersections with section S, (one point if the curve touches the section). Here, let $O_{2s}-x_{2s}y_{2s}z_{2s}$ denote the rectangular coordinate system with the origin, O_{2s} , namely the intersection between section S and z_2 axis, which is located on section S. Converting the coordinates, (x_{2q}, y_{2q}, z_{2q}) , of these intersections into the coordinates, (x_{2s}, y_{2s}) , on rectangular coordinate system $O_{2s}-x_{2s}y_{2s}z_{2s}$ gives the following equation.

$$(x_{2s}, y_{2s}) = (x_{2q} / \cos \gamma, y_{2q}) \quad (19)$$

Connecting the coordinates, (x_{2s}, y_{2s}) , thus obtained for a constant of z_{1s} , gives the curve of tooth profile on section S.

Let P denote the operating pitch point between the conical and cylindrical-external gears on section S. The distance from z_1 axis to pitch point P, namely the radius of pitch circle of the cylindrical-external gear, r_{p1} , is given by the following equation.

$$r_{p1} = a_{ws} \frac{\pm z_b}{z_{ae} \pm z_b} \quad (20)$$

where, a_{ws} is the distance between the two axes on section S, and $a_{ws} = a_w + z_{1s} \tan \gamma$.

Figure 6 shows the curve of tooth profile thus obtained for the conical-external gear described previously. Figure 6(a) shows an example of an obtained tooth profile curve, which was derived for $z_{1s} = 2$ mm and $\Delta z_1 = 0.002$ mm. Δz_1 was, however, made smaller near pitch point P than in the other area. In

(a) $z_{1s} = 2$ mm (b) $z_{1s} = -2 \sim 4$ mm

Fig. 6 The calculated tooth profiles of conical gear on plane S, and superimposed involute curves ($m = 1.0$, $z_a = 12$, $z_b = 17$, $x_b = 0.3$, $\alpha_c = 20^\circ$, $\gamma = 20^\circ$)

the figure, the given curve of involute tooth profile of the cylindrical-external gear was rotated so that pitch point P could be placed on x_0 axis for clarity, and the obtained curve of the tooth profile of conical-external gear was also rotated by a corresponding angle, which was calculated from the reduction ratio of the pair of cylindrical and conical-external gears. As is shown, the obtained curve of tooth profile shown with a white circle becomes a convex curve, which is very similar to an involute curve, shown as a solid line, touching the involute tooth profile curve of the cylindrical-external gear at pitch point P. Figure 6(b) shows the curves of tooth profile of conical-external gear synthesized for $z_{1s} = -2, 0, 2, 4$ mm, which are superimposed on the axis-perpendicular reference plane S_0 . In this figure, the synthesized curves of tooth profile are shown while keeping the curve of involute tooth profile of the cylindrical-external gear fixed at the initial position. As is shown, the position of the line of contact, the position of pitch point, and the operating pressure angle, change markedly according to the change in the position of the section of the cylindrical-

Fig. 7 The definition of involute curve to be superimposed on calculated tooth profiles

external gear along the direction of tooth width, z_{1s} .

(b) Comparison with involute curve

The characteristics of the curve of tooth profile obtained on section S are investigated here by comparison with a plane involute curve.

First, the radius of base circle of the plane involute curve r_{g3} is given by using the equivalent tooth number of conical-external gear z_{ae} as follows.

$$r_{g3} = z_{ae} m \cos \alpha_c / 2 \quad (21)$$

where, the center of the base circle is O_{2s} .

Next, an arbitrary point, $Q(x_{2s}, y_{2s})$, on the curve of tooth profile obtained on section S is chosen as shown in Fig. 7. The angle θ_q of the starting point Q_0 of the involute curve is then obtained from the characteristics of involute curve as follows.

$$Q_q = \text{inv } \alpha_{2s} + \theta_{2s} \quad (22)$$

where,

$$\alpha_{2s} = \cos^{-1} \frac{r_{g3}}{r_{2s}}, \quad \theta_{2s} = \tan^{-1} \frac{y_{2s}}{|x_{2s}|}, \quad r_{2s} = \sqrt{x_{2s}^2 + y_{2s}^2} \quad (23)$$

The coordinates, (x_q, y_q) , of the starting point Q_0 are given as follows.

$$(x_q, y_q) = (r_{g3} \cos \theta_q, r_{g3} \sin \theta_q) \quad (24)$$

Next, calculating the coordinates, (x', y') , of point Q' , which is on the plane involute curve and is located at the distance of r' from the center O_{2s} of the base circle, gives the following equation.

$$(x', y') = (r' \cos \theta', r' \sin \theta') \quad (25)$$

where,

$$\theta' = \theta_q - \text{inv } \alpha', \quad \alpha' = \cos^{-1} \frac{r_{g3}}{r'} \quad (26)$$

As is shown in Fig. 6(b), the synthesized curves of tooth profile, shown with white circles, coincide well with the plane involute curves, shown as solid lines, obtained by substituting various values of r' into Eq. (25). For instance, the difference in the direction of y_{2s} between the two curves is $0.05 \mu\text{m}$ at point P, $0.83 \mu\text{m}$ at point M, and $1.78 \mu\text{m}$ at point N for $z_{1s} = 2$ mm. It is also seen that r_{g3} is constant, since all four involute curves obtained for $z_{1s} = -2, 0, 2, 4$ mm were

Table 1 The calculated profile-shift coefficients of conical-external gear, x_a , and the values back-calculated from the center-to-center distance a_{ws} , x'_a

z_{1s} (mm)	x_q (mm)	y_q (mm)	α_b (deg.)	α'_b (deg.)	x_a	x'_a	a_{ws} (mm)
-2	-5.999	-0.102	14.3571	14.3567	-0.6895	-0.6895	14.438
-1	-6.000	0.003	19.0999	19.0998	-0.3810	-0.3810	14.802
0	-5.998	0.133	22.7394	22.7394	0	0	15.166
1	-5.993	0.283	25.7554	25.7554	0.4377	0.4377	15.530
2	-5.983	0.448	28.3537	28.3537	0.9222	0.9222	15.894
3	-5.967	0.627	30.6463	30.6462	1.4465	1.4465	16.258
4	-5.944	0.813	32.7020	32.7019	2.0056	2.0055	16.622
5	-5.913	1.016	34.5673	34.5671	2.5952	2.5952	16.986
6	-5.873	1.223	36.2751	36.2749	3.2122	3.2121	17.350
7	-5.825	1.437	37.8500	37.8497	3.8537	3.8536	17.714
8	-5.766	1.565	39.3102	39.3104	4.5172	4.5173	18.078

calculated using the value of base circle r_{g3} given by Eq. (21).

(c) Axial distribution of profile-shift coefficient

Let x_a denote the profile-shift coefficient of the plane involute tooth profile of conical-external gear obtained on section S, and let α_b be the operating pressure angle. The following relation is obtained from the formula for profile-shifted spur gears⁽¹⁰⁾.

$$\text{inv } \alpha_b = 2 \tan \alpha_c \cdot \frac{x_a + x_b}{z_{ae} + z_b} + \text{inv } \alpha_c \quad (27)$$

where, pressure angle α_b is equal to θ_q obtained in the previous section, and is given as follows.

$$\text{inv } \alpha_b = \tan^{-1} \left(\frac{y_q}{|x_q|} \right) \quad (28)$$

The profile-shift coefficient, x_a , of conical-external gear is finally given from Eqs. (27) and (28) by the following equation.

$$x_a = \frac{z_{ae} + z_b}{2 \tan \alpha_c} (\text{inv } \alpha_b - \text{inv } \alpha_c) - x_b \quad (29)$$

Next, calculating the operating pressure angle α'_b from the distance a_{ws} between the axes of the two gears on section S gives the following equation.

$$\cos \alpha'_b = \frac{(z_{ae} + z_b) m \cos \alpha_c}{2 a_{ws}} \quad (30)$$

Here, let x'_a denote the profile-shift coefficient, which is obtained from Eq. (29) by using the pressure angle α'_b calculated with the above equation.

Table 1 shows the comparison of the profile-shift coefficients x_a and x'_a obtained by using α_b and α'_b calculated from Eqs. (28) and (30). This comparison clearly shows that $\alpha'_b \approx \alpha_b$, $x'_a \approx x_a$, and hence that the profile-shift coefficient x_a of conical-external gear can be calculated from Eq. (29) by using the operating pressure angle α'_b which is calculated from Eq. (30).

Figure 8 shows the change in the profile-shift coefficient, x_a , of conical-external gear along the direction of the axis of the cylindrical-external gear. This figure shows that the profile-shift coefficient, x_a , of conical-external gear, shown as a solid line,

Fig. 8 The calculated profile-shift coefficients of conical gears, x_a and x_d

changes non-linearly according to the increase in axial distance, z_{1s} , and is zero for $z_{1s}=0$ mm, since $x_a=0$ on the axis-perpendicular reference plane S_0 .

From the above investigations, the basic characteristics of the synthesized curve of tooth profile of conical external gear are summarized with respect to the axis-perpendicular plane of the cylindrical-external gear as follows: (1) the curve of tooth profile coincides with a plane involute curve, (2) the base circle coincides with the base circle of a gear with an equivalent tooth number, (3) the radius of base circle remains constant at any axial position, (4) the profile-shift coefficient at an arbitrary section can be calculated based on the assumption that the center-to-center distance of the cylindrical and conical external gears is equal to the distance between z_1 and z_2 axes, which are the center axis of the cylindrical gear on the axis-perpendicular plane and the center axis of conical-external gear, respectively, and (5) the profile-shift coefficient changes non-linearly along the axis of the cylindrical-external gear.

4.2 Conical-internal gear

4.2.1 Ideal curved surface of tooth profile

The precise curved surface of tooth profile was similarly synthesized for a conical-internal gear with the following data: tooth number z_d is 42, profile-shift coefficient x_d is zero on the axis-perpendicular reference plane S_0 , and inclined angle γ is 20° . Figure 9 shows an example of the results obtained. Figure 9 (a) shows the given curve of involute tooth profile of the cylindrical-external gear, the projection of the synthesized and three-dimensional curved surface of tooth profile of the conical-internal gear on the axis-perpendicular reference plane S_0 , and the corresponding line of contact.

Figure 9(b) shows the projections of the three curves, which were synthesized for $z_1=0$ and ± 0.020 mm, on x_2z_2 plane. The center curve of tooth profile obtained for $z_1=0$ touches the axis-perpendicular

(a) Projection on x_0y_0 -plane (b) Projection on $x_2'y_2'$ -plane

(c) Oblique projection

Fig. 9 The calculated tooth profile of conical-internal gear

reference plane S_0 at pitch point P_0 .

Figure 9(c) shows the oblique projections of the curves of tooth profile of the conical-internal gear, which were calculated by changing z_1 from -1.0 mm to 1.0 mm with an interval of 0.25 mm. In the figure, plane ABCD is the curved surface of involute tooth profile of the cylindrical-external gear, plane A'B'C'D' is the synthesized and slightly skewed curved surface of tooth profile of the conical internal gear, and plane A''B''C''D'' is the slightly skewed operating surface of contact.

4.2.2 Curve of tooth profile on plane perpendicular to axis of cylindrical-external gear Figure 10 shows the curves of tooth profile of the conical-inter-

(a) $z_{1s} = 2$ mm (b) $z_{1s} = 0 \sim 6$ mmFig. 10 The calculated tooth profiles of conical-internal gear on plane S, and superimposed involute curves ($m=1.0$, $z_b=17$, $z_d=42$, $x_b=0.3$, $\gamma=20^\circ$)

nal gear, which were obtained by the same method as for the conical external gear. Figure 10(a) shows the results obtained for $z_{1s}=2$ mm, $\Delta z_1=0.002$ mm. As shown in the figure, the obtained curve of tooth profile, shown with white circles, becomes a concave curve, shown as a solid line, similar to an involute curve which touches the involute curve of tooth profile of the cylindrical-external gear at pitch point P. Figure 10(b) shows the curves of tooth profiles of the conical internal gear (white circles), which were synthesized for $z_{1s}=0, 2, 4, 6$ mm, and superimposed on the axis-perpendicular reference plane S_0 . It is shown in the figure that the position of the line of contact, the position of pitch point, and the operating pressure angle change markedly according to the change of the position, z_{1s} , of the section along the direction of the tooth width of the involute external gear.

Next, the plane involute curves obtained by a method similar to that for the conical-external gear using the same tooth number are superimposed on the plane involute curves shown with white circles in Fig. 10(b). As shown in this figure, these two curves coincide well with each other.

Next, the profile-shift coefficient, x_d , of the plane involute curve of the conical-internal gear on section S is given by the following equation in the same way

Table 2 The calculated profile-shift coefficients of conical-internal gear, x_d , and the values back-calculated from the center-to-center distance a_{ws} , x'_d

z_{1s} (mm)	x_g (mm)	y_g (mm)	α_b (deg.)	α'_b (deg.)	x_d	x'_d	a_{ws} (mm)
-1	-21.000	0.069	8.3032	8.3032	-0.2281	-0.2281	13.150
0	-20.999	0.147	15.6627	15.6627	0	0	13.514
1	-20.998	0.261	20.3450	20.3450	0.3309	0.3309	13.878
2	-20.996	0.398	23.9846	23.9846	0.7334	0.7334	14.242
3	-20.993	0.555	27.0150	27.0150	1.1922	1.1922	14.606
4	-20.987	0.728	29.6310	29.6310	1.6975	1.6975	14.970
5	-20.980	0.914	31.9407	31.9407	2.2425	2.2425	15.334
6	-20.971	1.112	34.0119	34.0119	2.8220	2.8220	15.698
7	-20.958	1.320	35.8906	35.8906	3.4318	3.4318	16.062
8	-20.944	1.538	37.6097	37.6097	4.0687	4.0687	16.426

as Eq.(29).

$$x_d = \frac{z_{de} - z_b}{2 \tan \alpha_c} (\text{inv } \alpha_b - \text{inv } \alpha_c) + x_b \quad (31)$$

Calculating the operating pressure angle, α'_b , from the center-to-center distance, a_{ws} , between the two cylindrical and conical gears on section S in a manner similar to that for the conical-external gear gives the following equation.

$$\cos \alpha'_b = \frac{(z_{de} - z_b)m \cos \alpha_c}{2a_{ws}} \quad (32)$$

Let x'_d denote the profile-shift coefficient obtained from Eq.(31) using the operating pressure angle, α'_b , calculated by the above equation. A comparison of the profile-shift coefficients, x_d and x'_d , obtained using the equations of α_b and α'_b , is shown in Table 2. This result clearly shows that $\alpha'_b \approx \alpha_b$, $x'_d \approx x_d$, and hence that the profile-shift coefficient, x_d , of conical-internal gear can be calculated from Eq.(31) using the pressure angle, α'_b , which is calculated from Eq.(32).

Figure 8 shows the change in the tooth profile-shift coefficient, x_d , as a broken line: the change was calculated in a manner similar to that for the conical-external gear. From the figure it is clear that the tooth profile-shift coefficient, x_d , of the conical-internal gear is zero for $z_{1s} = 0$ mm, and increases non-linearly according to the increase in axial distance z_{1s} .

From the above investigations, it has become clear that the curve of tooth profile of conical-internal gear with respect to the axis-perpendicular plane (section S) of cylindrical-external gear has the same characteristics as that of the conical external gear in section 4.1; in other words: (1) the curve of tooth profile coincides well with a plane involute curve, (2) the base circle coincides with the base circle of a gear with an equivalent tooth number, (3) the radius of base circle remains constant at any axial position, (4) the profile-shift coefficient at an arbitrary section can be calculated based on the assumption that the center-to-center distance of the cylindrical and conical-external gears is equal to the distance between z_1

and z_2 axes, which are the center axis of the cylindrical gear on the axis-perpendicular plane and the center axis of the conical-external gear, respectively, and (5) the profile-shift coefficient changes non-linearly along the axis of the cylindrical-external gear.

5. Conclusions

The authors have proposed a new synthesis method for tooth profiles using vectors to synthesize the ideal curves of tooth profile of conical-external and -internal gears which engage with a cylindrical-external gear over the entire tooth width, and have investigated the characteristics of the synthesized curves of tooth profile with respect to a section perpendicular to the axis of the cylindrical-external gear. The results obtained are as follows.

(1) The ideal curve of tooth profile of a conical-external or -internal gear, which engages with a cylindrical-external gear over the entire tooth width, can be precisely derived using the proposed synthesis method for tooth profile.

(2) The obtained curved surface of conical gear is slightly skewed; the simultaneous contact lines are not parallel to the tooth trace, and shift little by little; and the curved surface of contact is a slightly curved and skewed plane.

(3) The curves of tooth profile of conical-external and -internal gears with respect to an axis-perpendicular plane of cylindrical-external gear coincides well with a plane involute curve with the base circle determined using the equivalent tooth number, the module, and the pressure angle of basic rack. Also, the radius of this base circle does not change with respect to the axial position of base circle and remains at a constant value.

(4) The profile-shift coefficients of conical-external and -internal gears coincide well with the values, which are reversibly calculated from the distance between the axes of conical and cylindrical gears, taking the equivalent tooth number and the coefficient of the increase in center-to-center distance into account. Also, this profile-shift coefficient changes non-linearly along the axial direction of cylindrical-external gear.

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Appendix

A1. Tooth surface normal vector $n_{1,1}$

Let a tangent vector normal to the normal vector be $t_{1,1}$, a vector in the direction of z_1 axis, and let the unit vectors of rectangular coordinate system O_1 -

$x_1y_1z_1$ be i, j, k . The tooth surface normal vector $n_{1,1}$ at contact point K_1 is given by the following equation.

$$n_{1,1} = t_{1,1} \times l_{1,1} = \begin{vmatrix} i & j & k \\ \frac{\partial x_1}{\partial \alpha} & \frac{\partial y_1}{\partial \alpha} & \frac{\partial z_1}{\partial \alpha} \\ \frac{\partial x_1}{\partial z_1} & \frac{\partial y_1}{\partial z_1} & \frac{\partial z_1}{\partial z_1} \end{vmatrix} = \left\{ \frac{\partial y_1}{\partial \alpha}, -\frac{\partial x_1}{\partial \alpha}, 0 \right\}^T \quad (A1)$$

In the above equation, the equation of component expression has been obtained by expanding under the consideration that $\partial x_1/\partial z_1 = \partial y_1/\partial z_1 = 0$ among the elements of the determinant of Eq. (A1).

A2. Transforming matrices between coordinate systems

The transforming matrices with respect to rotation are shown below.

(1) Transforming matrix $[T]_{01}$ from static coordinate system $O-x_0y_0z_0$ to moving coordinate matrix $O_1-x_1y_1z_1$, and the reverse transforming matrix $[T]_{10}$.

$$[T]_{01} = \begin{bmatrix} \cos \phi_1 & \sin \phi_1 & 0 \\ -\sin \phi_1 & \cos \phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (A2)$$

$$[T]_{10} = [T]_{01}^T \quad (A3)$$

(2) Transforming matrix $[T]_{02}$ from static coordinate system $O-x_0y_0z_0$ to moving matrix $O_2-x_2y_2z_2$, and the reverse transforming matrix $[T]_{20}$.

$$[T]_{02} = \begin{bmatrix} \cos \phi_2 & \sin \phi_2 & 0 \\ -\sin \phi_2 & \cos \phi_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \gamma & 0 & -\sin \gamma \\ 0 & 1 & 0 \\ \sin \gamma & 0 & \cos \gamma \end{bmatrix} \quad (A4)$$

$$[T]_{20} = [T]_{02}^T \quad (A5)$$

(3) Transforming matrix $[T]_{12}$ from moving coordinate system $O_1-x_1y_1z_1$ to moving coordinate system $O_2-x_2y_2z_2$, and the reverse transforming matrix $[T]_{21}$.

$$[T]_{12} = [T]_{02} [T]_{10} \quad (A6)$$

$$[T]_{21} = [T]_{01} [T]_{20} \quad (A7)$$

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