The Ideal Tooth Profiles of Conical-External and -Internal Gears Meshing with Cylindrical-Involute Gears over the Entire Tooth Width*

Kohei HORI**, Iwao HAYASHI***
and Nobuyuki IWATSUKI***

A new synthesis method for tooth profiles using vector equations is proposed to precisely derive the curve of tooth profile of conical-external and -internal gears which engage a cylindrical-involute gear over the entire tooth width. As an example, the ideal tooth profiles of conical-external and -internal gears which engage with an inclined cylindrical-involute gear over the entire tooth width have been derived, and their characteristics with respect to a plane perpendicular to the axis of the inclined cylindrical gear have been investigated. The results obtained are as follows. (1) The tooth profiles of conical-external and -internal gears favorably coincide with plane involute curves. (2) The radius of the base circle is given by using the equivalent tooth number of the conical-external or -internal gear. (3) The profile-shift coefficient is given by the center-to-center distance between the inclined involute gear and the conical-external gear, or between the cylindrical gear and the conical-internal gear. (4) The profile-shift coefficient of conical-external or -internal gear changes nonlinearly along the axis of the cylindrical-involute gear.

Key Words: Machine Element, Gear, Design, Involute Conical Gear, Tooth Profile Synthesis, Planetary Gear Drive, Mechanical Paradox Gear, Profile-Shift Coefficient

1. Introduction

Traditional 3K-type mechanical paradox planetary gear drives, which are composed of cylindrical gears only, have a high reduction ratio and high efficiency\(^{(1)-(9)}\). In the field of industrial robotics, however, gear drives with a low reduction ratio of 1/30 to 1/60 and with high efficiency are now required to drive industrial robots at high speed. The authors have hence proposed a new mechanical paradox planetary gear drive in which the cylindrical planet gears are inclined at an angle and are engaged with a conical sun gear and two conical internal gears so that the difference between the tooth numbers of the two internal gears can be increased, thereby obtaining a low reduction ratio\(^{(10)}\). This new mechanical paradox planetary gear drive has many advantages, such as high efficiency\(^{(10)}\), adjustable backlash, and low noise since the tooth engagement is similar to that of helical gears\(^{(10)}\).

This gear drive, however, has the disadvantage of a small load capacity because the tooth engagements between the cylindrical planet gears and the conical-internal gears are point contact\(^{(17)}\). It is therefore an important issue to realize line contact tooth engagement. The authors hence attempted to cut and generate a conical gear by moving the cutting hob not along a straight line, as in the traditional method, but along a circle of large radius so that the conical gear could
engage with a cylindrical gear over the entire tooth width. However the proposed method could not be evaluated adequately, since it was not theoretically derived by taking line contact conditions into consideration, and the experimental inspection of line contact was also very difficult.

On the other hand, a synthesis method for the three-dimensional curved surface of tooth profile has already been proposed to derive the tooth profiles of special skew gears, taking line contact conditions into consideration. But this method has not been applied to other cases.

In this paper, the authors propose a new synthesis method for tooth profile using vector equations by which the curved surface of tooth profile of one gear can easily be derived from the given curved surface of tooth profile of the other mating gear, derive the precise curved surfaces of tooth profile of conical-external and -internal gears which engage with a cylindrical-involute external gear over the entire tooth width, and discuss the basic characteristics of the synthesized curves with respect to a section perpendicular to the axis of the cylindrical involute external gear.

2. Mechanical Paradox Planetary Gear Drive with Inclined Planet Gears

2.1 Construction of Inclined Planet Gears

Figure 1 shows the construction of a mechanical paradox planetary gear drive which is composed of three inclined cylindrical-involute planet gears, a conical sun gear fixed on the input axle, a fixed conical-internal gear, and a rotatory conical-internal gear fixed on the output axle. The sun and the two internal gears simultaneously engage with the three planet gears. The planet gear axis crosses the center axis of the reduction gear drive; the center-to-center distances between the planet and sun gears, between the planet and fixed internal gears, and between the planet and rotatory internal gears hence vary linearly along the center axis of the reduction gear drive.

2.2 Characteristics of the Reduction Gear Drive

This reduction gear drive can provide a lower reduction ratio because the diameters of the two internal gears are different due to the inclination of planet gears, and the difference between their tooth numbers can then be increased by more than three, which is the difference in tooth number for traditional mechanical paradox planetary gear drives composed of cylindrical-involute gears only. For instance, suppose that the number of planet gears is three, the difference between the tooth numbers of two internal gears is two times larger than three, namely six, and the tooth numbers of sun and two internal gears are equal to an integral value times the number of planet gears. The assembly conditions of the gear drive can thereby be satisfied since the tooth phases and tooth spaces coincide with each other for the engagement of a pair of planet-sun gears, a pair of planet-fixed internal gears, and a pair of planet-rotatory internal gears, respectively. The reduction ratio, in this case, is one half the value of a traditional mechanical paradox planetary gear drive composed of cylindrical-involute gears only, and also can be lower than one sixtieth. Other advantages such as high efficiency, adjustable backlash, and low noise can be expected.

For the practical application of reduction gear drives which have the above-mentioned advantages, it is necessary to make the teeth of the gears contact on a line on the tooth surfaces, so that the gears can engage with each other over the entire tooth width and thus can have a sufficiently large load capacity for practical use. In chapter 3, the authors propose a new synthesis method for tooth profiles to synthesize the precise tooth profile curves of conical-external and -internal gears, and, in chapter 4, synthesize them and investigate their characteristics.


3.1 Coordinate systems and definitions of vectors for describing points on tooth surfaces

Figure 2 shows the three right-hand rectangular coordinate systems defined as follows. (1) Static coordinate system $O-x_0y_0z_0$: $z_0$ axis coincides with the axis of cylindrical-external gear; an arbitrary point on $z_0$ axis is origin O. A plane which passes through origin O and is perpendicular to $z_0$ axis is designated as the axis-perpendicular reference plane $S_0$. $x_0$ axis is $OO_2$ axis, where $O_2$ is the intersection between the axis-perpendicular reference plane and the axis of conical gear. (2) Moving coordinate system $O_1-x_1y_1z_1$: the coordinate system fixed on the cylindrical gear. Origin $O_1$ is the same as origin O defined above, and $z_1$ axis is also the same as $z_0$ axis defined above. In the initial state, $x_1$ and $y_1$ axes are superimposed on...
Fig. 2 The coordinate system for the tooth profile synthesis of cylindrical and conical-external and -internal gears

{x₀, y₀} axes, respectively. (3) Moving coordinate system O₀-x₀y₀z₀: the coordinate system fixed on the conical gear. The origin is the intersection, O₀, defined above. z₀ axis is the axis of conical gear which is inclined by an angle γ to the axis of the cylindrical gear. In the initial state, x₀ axis is located in plane x₀z₀.

The position vectors of a point on a tooth surface are here defined. Let K₁ denote a point on the tooth surface of cylindrical-external gear, also let r₁,1 denote the position vector of point K₁, and let r₁,1 = (x₁, y₁, z₁)ᵀ, which is described on moving coordinate system O₁-x₁y₁z₁, from origin O₁. The first subscript of position vector r indicates the gear (1: cylindrical gear, 2: conical gear) and the second subscript indicates the coordinate on which the position vector is described (1: moving coordinate system O₁-x₁y₁z₁, 2: moving coordinate system O₀-x₀y₀z₀). The superscript T in the component expression of a vector indicates the transpose of a vector. Let r₁,0 denote the position vector of the same point K₁, also let r₁,0 = (x₀, y₀, z₀)ᵀ, which is described on static coordinate system O₀-x₀y₀z₀ from origin O₀.

Next, let K₂ denote a point on the tooth surface of conical gear; let r₁,2 denote the position vector of point K₂; let r₁,2 = (x₂, y₂, z₂)ᵀ, which is described on moving coordinate system O₂-x₂y₂z₂, from origin O₂; and also let r₂,1 denote the position vector of this point K₂, which is described on moving coordinate system O₁-x₁y₁z₁, from origin O₁.

3.2 Equation of tooth engagement and its solution

Suppose that the cylindrical gear (angular velocity ω₁) and the conical gear (angular velocity ω₂) engage with each other and rotate with a reduction ratio ω = ω₁/ω₂, and also that point K₁ contacts point K₂ at point K when the cylindrical gear rotates by angle φ₁ and the conical gear rotates by the corresponding angle φ₂. In this case, the normal-to-tooth surface component of the relative velocity of the mating cylindrical and conical gears should be zero, since their tooth surfaces contact and rotate without separation. Consideration of this condition on moving coordinate O₀-x₀y₀z₀, gives the following equation of tooth engagement.

\[ (v_{1,1} - v_{1,1}) \cdot \hat{r}_{1,1} = 0 \]  (1)

where \( v_{1,1} \) and \( v_{1,1} \) are the velocity vectors of points K₁ and K₂, respectively, which are described on moving coordinate O₀-x₀y₀z₀; \( \hat{r}_{1,1} \) is the normal vector on the tooth surface of cylindrical gear at point K₁ (refer to Appendix A1); and indicates the inner product of vectors. \( v_{1,1} \) and \( v_{1,1} \) are given as follows.

\[ v_{1,1} = \omega_{1,1} \times \hat{r}_{1,1} \]
\[ v_{2,1} = [T]_{21} v_{2,1} = \omega_{2,1} \times \hat{r}_{2,1} \]  (2)

where, \([T]_{21}\) is the transformation matrix from moving coordinate system O₁-x₁y₁z₁ to moving coordinate system O₀-x₀y₀z₀ (refer to Appendix A2); \( \omega_{1,1} \) is the velocity vector of point K₁ described on moving coordinate system O₀-x₀y₀z₀, and \( \omega_{2,1} = \{0, 0, \omega_{2}\}^{T} \); \( \omega_{2,1} = \{0, 0, \omega_{2}\}^{T} \); and \( \times \) indicates the outer product of vectors.

Next, let \( r_{n,0} \) denote the position vector of point O₂, and \( r_{n,0} = \{a_n, 0, 0\}^{T} \), which is described on static coordinate system O₀-x₀y₀z₀ from origin O₀, where \( a_n \) is the center-to-center distance of the two mating gears on axis-perpendicular reference plane S₀. The position vector, \( r_{n,1} \), of point K₂ is given as follows.

\[ r_{n,1} = [T]_{10} [T]_{01} r_{n,0} \]  (3)

where, \([T]_{10}\) and \([T]_{01}\) are the transformation matrices from moving coordinate system O₁-x₁y₁z₁ to moving coordinate system O₀-x₀y₀z₀ and from static coordinate system O₀-x₀y₀z₀ to moving coordinate system O₁-x₁y₁z₁, respectively (refer to Appendix A2). Since points K₁ and K₂ contact each other, the following relation is applied.

\[ r_{n,1} = r_{n,2} \]  (4)

Substituting this relation into Eq.(3) gives position vector \( r_{n,2} \) in the following form.

\[ r_{n,2} = [T]_{10} [T]_{01} r_{n,0} \]  (5)

Substituting Eq.(5) into the second equation of Eq.(2), substituting Eq.(2) into Eq.(1), and rearranging the results by using the normal vectors and also the component expressions of transformation matrices shown in Appendices A1 and A2 gives the following equations which have φ₁ as an unknown variable.

\[ -\omega_{1} y_{1} + \omega_{2} z_{2} \sin \gamma \sin \phi_{1} \]
\[ + \omega_{2} (y_{1} + a_{n,0} \sin \phi_{1}) \cos \gamma \frac{d \gamma}{d a} \]
\[ + [\omega_{1} x_{1} + \omega_{2} z_{2} \sin \gamma \cos \phi_{1}] \frac{d \omega_{2}}{d a} \]
\[ - \omega_{2} (x_{1} - a_{n,0} \cos \phi_{1}) \cos \gamma \left( \frac{d x_{1}}{d a} \right) = 0 \]  (6)

Here, however, it is assumed that reduction ratio \( \omega \) is given previously, and that the components, \( x_{1}, y_{1}, \) and \( z_{1} \), of the position vector of point K₁, which is a point
on the curve of tooth profile of cylindrical gear, are also given previously. Dividing Eq. (6) by $\omega$ and by $-\frac{dx}{da}$, and rearranging the results gives the following equation,

$$-D \sin \phi_i + E \cos \phi_i + F = 0$$  \hspace{1cm} (7)

where,

$$D = a_w \cos \gamma + z_0 \sin \gamma \ \frac{dy_1}{dx_1}$$
$$E = a_w \cos \gamma + z_0 \sin \gamma$$
$$F = \left( x_1 + y_1 \frac{dy_1}{dx_1} \right) \left( 1 - u \cos \gamma \right)$$

Transferring the second term, $E \cos \phi_i$, on the left-hand side of Eq. (7) to the right-hand side and squaring both sides of the equation gives the following quadratic equation with respect to $\sin \phi_i$.

$$(D^2 + E^2) \sin^2 \phi_i - 2DF \sin \phi_i + F^2 - E^2 = 0$$  \hspace{1cm} (8)

Solving this equation and deriving $\phi_i$ gives the following equation.

$$\phi_i = \sin^{-1} \left( \frac{DF \pm \sqrt{(DF)^2 - (D^2 + E^2)(F^2 - E^2)}}{D^2 + E^2} \right)$$  \hspace{1cm} (9)

Next, substituting $\phi_i$ obtained from Eq. (9) and $\phi_i = u \phi_i$ into Eq. (5) gives the position vector, $\mathbf{r}_{2x}$, of point $\mathbf{K}_5$, which is a point on the tooth surface of conical gear, as follows.

$$\mathbf{r}_{2x} = \left( x_0, y_0, z_0 \right)^T$$

$$\begin{bmatrix} x_0 - a_w \\ y_0 \\ z_0 \end{bmatrix} \cos \gamma \cos \phi_i + y_0 \sin \phi_i$$

$$\begin{bmatrix} x_0 - a_w \\ y_0 \\ z_0 \end{bmatrix} \cos \gamma \sin \phi_i$$

$$\begin{bmatrix} x_0 - a_w \\ y_0 \\ z_0 \end{bmatrix} \sin \gamma + z_0 \cos \gamma$$

where, $x_0$, $y_0$, and $z_0$ are the components of position vector $\mathbf{r}_{2x}$. $\mathbf{r}_{2x}$ is given by transforming $\mathbf{r}_{1x}$ from moving coordinate system $O_{1x} - x_{1y}, y_{1y}, z_{1y}$ to static coordinate system $O - x_0, y_0, z_0$ as follows.

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = [T]_{10} \mathbf{r}_{1x}$$  \hspace{1cm} (11)

where, $[T]_{10}$ is the transforming matrix (refer to Appendix A2). Calculating $\mathbf{r}_{2x}$ and representing the obtained result in the form of component expression gives the following equation.

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cos \phi_i - y_1 \sin \phi_i$$

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \sin \phi_i + y_1 \cos \phi_i$$

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \sin \gamma + z_1 \cos \gamma$$

3.3 Ideal curved surface of tooth profile of conical gear

(1) Curved surface of tooth profile of conical gear

The center-to-center distance of the cylindrical and conical gears, $O$ to $O_0$, on axis-perpendicular reference plane $S_0$ is given by the following equation.

$$a_w = \left( \frac{2a_e + z_0}{2} + y_0 \right) m$$  \hspace{1cm} (13)

where, $a_{we}$ is the equivalent tooth number of conical gear, and $2a_e = a_{e}/\cos \gamma$, $z_0$ is the tooth number of conical gear, $z_0$ is the tooth number of cylindrical gear, $y_0$ is the coefficient of center-to-center distance increase, and $m$ is module. For the double signature in the equation, plus should be chosen for the pair of external gears and minus should be chosen for the pair of internal gears (similar hereinafter). $y_0$ in Eq. (13) is given by the following equation.

$$y_0 = \frac{a_{we} + z_0}{2} \left( \frac{\cos \alpha_c}{\cos \alpha_{bo}} - 1 \right)$$  \hspace{1cm} (14)

where, $\alpha_c$ is the pressure angle of basic rack; $\alpha_{bo}$ is the operating pressure angle, and is calculated by substituting zero into $x_0$, which is the profile-shifted coefficient of conical gear on the axis-perpendicular reference plane, in other words, $x_{bo}=0$, and also substituting the previously obtained value of the tooth profile-shift coefficient of cylindrical gear, $x_{0}$, into the following equation.

$$\text{inv} \ a_{bo} = 2 \tan \alpha_c \frac{x_{0}}{z_{0}} + \text{inv} \ \alpha_c$$

Next, let $P_1$ denote the pitch point of involute tooth profile curve on the plane which passes through the origin, $O$, of moving coordinate system $O_{1x} - x_{1y}, y_{1y}, z_{1y}$ and is perpendicular to $z_1$ axis as shown in Fig. 3, let $O_{1x}P_1$ be $x_1$ axis, and let $r_1$ be the distance from point $K_1$ to $z_1$ axis. If coordinate $z_1$ is given, the position vector, $\mathbf{r}_{1x}$, of point $K_1$ and the value of $(dy_{1x}/dx_{1x})$ at this point are given by the following equations.

$$\mathbf{r}_{1x} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \pm r_1 \cos \left( \text{inv} \ a - \text{inv} a_{bo} \right) \\ \pm r_1 \sin \left( \text{inv} \ a - \text{inv} a_{bo} \right) \\ z_1 \end{bmatrix}$$

$$\frac{dy_1}{dx_1} = \tan \left( \frac{r_{11}^2 - r_{01}^2}{r_{01}} \right)$$

where, $r = \cos^{-1} \left( r_{01}/r_1 \right)$, and $r_{01}$ is the radius of base circle of cylindrical gear. Calculating rotational angle $\phi_i$ from Eq. (9) and using Eq. (10) gives the position vector, $\mathbf{r}_{2x}$, of the corresponding point $K_2$ on the tooth surface of conical gear, where $\mathbf{r}_{2x} = \left( x_2, y_2, z_2 \right)^T$.

The coordinates, $(x_1, y_1, z_1)$, of a point on the

![Fig. 3 Position vector $\mathbf{r}_{1x}$ and normal vector $\mathbf{n}_{1x}$ on a cylindrical-involute surface](image-url)
curved surface of tooth profile of cylindrical-involute gear, and the coordinates, \((x_2, y_2, z_2)\), of the corresponding point on the curved surface of tooth profile of conical gear are thus determined. The precise curved surface of tooth profile of conical gear can then be obtained by changing the value of radius \(r_1\) from the radius of base circle, \(r_{bas}\), to the desired length, and also by changing the value of coordinate, \(z_1\), from one end to the other of tooth width successively.

(2) Simultaneous contact line

The simultaneous contact line on the tooth surface of cylindrical gear is obtained by connecting the coordinates, \((x_1, y_1, z_1)\), which have the same rotational angle, \(\phi_x\). The corresponding simultaneous contact line on the tooth surface of conical gear is also obtained by connecting the coordinates, \((x_2, y_2, z_2)\), which have the same rotational angle, \(\phi_2 (=u\phi_x)\).

(3) Curved surface of contact

Let \(r_{1,1}\) denote the position vector of contact point \(K_1\) which corresponds to a constant coordinate value, \(z_1\), and also let \(r_{1,0}\) denote the position vector obtained by representing \(r_{1,1}\) on the static coordinate system (refer to Eqs. (11) and (12)). Position vector \(r_{1,0}\) then represents the line of contact of the engaging pair of conical- and cylindrical-gears, since the line of contact is the trajectory of contact point, which appears on the static coordinate system due to the rotation of the pair of gears. The curved surface of contact of the engaging pair of conical- and cylindrical-gears is thus obtained by calculating the line of contact which corresponds to each coordinate value \(z_1\), from one end to the other end of tooth width successively and then connecting the obtained lines of contact.

(4) Definition of pitch point

Here, it is defined that the point which is distant \(r_{p0}\) by from origin \(O\) on \(z_0\) axis is the pitch point, \(P_0\), of the engaging pair of conical and cylindrical gears. \(r_{p0}\), \(r_{p10}\) and \(r_{p20}\), where \(r_{p10}\) is the length of the perpendicular segment from pitch point \(P_0\) on \(z_0\) axis, are given by the following equations.

\[
r_{p10} = a_w \frac{\pm z_0}{z_{ax} \pm z_0}, \quad r_{p20} = a_w \frac{z_{pe}}{z_{ax} \pm z_0} \cos \gamma
\]

In the initial state where \(\phi_1 = \phi_2 = 0\), pitch point \(P_1\) which was used when \(x_1\) axis was defined, coincides with this pitch point \(P_0\).

The method described in this chapter is the proposed synthesis method for tooth profile of a conical-external or -internal gear which engages with an involute cylindrical gear over the entire tooth width. If plus is chosen for the double signatures in the given equations, the curve of tooth profile is obtained for a pair of external gears; and if minus is chosen, the curve of tooth profile is obtained for a pair of internal gears.

4. Ideal Curved Surface of Tooth Profile of a Conical Gear

As an example, the precise curved surfaces of tooth profile of conical-external and internal gears, which engage with a cylindrical-involute gear with module \(m = 1.0\), the pressure angle of basic rack \(\alpha_r = 20^\circ\), tooth number \(z_a = 17\), and tooth profile-shift coefficient \(x_0 = 0.3\), are synthesized, and the characteristics of the obtained curved surfaces are investigated in the following sections.

4.1 Conical-external gear

4.1.1 Ideal curved surface of tooth profile

The ideal curved surface of tooth profile was synthesized for a conical-external gear with the following data: tooth number \(z_a = 12\), the tooth profile-shift coefficient on axis-perpendicular reference plane \(S_0\), \(x_a = 0\), and inclined angle \(\gamma = 20^\circ\). Figures 4(a), 4(b), and 4(c) show an example of the obtained curved surface of tooth profile. Figure 4(a) shows the involute tooth profile curve of cylindrical-external gear given on axis-perpendicular reference plane \(S_0\), the projection of the synthesized three-dimensional curved surface of conical-external gear on axis-perpendicular reference plane \(S_0\), and the line of contact. The bottom left of the tooth profile curve of conical-external gear corresponds to the tooth tip, the top right of the curve corresponds to the tooth root, and the synthesized curve touches the given involute curve at pitch point \(P_0\), which is indicated with a black filled circle. The line of contact is a slightly curved line, although it appears to be a straight line in the figure. Figure 4(b) shows the projections of the three synthesized tooth profile curves of conical-external gear on plane \(x_2z_2\): the curves were calculated for \(z_1 = 0, \pm 0.020\) mm. They are all slightly curved and downward convex lines; the center curve, which corresponds to \(z_1 = 0\), touches axis-perpendicular reference plane \(S_0\) at pitch point \(P_0\). Figure 4(c) shows an oblique projection of the curved surface of tooth profile of conical-external gear which was synthesized by changing \(z_1\) from \(-1.0\) mm to \(1.0\) mm with an interval of \(0.25\) mm. Plane ABCD is the cylindrical-involute surface of the cylindrical-external gear; plane \(A'B'C'D'\) is the synthesized, slightly skewed and curved surface of tooth profile of conical external gear. As is shown in the figure, the simultaneous contact points are not the same point on the involute curve, but shift slightly according to the value of \(z_1\), and the positions and angles of the lines of contact also shift slightly accordingly, since each of the lines of contact intersects the involute curve at the simultaneous contact point at a right angle. Plane \(A'B'C'D'\),
which was obtained by connecting the lines of contact, hence became a slightly curved and skewed surface as shown in the figure.

4.1.2 The curve of tooth profile on a section perpendicular to the axis of cylindrical-external gear

(a) Calculation of the curve of tooth profile

Let section S denote an axis-perpendicular plane separated by \( z_s \) from axis-perpendicular reference plane \( S_0 \) as shown in Fig. 5. The equation for calculating the curve of tooth profile on section S is derived here. The equation for section S on coordinate system \( O_x-x_y-y_z \) is given below:

\[
z_{2s} = z_{1s} \tan \gamma + z_{bs} \cos \gamma
\]  

Next, put \( z_i = z_{is} - (i-1)\Delta z_i \) where, \( \Delta z_i \) is the increase in coordinate \( z_i \) and \( i=1, 2\cdots, n \). The precise curve of tooth profile, \( l_i \), of conical-external gear is then derived for a given \( z_i \) by the synthesis method for tooth profiles described in chapter 3 giving the coordinates of the involute tooth profile curve of the cylindrical-external gear. Searching the points which satisfy Eq. (18) on the obtained curve of tooth profile, \( l_i \), gives \( Q_{1i} \) and \( Q_{2i} \) which are the two intersections with section S, (one point if the curve touches the section). Here, let \( O_{x_{is}}-x_{ys}-y_{zs} \) denote the rectangular coordinate system with the origin, \( O_{x_{is}} \), namely the intersection between section S and \( z_s \) axis, which is located on section S. Converting the coordinates, \( (x_{is}, y_{is}, z_{is}) \), of these intersections into the coordinates, \( (x_{es}, y_{es}) \), on rectangular coordinate system \( O_{x_{es}}-x_{es}-y_{es}-z_{es} \) gives the following equation.

\[
(x_{es}, y_{es}) = \left( \frac{x_{is}}{ \cos \gamma}, y_{is} \right)
\]  

Connecting the coordinates, \( (x_{es}, y_{es}) \), thus obtained for a constant of \( z_{es} \), gives the curve of tooth profile on section S.

Let P denote the operating pitch point between the conical and cylindrical-external gears on section S. The distance from \( z_s \) axis to pitch point P, namely the radius of pitch circle of the cylindrical-external gear, \( r_{pi} \), is given by the following equation.

\[
r_{pi} = \frac{a_{o∪} - \frac{z_{bs}}{2}}{a_{o∪} + \frac{z_{bs}}{2}}
\]  

where, \( a_{o∪} \) is the distance between the two axes on section S, and \( a_{o∪} = a_{bs} + z_{bs} \tan \gamma \).

Figure 6 shows the curve of tooth profile thus obtained for the conical-external gear described previously. Figure 6(a) shows an example of an obtained tooth profile curve, which was derived for \( z_s = 2 \) mm and \( \Delta z_i = 0.002 \) mm. \( \Delta z_i \) was, however, made smaller near pitch point P than in the other area. In
the figure, the given curve of involute tooth profile of the cylindrical-external gear was rotated so that pitch point P could be placed on x₀ axis for clarity, and the obtained curve of the tooth profile of conical-external gear was also rotated by a corresponding angle, which was calculated from the reduction ratio of the pair of cylindrical and conical-external gears. As is shown, the obtained curve of tooth profile shown with a white circle becomes a convex curve, which is very similar to an involute curve, shown as a solid line, touching the involute tooth profile curve of the cylindrical-external gear at pitch point P. Figure 6(b) shows the curves of tooth profile of conical-external gear synthesized for z₁₈ = -2, 0, 2, 4 mm, which are superimposed on the axis-perpendicular reference plane S₀. In this figure, the synthesized curves of tooth profile are shown while keeping the curve of involute tooth profile of the cylindrical-external gear fixed at the initial position. As is shown, the position of the line of contact, the position of pitch point, and the operating pressure angle, change markedly according to the change in the position of the section of the cylindrical-

(b) Comparison with involute curve

The characteristics of the curve of tooth profile obtained on section S are investigated here by comparison with a plane involute curve.

First, the radius of base circle of the plane involute curve rₛₐ is given by using the equivalent tooth number of conical-external gear zₑₐ as follows.

\[ rₛₐ = zₑₐ m \cos \alphaₑ/2 \]  

(21)

where, the center of the base circle is Oₛₑ.

Next, an arbitrary point, Q(xₛₑ, yₛₑ), on the curve of tooth profile obtained on section S is chosen as shown in Fig. 7. The angle θₑ of the starting point Q₀ of the involute curve is then obtained from the characteristics of involute curve as follows.

\[ Q₀ = \text{inv } αₑ + θₑ \]  

(22)

where,

\[ αₑ = \cos^{-1} \frac{rₑₐ}{rₛₐ}, \quad θₑ = \tan^{-1} \frac{yₑₐ}{xₑₐ}, \quad rₑₐ = \sqrt{xₑₐ^2 + yₑₐ^2} \]  

(23)

The coordinates, (xₑₐ, yₑₐ), of the starting point Q₀ are given as follows.

\[ (xₑₐ, yₑₐ) = (rₑₐ \cos θₑ, rₑₐ \sin θₑ) \]  

(24)

Next, calculating the coordinates, (xₑ’, yₑ’), of point Q’ which is on the plane involute curve and is located at the distance of r’ from the center Oₑₑ of the base circle, gives the following equation.

\[ (xₑ’, yₑ’) = (rₑ’ \cos θ’, rₑ’ \sin θ’) \]  

(25)

where,

\[ θ’ = θₑ - \text{inv } α’, \quad α’ = \cos^{-1} \frac{rₑₑ}{rₑₐ} \]  

(26)

As is shown in Fig. 6(b), the synthesized curves of tooth profile, shown with white circles, coincide well with the plane involute curves, shown as solid lines, obtained by substituting various values of r’ into Eq. (25). For instance, the difference in the direction of yₑₑ between the two curves is 0.05 µm at point P, 0.83 µm at point M, and 1.78 µm at point N for z₁₈ = 2 mm. It is also seen that rₑₑ is constant, since all four involute curves obtained for z₁₈ = -2, 0, 2, 4 mm were
Table 1 The calculated profile-shift coefficients of conical-external gear, $x_a$, and the values back-calculated from the center-to-center distance $a_{os}$, $x_a$

<table>
<thead>
<tr>
<th>$z_2$ (mm)</th>
<th>$x_2$ (mm)</th>
<th>$y_2$ (mm)</th>
<th>$a_x$ (deg)</th>
<th>$a_y$ (deg)</th>
<th>$x_a$</th>
<th>$x'_a$</th>
<th>$a_{os}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-5.999</td>
<td>-0.102</td>
<td>14.3571</td>
<td>14.3567</td>
<td>-0.6895</td>
<td>0.6895</td>
<td>14.438</td>
</tr>
<tr>
<td>-1</td>
<td>-6.000</td>
<td>0.003</td>
<td>19.0999</td>
<td>19.0998</td>
<td>-0.3810</td>
<td>-0.3810</td>
<td>14.802</td>
</tr>
<tr>
<td>0</td>
<td>-5.998</td>
<td>0.133</td>
<td>22.7394</td>
<td>22.7394</td>
<td>0</td>
<td>0</td>
<td>15.166</td>
</tr>
<tr>
<td>1</td>
<td>-5.993</td>
<td>0.283</td>
<td>25.7554</td>
<td>25.7554</td>
<td>0.4377</td>
<td>0.4377</td>
<td>15.530</td>
</tr>
<tr>
<td>2</td>
<td>-5.983</td>
<td>0.448</td>
<td>28.3337</td>
<td>28.3337</td>
<td>0.9222</td>
<td>0.9222</td>
<td>15.894</td>
</tr>
<tr>
<td>3</td>
<td>-5.967</td>
<td>0.627</td>
<td>30.6463</td>
<td>30.6462</td>
<td>1.4465</td>
<td>1.4465</td>
<td>16.258</td>
</tr>
<tr>
<td>4</td>
<td>-5.944</td>
<td>0.815</td>
<td>32.7020</td>
<td>32.7019</td>
<td>2.0056</td>
<td>2.0055</td>
<td>16.622</td>
</tr>
<tr>
<td>5</td>
<td>-5.913</td>
<td>1.016</td>
<td>34.5673</td>
<td>34.5671</td>
<td>2.5952</td>
<td>2.5952</td>
<td>16.986</td>
</tr>
<tr>
<td>6</td>
<td>-5.873</td>
<td>1.223</td>
<td>36.2751</td>
<td>36.2749</td>
<td>3.2122</td>
<td>3.2121</td>
<td>17.350</td>
</tr>
<tr>
<td>7</td>
<td>-5.825</td>
<td>1.437</td>
<td>37.8500</td>
<td>37.8497</td>
<td>3.8537</td>
<td>3.8536</td>
<td>17.714</td>
</tr>
</tbody>
</table>

Translated using the value of base circle $r_2$ given by Eq. (21).

(c) Axial distribution of profile-shift coefficient

Let $x_2$ denote the profile-shift coefficient of the plane involute tooth profile of conical-external gear obtained on section S, and let $a_x$ be the operating pressure angle. The following relation is obtained from the formula for profile-shift spur gears:\(^{10}\)

$$\text{inv} \ a_x = 2 \tan \ a_x \cdot \frac{x_2 + x_2'}{x_2 + x_2'} + \text{inv} \ a_d$$ \hspace{1cm} (27)

where, pressure angle $a_d$ is equal to $a_d$ obtained in the previous section, and is given as follows.

$$\text{inv} \ a_d = \tan^{-1} \left( \frac{y_2}{x_2} \right)$$ \hspace{1cm} (28)

The profile-shift coefficient, $x_a$, of conical-external gear is finally given from Eqs. (27) and (28) by the following equation.

$$x_a = \frac{x_2 + x_2'}{2 \tan \ a_x} \text{inv} \ a_x - \text{inv} \ a_d \hspace{1cm} (29)$$

Next, calculating the operating pressure angle $a_x$ from the distance $a_{os}$ between the axes of the two gears on section S gives the following equation.

$$\cos \ a_x = \frac{x_2 + x_2'}{2 \ a_{os}} \hspace{1cm} (30)$$

Here, let $x'_2$ denote the profile-shift coefficient, which is obtained from Eq. (29) by using the pressure angle $a_x$ calculated with the above equation.

Table 1 shows the comparison of the profile-shift coefficients $x_a$ and $x'_a$ obtained by using $a_x$ and $a_x'$ calculated from Eqs. (28) and (30). This comparison clearly shows that $a_x \approx a_x'$, $x_a \approx x_a$, and hence that the profile-shift coefficient $x_a$ of conical-external gear can be calculated from Eq. (29) by using the operating pressure angle $a_x$ which is calculated from Eq. (30).

Figure 8 shows the change in the profile-shift coefficient, $x_a$, of conical-external gear along the direction of the axis of the cylindrical-external gear. This figure shows that the profile-shift coefficient, $x_a$, of conical-external gear, shown as a solid line, changes non-linearly according to the increase in axial distance, $x_a$, and is zero for $x_a = 0$ mm, since $x_a = 0$ on the axis-perpendicular reference plane $S_0$.

From the above investigations, the basic characteristics of the synthesized curve of tooth profile of conical external gear are summarized with respect to the axis-perpendicular plane of the cylindrical-external gear as follows: (1) the curve of tooth profile coincides with a plane involute curve, (2) the base circle coincides with the base circle of a gear with an equivalent tooth number, (3) the radius of base circle remains constant at any axial position, (4) the profile-shift coefficient at an arbitrary section can be calculated based on the assumption that the center-to-center distance of the cylindrical and conical external gears is equal to the distance between $z_1$ and $z_2$ axes, which are the center axes of the cylindrical gear on the axis-perpendicular plane and the center axis of conical-external gear, respectively, and (5) the profile-shift coefficient changes non-linearly along the axis of the cylindrical-external gear.

4.2 Conical-internal gear

4.2.1 Ideal curved surface of tooth profile

The precise curved surface of tooth profile was similarly synthesized for a conical-internal gear with the following data: tooth number $z_2$ is 42, profile-shift coefficient $x_a$ is zero on the axis-perpendicular reference plane $S_0$, and inclined angle $g$ is 20°. Figure 9 shows an example of the results obtained. Figure 9(a) shows the given curve of involute tooth profile of the cylindrical-external gear, the projection of the synthesized and three-dimensional curved surface of tooth profile of the conical-internal gear on the axis-perpendicular reference plane $S_0$, and the corresponding line of contact.

Figure 9(b) shows the projections of the three curves, which were synthesized for $z_1 = 0$ and $\pm 0.020$ mm, on $x'z_2'$ plane. The center curve of tooth profile obtained for $z_1 = 0$ touches the axis-perpendicular
reference plane $S_0$ at pitch point $P_0$.

Figure 9(c) shows the oblique projections of the curves of tooth profile of the conical-internal gear, which were calculated by changing $z_i$ from $-1.0$ mm to $1.0$ mm with an interval of $0.25$ mm. In the figure, plane ABCD is the curved surface of involute tooth profile of the cylindrical-external gear, plane $A'B'C'D'$ is the synthesized and slightly skewed curved surface of tooth profile of the conical internal gear, and plane $A''B''C''D''$ is the slightly skewed operating surface of contact.

4.2.2 Curve of tooth profile on plane perpendicular to axis of cylindrical-external gear Figure 10 shows the curves of tooth profile of the conical-internal gear, which were obtained by the same method as for the conical external gear. Figure 10(a) shows the results obtained for $z_{i1}=2$ mm, $\Delta z_i=0.002$ mm. As shown in the figure, the obtained curve of tooth profile, shown with white circles, becomes a concave curve, shown as a solid line, similar to an involute curve which touches the involute curve of tooth profile of the cylindrical-external gear at pitch point P. Figure 10(b) shows the curves of tooth profiles of the conical internal gear (white circles), which were synthesized for $z_{i1}=0, 2, 4, 6$ mm, and superimposed on the axis-perpendicular reference plane $S_0$. It is shown in the figure that the position of the line of contact, the position of pitch point, and the operating pressure angle change markedly according to the change of the position, $z_{i1}$, of the section along the direction of the tooth width of the involute external gear.

Next, the plane involute curves obtained by a method similar to that for the conical-external gear using the same tooth number are superimposed on the plane involute curves shown with white circles in Fig. 10(b). As shown in this figure, these two curves coincide well with each other.

Next, the profile-shift coefficient, $x_{as}$, of the plane involute curve of the conical-internal gear on section S is given by the following equation in the same way.
Table 2  The calculated profile-shift coefficients of conical-internal gear, \( x_d \), and the values back-calculated from the center-to-center distance \( a_{sw} \), \( x_d \)

<table>
<thead>
<tr>
<th>( x_{12} ) (mm)</th>
<th>( x_d ) (mm)</th>
<th>( y_{1d} ) (mm)</th>
<th>( \alpha_b ) (deg.)</th>
<th>( \beta_b ) (deg.)</th>
<th>( x_{1d} ) (mm)</th>
<th>( y_{1d} ) (mm)</th>
<th>( a_{sw} ) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>21.000</td>
<td>0.069</td>
<td>8.3032</td>
<td>8.3032</td>
<td>-0.2281</td>
<td>-0.2281</td>
<td>13.150</td>
</tr>
<tr>
<td>0</td>
<td>0.000</td>
<td>0.117</td>
<td>5.6072</td>
<td>5.6072</td>
<td>0</td>
<td>0</td>
<td>15.514</td>
</tr>
<tr>
<td>1</td>
<td>20.998</td>
<td>0.261</td>
<td>20.3402</td>
<td>20.3402</td>
<td>0.3309</td>
<td>0.3309</td>
<td>13.878</td>
</tr>
<tr>
<td>2</td>
<td>20.996</td>
<td>0.393</td>
<td>23.9484</td>
<td>23.9484</td>
<td>0.7334</td>
<td>0.7334</td>
<td>14.242</td>
</tr>
<tr>
<td>3</td>
<td>20.993</td>
<td>0.555</td>
<td>27.0105</td>
<td>27.0105</td>
<td>1.1922</td>
<td>1.1922</td>
<td>14.606</td>
</tr>
<tr>
<td>4</td>
<td>20.987</td>
<td>0.728</td>
<td>29.6310</td>
<td>29.6310</td>
<td>1.6975</td>
<td>1.6975</td>
<td>14.970</td>
</tr>
<tr>
<td>5</td>
<td>20.980</td>
<td>0.914</td>
<td>31.9407</td>
<td>31.9407</td>
<td>2.2425</td>
<td>2.2425</td>
<td>15.334</td>
</tr>
<tr>
<td>6</td>
<td>20.971</td>
<td>1.112</td>
<td>34.0119</td>
<td>34.0119</td>
<td>2.8220</td>
<td>2.8220</td>
<td>15.698</td>
</tr>
<tr>
<td>7</td>
<td>20.958</td>
<td>1.320</td>
<td>35.8066</td>
<td>35.8066</td>
<td>3.4318</td>
<td>3.4318</td>
<td>16.062</td>
</tr>
<tr>
<td>8</td>
<td>20.944</td>
<td>1.538</td>
<td>37.6097</td>
<td>37.6097</td>
<td>4.0687</td>
<td>4.0687</td>
<td>16.425</td>
</tr>
</tbody>
</table>

as Eq. (29).

\[
x_d = \frac{x_{de} - z_{2o}}{2 \tan \alpha_c} (\tan \alpha_c - \tan \alpha_c) + x_b \tag{31}
\]

Calculating the operating pressure angle, \( \alpha_c \), from the center-to-center distance, \( a_{sw} \), between the two cylindrical and conical gears on section S in a manner similar to that for the conical-external gear gives the following equation.

\[
\cos \alpha_c = \frac{(x_{de} - z_{2o})m \cos \alpha_c}{2a_{sw}} \tag{32}
\]

Let \( x_d \) denote the profile-shift coefficient obtained from Eq. (31) using the operating pressure angle, \( \alpha_c \), calculated by the above equation. A comparison of the profile-shift coefficients, \( x_d \) and \( x_{1d} \), obtained using the equations of \( \alpha_c \) and \( \alpha_c \), is shown in Table 2. This result clearly shows that \( \alpha_c \approx \alpha_c \), \( \alpha_c \approx \alpha_c \), and hence that the profile-shift coefficient, \( x_d \), of conical-internal gear can be calculated from Eq.(31) using the pressure angle, \( \alpha_c \), which is calculated from Eq.(32).

Figure 8 shows the change in the tooth profile-shift coefficient, \( x_d \), as a broken line: the change was calculated in a manner similar to that for the conical-external gear. From the figure it is clear that the tooth profile-shift coefficient, \( x_d \), of the conical-internal gear is zero for \( z_{1d} = 0 \) mm, and increases non-linearly according to the increase in axial distance \( z_{1d} \).

From the above investigations, it has become clear that the curve of tooth profile of conical-internal gear with respect to the axis-perpendicular plane (section S) of cylindrical-external gear has the same characteristics as that of the conical external gear in section 4.1; in other words: (1) the curve of tooth profile coincides well with a plane involute curve, (2) the base circle coincides with the base circle of a gear with an equivalent tooth number, (3) the radius of base circle remains constant at any axial position, (4) the profile-shift coefficient at an arbitrary section can be calculated based on the assumption that the center-to-center distance of the cylindrical and conical-external gears is equal to the distance between \( z_1 \) and \( z_2 \) axes, which are the center axis of the cylindrical gear on the axis-perpendicular plane and the center axis of the conical-external gear, respectively, and (5) the profile-shift coefficient changes non-linearly along the axis of the cylindrical-external gear.

5. Conclusions

The authors have proposed a new synthesis method for tooth profiles using vectors to synthesize the ideal curves of tooth profile of conical-external and -internal gears which engage with a cylindrical-external gear over the entire tooth width, and have investigated the characteristics of the synthesized curves of tooth profile with respect to a section perpendicular to the axis of the cylindrical-external gear. The results obtained are as follows.

(1) The ideal curve of tooth profile of a conical-external or -internal gear, which engages with a cylindrical-external gear over the entire tooth width, can be precisely derived using the proposed synthesis method for tooth profile.

(2) The obtained curved surface of conical gear is slightly skewed; the simultaneous contact lines are not parallel to the tooth trace, and shift little by little; and the curved surface of contact is a slightly curved and skewed plane.

(3) The curves of tooth profile of conical-external and -internal gears with respect to an axis-perpendicular plane of cylindrical-external gear coincides well with a plane involute curve with the base circle determined using the equivalent tooth number, the module, and the pressure angle of basic rack. Also, the radius of this base circle does not change with respect to the axial position of base circle and remains at a constant value.

(4) The profile-shift coefficients of conical-external and -internal gears coincide well with the values, which are reversibly calculated from the distance between the axes of conical and cylindrical gears, taking the equivalent tooth number and the coefficient of the increase in center-to-center distance into account. Also, this profile-shift coefficient changes non-linearly along the axial direction of cylindrical-external gear.

The authors would like to thank Dr. Shitta Shingu of the Research and Development Center, Toshiba Corporation, for translating the reference written in Russian and for his valuable advice.

Appendix

A1. Tooth surface normal vector \( n_{1,1} \)

Let a tangent vector normal to the normal vector be \( n_{1,1} \), a vector in the direction of \( z_1 \) axis, and let the unit vectors of rectangular coordinate system \( O_1- \)
the tooth surface normal vector \( n_{11} \) at contact point \( K_1 \) is given by the following equation.

\[
n_{i1} = t_{i1} \times l_{i1} = \begin{bmatrix} i & j & k \\ \frac{\partial x_1}{\partial \alpha} & \frac{\partial y_1}{\partial \alpha} & \frac{\partial z_1}{\partial \alpha} \\ \frac{\partial x_1}{\partial \beta} & \frac{\partial y_1}{\partial \beta} & \frac{\partial z_1}{\partial \beta} \end{bmatrix} = \left( \frac{\partial y_1}{\partial \alpha}, -\frac{\partial x_1}{\partial \alpha}, 0 \right)^	op
\]

(A1)

In the above equation, the equation of component expression has been obtained by expanding under the consideration that \( \frac{\partial x_1}{\partial \beta} = \frac{\partial y_1}{\partial \beta} = 0 \) among the elements of the determinant of Eq. (A1).

A2. Transforming matrices between coordinate systems

The transforming matrices with respect to rotation are shown below.

(1) Transforming matrix \([T]_{01}\) from static coordinate system \(O-x'y'z'\) to moving coordinate matrix \(O_1-x_1y_1z_1\), and the reverse transforming matrix \([T]_{10}\).

\[
[T]_{01} = \begin{bmatrix} \cos \phi_1 & \sin \phi_1 & 0 \\ -\sin \phi_1 & \cos \phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

(A2)

\[
[T]_{10} = [T]_{01}^{-1}
\]

(A3)

(2) Transforming matrix \([T]_{00}\) from static coordinate system \(O-x'y'z'\) to moving matrix \(O_2-x_2y_2z_2\), and the reverse transforming matrix \([T]_{20}\).

\[
[T]_{00} = \begin{bmatrix} \cos \phi_2 & \sin \phi_2 & 0 & \cos \gamma & 0 & -\sin \gamma \\ -\sin \phi_2 & \cos \phi_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \sin \gamma & 0 & \cos \gamma \end{bmatrix}
\]

(A4)

\[
[T]_{20} = [T]_{00}^{-1}
\]

(A5)

(3) Transforming matrix \([T]_{12}\) from moving coordinate system \(O_1-x_1y_1z_1\) to moving coordinate system \(O_2-x_2y_2z_2\), and the reverse transforming matrix \([T]_{21}\).

\[
[T]_{12} = [T]_{00}[T]_{01}
\]

(A6)

\[
[T]_{21} = [T]_{12}[T]_{10}
\]

(A7)

References


