Design of Observer-Based Controller for Perturbed Time-Delay Systems*

Sangchul WON** and Ju H. PARK***

In this paper, we present a robust observer-based controller for perturbed time-delay systems, in which the full state cannot be measured. Lyapunov method with linear matrix inequality (LMI) technique is used for synthesis of the control law to stabilize the system. The time-varying perturbations are assumed to be within a given compact set.

Key Words: Time-Delay, Observer-Based Controller, Linear Matrix Inequality

1. Introduction

Over the past decades, numerous papers have been concerned with the problems of stabilizing perturbed time-delay systems using state feedback control⁽¹⁾⁻⁽³⁾. However, in many cases, it may not be possible to measure all the states of the system. For this reason, the problem of designing an observer-based control law for stabilization of perturbed dynamic systems has received some attention in recent years^{(4),(5)}. But the studies are only limited to systems without time-delay. With the motivation, we extend the concept of designing observer-based controller to the perturbed time-delay systems of which the full states are not available. The proposed design technique is based on the LMI method combining with Lyapunov second method.

The perturbations considered are time-varying and bounded in magnitude, and the matching condition⁽²⁾ of perturbations are not necessary in this design. From the positive-definite solutions of two LMIs for the given system, we can obtain gain matrices of the observer and thus feedback control law which stabilize the perturbed time-delay systems.

This paper is organized as follows. Section 2 defines the notations. In section 3, we state the problem of observer-based controller for perturbed time-delay systems. Sufficient condition for stabilization of the system is given in section 4. In section 5, a numerical example is presented to show the effectiveness of the method developed in the paper. Finally, section 6 gives conclusions.

2. Notation

- $\mathcal{R}$: real number field
- $\mathcal{C}$: complex number field
- x : state vector, $x \in \mathcal{R}^n$
- A^T : transpose of matrix A
- x^* : complex conjugate transpose of vector x
- z : complex number
- I : identity matrix
- $A < B$: the matrix $A - B$ is negative definite matrix

3. Problem Formulation

Consider a class of perturbed time-delay system

$$\begin{aligned} \dot{x}(t) &= [A + \Delta A(r(t))]x(t) + [A_d \\ &\quad + \Delta A_d(v(t))]x(t-h) \\ &\quad + [B + \Delta B(s(t))]u(t), \\ y(t) &= Cx(t) \end{aligned} \quad (1)$$

where $x(t)$ is the state, $u(t)$ is the control, $y(t)$ is the measured output. A, A_d, B , and C are constant matrices with appropriate dimensions, the delay duration h is a positive constant, and $\Delta A(\cdot)$, $\Delta A_d(\cdot)$ and $\Delta B(\cdot)$ are parameter perturbations of the system. And, the pairs (A, B) and (A, C) are controllable and

* Received 12th September, 1997

** Department of Electrical Engineering, Pohang University of Science and Technology, San 31 Hyoja-Dong, Pohang, 790-784, Korea. E-mail: won@postech.ac.kr

*** Automation Research Center, Pohang University of Science and Technology, Korea

observable, respectively. $r(t) \in \mathcal{R}^k$, $v(t) \in \mathcal{R}^l$, and $s(t) \in \mathcal{R}^q$ are vectors of uncertain parameters. It is assumed that $r(\cdot)$, $v(\cdot)$, and $s(\cdot)$ are Lebesgue-measurable and range is in known compact sets $R \triangleq \{r : |r_i(t)| \leq \bar{r}, i=1, 2, \dots, k\}$, $V \triangleq \{v : |v_i(t)| \leq \bar{v}, i=1, 2, \dots, l\}$, $S \triangleq \{s : |s_i(t)| \leq \bar{s}, i=1, 2, \dots, q\}$.

Furthermore, we assume that the perturbations are the 'rank-1' type^{(1),(3),(4)}, i.e.,

$$\begin{aligned} \Delta A(r(t)) &= \sum_{i=1}^k A_i r_i(t), \quad \Delta A_d(v(t)) = \sum_{i=1}^l A_{di} v_i(t), \\ \Delta B(s(t)) &= \sum_{i=1}^q B_i s_i(t) \end{aligned} \quad (2)$$

where

$$A_i = d_i e_i^T, \quad B_i = f_i g_i^T, \quad A_{di} = h_i w_i^T$$

with $d_i, e_i, f_i, w_i, h_i \in \mathcal{R}^n$ and $g_i \in \mathcal{R}^m$

Consider a full state observer for nominal part of the system given in (1)

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + A_d\tilde{x}(t-h) + Bu(t) \\ &\quad + L(y(t) - C\tilde{x}(t)), \end{aligned} \quad (3)$$

and the control law for the perturbed system

$$u(t) = -K\tilde{x}(t) \quad (4)$$

in which $\tilde{x}(t)$ is the observer state, and K and L are the constant feedback matrix and observer gain with appropriate dimensions.

Then, the problem is to find the feedback gain matrix K and the observer gain matrix L of the controller which stabilize the system given in (1).

4. Controller Design

Now, we derive a sufficient condition for robust stabilization of the time-delay systems by combining Lyapunov method with LMI technique.

Before proceeding our approach, some preliminary results are given below.

Lemma 1⁽⁶⁾ The system (1) without control input and perturbation (i.e. $x(t) = Ax(t) + A_dx(t-h)$) is asymptotically stable independent of delay if and only if for any given positive-definite Hermitian matrix $Q(z)$, there exists a unique positive definite hermitian matrix $P(z)$ which satisfies the complex Lyapunov matrix equation

$$\begin{aligned} A^*(z)P(z) + P(z)A(z) &= -Q(z) \\ \forall |z|=1 \end{aligned} \quad (5)$$

where

$$A(z) = A + zA_d \quad (6)$$

and $z = \exp(jw)$, $j = \sqrt{-1}$ and $w \in [0, 2\pi]$.

Lemma 2⁽⁷⁾ Lemma 1 may be viewed as the asymptotic stability conditions of a generalized linear system (Lee *et al.*⁽⁷⁾) described by

$$\dot{y}(t) = A(z)y(t), \quad \forall |z|=1 \quad (7)$$

where $y \in \mathcal{C}^n$.

For simplicity, we define

$$T = \bar{r} \sum_{i=1}^k d_i d_i^T, \quad U = \bar{r} \sum_{i=1}^k e_i e_i^T$$

$$\begin{aligned} W &= \bar{s} \sum_{i=1}^l f_i f_i^T, \quad V = \bar{s} \sum_{i=1}^l g_i g_i^T \\ Y &= \bar{v} \sum_{i=1}^q h_i h_i^T, \quad Z = \bar{v} \sum_{i=1}^q w_i w_i^T \end{aligned} \quad (8)$$

and define the following matrix functions which play an important role to get main results

$$\begin{aligned} S_1(X_c, \gamma_c) &= X_c A^T + A X_c - \gamma_c (B B^T \\ &\quad - 2W - 2BVB^T) + T + Y + A_d A_d^T \\ &\quad + X_c (2U + 2Z + I) X_c \end{aligned} \quad (9)$$

$$\begin{aligned} S_2(X_o, \gamma_o) &= A^T X_o + X_o A - \{2\gamma_o C^T C \\ &\quad - I - P_c B (2\gamma_c V + I) B^T P_c\} \\ &\quad + X_o (2\gamma_c W + A_d A_d^T + T + Y) X_o \end{aligned} \quad (10)$$

where γ_c and γ_o are positive scalars, X_c and X_o are positive-definite matrices. And also, we define $P_c = X_c^{-1}$ and $P_o = X_o^{-1}$

Then we have a following theorem.

Theorem :

Consider the time-delay system (1) with the observer-based controller in (3) and (4) with gain matrices K and L where $K = \gamma_c B^T P_c$ and $L = \gamma_o P_o C^T$. Then, the closed-loop system is guaranteed to be stabilized asymptotically if there exist positive definite matrices X_c and X_o , and positive scalars γ_c and γ_o which satisfy the following LMIs

$$\begin{aligned} \Omega_1(X_c, \gamma_c) &= \begin{bmatrix} Q_1(X_c, \gamma_c) + T + Y & & & \\ U^{1/2} X_c & & & \\ Z^{1/2} X_c & & & \\ X_c & & & \end{bmatrix} \\ &\quad \begin{bmatrix} X_c U^{1/2} & X_c Z^{1/2} & X_c \\ -\frac{1}{2}I & 0 & 0 \\ 0 & -\frac{1}{2}I & 0 \\ 0 & 0 & -I \end{bmatrix} < 0 \end{aligned} \quad (11)$$

$$\begin{aligned} \Omega_2(X_o, \gamma_o) &= \begin{bmatrix} Q_2(X_o, \gamma_o) & X_o W^{1/2} \\ W^{1/2} X_o & -\frac{1}{2}\gamma_c^{-1}I \\ A_d^T X_o & 0 \\ T^{1/2} X_o & 0 \\ Y^{1/2} X_o & 0 \end{bmatrix} \\ &\quad \begin{bmatrix} X_o A_d & X_o T^{1/2} & X_o Y^{1/2} \\ 0 & 0 & 0 \\ -I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} < 0 \end{aligned} \quad (12)$$

where

$$\begin{aligned} Q_1(X_c, \gamma_c) &= X_c A^T + A X_c - \gamma_c (B B^T \\ &\quad - 2W - 2BVB^T) + A_d A_d^T \\ Q_2(X_o, \gamma_o) &= A^T X_o + X_o A - (2\gamma_o C^T C \\ &\quad - I - P_c B (2\gamma_c V + I) B^T P_c). \end{aligned}$$

Proof :

Define error vector as $e(t) = x(t) - \tilde{x}(t)$. Then,

combining (1) and (3),

$$\dot{e}(t) = (A - LC)e(t) + \Delta Ax(t) + A_d e(t-h) + \Delta A_d x(t-h) - \Delta BK(x-e). \quad (13)$$

Using the control law of (4), (1) and (13) can be combined into

$$\dot{X}(t) = \begin{bmatrix} A_f + \Delta A - \Delta BK & BK + \Delta BK \\ \Delta A - \Delta BK & A_o + \Delta BK \end{bmatrix} X + \begin{bmatrix} A_d + \Delta A_d & 0 \\ \Delta A_d & A_d \end{bmatrix} X(t-h) \quad (14)$$

where $X = [x^T e^T]^T$, $A_f = A - BK$ and $A_o = A - LC$. In the light of Lemma 1 and 2, the stability of the system (14) mutually implies that of the following system

$$\dot{Y}(t) = \begin{bmatrix} A_f + \Delta A + A_d z + \Delta A_d z - \Delta BK \\ \Delta A + \Delta A_d z - \Delta BK \\ BK + \Delta BK \\ A_o + A_d z + \Delta BK \end{bmatrix} Y(t), \quad \forall |z|=1 \quad (15)$$

where $Y(t) = [y_1^*(t) \ y_2^*(t)]^T \in \mathcal{C}^{2n}$

Let us define a Lyapunov functions as

$$V(Y(t)) = Y^*(t) \begin{bmatrix} P_c & 0 \\ 0 & P_o^{-1} \end{bmatrix} Y(t). \quad (16)$$

Taking the derivative of $V(Y(t))$ along the trajectory of (15) yields

$$\begin{aligned} \dot{V} &= \dot{y}_1^*(t) P_c y_1(t) + y_1^*(t) P_c \dot{y}_1(t) \\ &+ \dot{y}_2^*(t) P_o^{-1} y_2(t) + y_2^*(t) P_o^{-1} \dot{y}_2(t) \\ &= y_1^*(A_f^T P_c + P_c A_f + P_c \Delta A + \Delta A^T P_c \\ &+ z^* A_d^T P_c + P_c A_d z + z^* \Delta A_d^T P_c \\ &+ P_c \Delta A_d z - K^T \Delta B^T P_c - P_c \Delta BK) y_1 \\ &+ y_1^*(P_c BK + P_c \Delta BK) y_2 \\ &+ y_2^*(K^T B^T P_c + K^T \Delta B^T P_c) y_1 \\ &+ y_2^*(P_o^{-1} \Delta A + P_o^{-1} \Delta A_d z - P_o^{-1} \Delta BK) y_1 \\ &+ y_1^*(\Delta A^T P_o^{-1} + z^* \Delta A_d^T P_o^{-1} \\ &- K^T \Delta B^T P_o^{-1}) y_2 + y_2^*(A_o^T P_o^{-1} + P_o^{-1} A_o \\ &+ z^* A_d^T P_o^{-1} + P_o^{-1} A_d z + K^T \Delta B^T P_o^{-1} \\ &+ P_o^{-1} \Delta BK) y_2. \end{aligned} \quad (17)$$

Using the inequality $2|x^T y| \leq \beta x^T x + y^T y / \beta$, where x and y are any vectors with appropriate dimensions and $\beta > 0$, we can obtain

$$\begin{aligned} &y_1^*(P_c \Delta A + \Delta A^T P_c) y_1 \\ &= y_1^* P_c \sum_{i=1}^k d_i e_i^T r_i(t) y_1 + y_1^* \sum_{i=1}^k r_i(t) e_i d_i^T P_c y_1 \\ &\leq \bar{r} y_1^* P_c \sum_{i=1}^k d_i d_i^T P_c y_1 + \bar{r} y_1^* \sum_{i=1}^k e_i e_i^T y_1 \\ &= y_1^* P_c T P_c y_1 + y_1^* U y_1 \\ &y_1^*(z^* A_d^T P_c + P_c A_d z) y_1 \leq y_1^* P_c A_d A_d^T P_c y_1 \\ &+ y_1^* z^* z y_1 = y_1^* P_c A_d A_d^T P_c y_1 + y_1^* y_1 \\ &y_1^*(z^* \Delta A_d^T P_c + P_c \Delta A_d z) y_1 \\ &\leq \bar{v} y_1^* P_c \sum_{i=1}^l h_i h_i^T P_c y_1 + \bar{v} y_1^* \sum_{i=1}^l w_i w_i^T y_1 \\ &= y_1^* P_c Y P_c y_1 + y_1^* Z y_1. \end{aligned} \quad (18)$$

Similarly,

$$\begin{aligned} &-y_1^*(K^T \Delta B^T P_c + P_c \Delta BK) y_1 \\ &\leq \gamma_c (y_1^* P_c W P_c y_1 + y_1^* P_c B V B^T P_c y_1) \\ &y_2^*(z^* A_d^T P_o^{-1} + P_o^{-1} A_d z) y_2 \end{aligned}$$

$$\begin{aligned} &\leq y_2^* P_o^{-1} A_d A_d^T P_o^{-1} y_2 + y_2^* y_2 \\ &y_2^*(K^T \Delta B^T P_o^{-1} + P_o^{-1} \Delta BK) y_2(t) \\ &\leq \gamma_c (y_2^* P_o^{-1} W P_o^{-1} y_2 + y_2^* P_c B V B^T P_c y_2) \\ &y_1^* P_c B K y_2 + y_2^* K^T B^T P_c y_1 \\ &\leq \gamma_c (y_1^* P_c B B^T P_c y_1 + y_2^* P_c B B^T P_c y_2) \\ &y_1^* P_c \Delta B K y_2 + y_2^* K^T \Delta B^T P_c y_1 \\ &\leq \gamma_c (y_1^* P_c W P_c y_1 + y_2^* P_c B V B^T P_c y_2) \\ &y_2^* P_o^{-1} \Delta A y_1 + y_1^* \Delta A^T P_o^{-1} y_2 \\ &\leq y_2^* P_o^{-1} T P_o^{-1} y_2 + y_1^* U y_1 \\ &y_2^* P_o^{-1} \Delta A_d z y_1 + y_1^* z^* \Delta A_d^T P_o^{-1} y_2 \\ &\leq y_2^* P_o^{-1} Y P_o^{-1} y_2 + y_1^* Z y_1 \\ &- y_2^* P_o^{-1} \Delta B K y_1 - y_1^* K^T \Delta B^T P_o^{-1} y_2 \\ &\leq \gamma_c (y_2^* P_o^{-1} W P_o^{-1} y_2 + y_1^* P_c B V B^T P_c y_1). \end{aligned} \quad (19)$$

Therefore, using inequalities (18) and (19), (17) becomes

$$\begin{aligned} \dot{V} &\leq y_1^* [A^T P_c + P_c A - P_c \{ \gamma_c (B B^T \\ &- 2 W - 2 B V B^T) - T - Y \\ &- A_d A_d^T \} P_c + 2 U + 2 Z + I] y_1 \\ &+ y_2^* P_o^{-1} [P_o A^T + A P_o - P_o \{ 2 \gamma_o C^T C - I \\ &- P_c B (2 \gamma_c V + I) B^T P_c \} P_o \\ &+ (2 \gamma_c W + A_d A_d^T + T + Y)] P_o^{-1} y_2 \\ &= y_1^* X_c^{-1} S_1 X_c^{-1} y_1 + y_2^* S_2 y_2 \end{aligned} \quad (20)$$

Then, the sufficient condition for the stability of the system in (15) is

$$S_1 < 0 \text{ and } S_2 < 0. \quad (21)$$

By Schur complement, inequalities (11) and (12) are equivalent to the inequalities (21). Thus, the system in (15) is asymptotically stable independent of delay under the condition (11) and (12). Therefore, by Lemma 1 and 2, the time-delay system (1) is also asymptotically stable. This completes the proof.

5. Example

To illustrate the preceding result, we consider the following linear perturbed time-delay system:

$$\begin{aligned} \dot{x}(t) &= [A + \Delta A(r(t))] x(t) + [A_d + \Delta A_d(v(t))] x(t-h) \\ &- 1 + [B + \Delta B(s(t))] u(t), \\ y(t) &= C x(t) \end{aligned}$$

where

$$A = \begin{bmatrix} -4 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 1 \\ 0.1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad C = [1 \quad 3],$$

$$\Delta A(\cdot) = \begin{bmatrix} 0.3 \sin t & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta A_d(\cdot) = \begin{bmatrix} 0.2 \sin t & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Delta B(\cdot) = \begin{bmatrix} 0 \\ 0.1 \sin t \end{bmatrix}.$$

Take $A_1 = d_1 e_1^T$, $B_1 = f_1 g_1^T$, and $A_{d1} = h_1 w_1^T$, where

$$d_1 = e_1 = \begin{bmatrix} 0.5477 \\ 0 \end{bmatrix}, \quad h_1 = w_1 = \begin{bmatrix} 0.4472 \\ 0 \end{bmatrix},$$

$$f_1 = h_1 = \begin{bmatrix} 0 \\ 0.3162 \end{bmatrix}, \quad g_1 = 0.3162$$

and $\bar{r} = \bar{v} = \bar{s} = 1$.

Using Matlab-LMI toolbox, the positive solutions

Fig. 1 Time responses of closed-loop system

of the LMI in (11) are obtained as

$$P_c = \begin{bmatrix} 0.3271 & -0.0284 \\ -0.0284 & 0.3108 \end{bmatrix}, \quad \gamma_c = 4.6931.$$

Then, utilizing the above solutions P_c and γ_c , the solutions of the LMI given in (12) are

$$P_o = \begin{bmatrix} 4.3867 & 0.4068 \\ 0.4068 & 1.6589 \end{bmatrix}, \quad \gamma_o = 0.2741$$

and then $u(t) = -1.1358 \tilde{x}_1(t) - 4.2421 \tilde{x}_2(t)$ is a stabilizing control.

The simulation result with initial value, $x_0 = [1 \ 3]^T$ for $-1 \leq t \leq 0$, is shown in Fig. 1.

6. Conclusion

Using Lyapunov method and LMI technique, we derived the sufficient condition for robust stability of perturbed time-delay systems. The proposed technique does not require a matching condition, and the gain matrices of the controller are easily found using the LMI method. Finally, an example was used to demonstrate the main result.

References

- (1) Shen, J., Chen, B. and Kung, F., Memoryless Stabilization of Uncertain Dynamic Delay Sys-

tems: Riccati Equation Approach, IEEE Trans. Aut. Control, Vol. 36 (1991), p. 638.

- (2) Cheres, E., Gutman, S. and Palmor, Z., Stabilization of Uncertain Dynamic Systems Including State Delay, IEEE Trans. Aut. Control, Vol. 34 (1989), p. 1198.
- (3) Choi, H. H. and Chung, M. J., Memoryless Stabilization of Uncertain Dynamic Systems with Time-Varying Delayed States and Controls, Automatica, Vol. 31 (1995), p. 1349.
- (4) Petersen, I. R., A Riccati Equation Approach to the Design of Stabilizing Controllers and Observers for a Class of Uncertain Linear Systems, IEEE Trans. Aut. Control, Vol. 30 (1985), p. 904.
- (5) Jabbari, F. and Schmitendorf, W. E., Robust Linear Controllers Using Observers, IEEE Trans. Aut. Control, Vol. 36 (1991), p. 1509.
- (6) Brierley, S. D., Chiasson, J. N., Lee, E. B. and Zak, S. H., On Stability Independent of Delay for Linear Systems, IEEE Trans. Aut. Control, Vol. 27 (1982), p. 252.
- (7) Hmamed, A., Further Results on the Delay-Independent Asymptotic Stability of Linear Systems, Int. J. Syst. Sci., Vol. 22 (1991), p. 1127.