Model Conversions of Uncertain Linear System via the Interval Tuning Bilinear Approximation Method*

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This paper presents a new tuning bilinear approximation method together with the interval arithmetic operation to convert a continuous-time (discrete-time) uncertain system to an equivalent discrete-time (continuous-time) uncertain model. Based on the bilinear approximation method, we induct one parameter for tuning the error bound between the exact and approximate uncertain system matrices. The bounds of interval system matrices via the proposed interval tuning bilinear approximation method are narrower than those of the existing interval bilinear and the interval pade approximation methods⁽¹⁾. The resulting interval models (the enclosing interval models) are able to tightly enclose the exact uncertain models. Also the approximate discrete-time interval solution is able to tightly enclose the exact interval solution of the continuous-time uncertain state-space equation. The proposed method will be helpful for analysis and synthesis of uncertain systems.

Key Words: Bilinear Approximation Method, Tuning Bilinear Approximation Method, Pade Approximation Method

1. Introduction

Most real physical dynamic plants are often formulated by a continuous-time uncertain framework because parameter variation, sensor noise, actuator constrains, load disturbance and nonlinear effects will arise randomly. Naturally, nominal model-conversion problems will become uncertain model-conversion problems. The main purpose of this paper is to perform model conversions of uncertain system, whose system matrices characterizing the state-space representation are assumed to be interval matrices, via a newly proposed method, the tuning bilinear approximation method together with the interval

arithmetic operation. So the resulting interval models (the enclosing interval models) are able to tightly enclose the exact uncertain models. Also, the approximate discrete-time interval solution is able to tightly enclose the exact interval solution of the continuous-time uncertain state-space equation. The solution of the continuous-time uncertain state-space equation has not yet been fully developed. As a result, the proposed method will be helpful for analysis and control of uncertain system, such as the state-matching digital redesign of sampled-data uncertain systems⁽²⁾.

Only a few methods for model conversions of uncertain linear systems have been reported. Ezzine and Johnson⁽⁵⁾ took a perturbation approach to convert a continuous-time uncertain linear system to an equivalent discrete-time uncertain model, but they did not solve the converse of the problem. Recently, Shieh et al. proposed the interval Pade approximation⁽¹⁾ to convert a continuous-time (discrete-time) uncertain system to an equivalent discrete-time (continuous-time) uncertain model, which is quite useful for digital simulation and adaptive control of continuous-time uncertain systems. In their work⁽¹⁾, Shieh et al. used

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the error bounds of approximate uncertain models with respect to their exact uncertain models to modify the interval Pade's approximants, so that the inclusion property is guaranteed. To meet some specified dynamic performance requirements⁽⁴⁾⁻⁽⁶⁾ for all possible variation of a set of parameters in designing dynamic uncertain systems, the requirement of the inclusion property is of fundamental importance. For example, to satisfy the actuator saturation, the value of the worst-case tolerated overshoot/undershoot of the dynamic uncertain system should not exceed a prescribed threshold value of the actuator. Hence, it is important to develop an enclosing approximate discrete-time uncertain model which can tightly enclose the exact discrete-time uncertain model of the original continuous-time uncertain system⁽⁷⁾. Since the error bound proposed in Shieh et al.⁽¹⁾ is fixed, a narrower error bound obtained via a simpler formula will be more helpful for analysis and design of the original uncertain system.

In this paper, based on the "Pade and inverse-Pade method⁽⁸⁾" and the "bilinear and inverse-bilinear method⁽⁹⁾", we induct one parameter to tune error bounds of approximate uncertain models with respect to their exact uncertain models so that the obtained error bounds of uncertain interval models are narrower than those obtained via the interval Pade approximate method⁽¹⁾.

2. Interval Analysis Preliminaries⁽¹⁰⁾

Let the interval number $[a, b]$ be the set of $x \in \mathcal{R}$ such that $a \leq x \leq b$. The arithmetic operations on the intervals are defined as follows:

$$\begin{aligned} [a, b] + [c, d] &= [a + c, b + d], \\ [a, b] - [c, d] &= [a - d, b - c], \\ [a, b] \times [c, d] &= [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)], \end{aligned}$$

$$[a, b] \div [c, d] = [a, b] \times \left[\frac{1}{d}, \frac{1}{c} \right] \quad \text{iff } 0 \notin [c, d]$$

Note that the interval addition and multiplication are associative and commutative, and the distributive law becomes a subdistributive law, i.e.,

$$[a, b] \times ([c, d] + [e, f]) \subseteq [a, b] \times [c, d] + [a, b] \times [e, f]$$

Alternatively, the interval number x can be expressed as

$$\begin{aligned} x &= [\underline{x}, \bar{x}] \triangleq \{ \hat{x} \in \mathcal{R} | \underline{x} \leq \hat{x} \leq \bar{x} \} \\ &= [x_0 - \Delta x, x_0 + \Delta x], \end{aligned}$$

where the nominal value $x_0 = (\underline{x} + \bar{x})/2$ and the uncertainty $\Delta x = (\bar{x} - \underline{x})/2$. The interval real number $\hat{x}$ in x is called a degenerate (real) interval number.

For all $n \times n$ interval real ($\mathcal{IR}$) matrices, $F^I = [f_{ij}] \in \mathcal{IR}^{n \times n}$ and $G^I = [g_{ij}] \in \mathcal{IR}^{n \times n}$, the addition, subtraction, and multiplication operations can be

written as

$$\begin{aligned} F^I \pm G^I &= [f_{ij} \pm g_{ij}] \in \mathcal{IR}^{n \times n}, \\ F^I G^I &= [f_{ij}][g_{ij}] = \left[\sum_{k=1}^n f_{ik} \times g_{kj} \right] \in \mathcal{IR}^{n \times n} \end{aligned}$$

The interval matrix addition and subtraction operations are associative and commutative. However, the interval matrix multiplication operation, in general, is not associative, and the distributive law becomes a subdistributive law. An important inclusive theorem of the interval arithmetic is described as follows.

Theorem 2. 1⁽¹⁰⁾: If $F(x_1, x_2, \dots, x_n)$ is a rational expression or an irrational monotonic real function in the interval variables $x_1, x_2, \dots, x_n$, i.e., a finite combination of $x_1, x_2, \dots, x_n$ and a finite set of constant intervals with interval arithmetic operations, then $x'_1 \subset x_1, \dots, x'_n \subset x_n$ implies $F(x'_1, x'_2, \dots, x'_n) \subset F(x_1, x_2, \dots, x_n)$ for every set of interval numbers $x_1, x_2, \dots, x_n$, for which the interval arithmetic operations in F are defined.

In this paper, we use the Hansen's method⁽¹⁰⁾⁻⁽¹²⁾ to estimate the inverse of an interval matrix as shown below.

Let an interval matrix A^I be $A^I = [a_{ij}] \in \mathcal{IR}^{n \times n}$ and an $n \times n$ real matrix $A_0 = \{a_{0ij}\}$ be a degenerate interval matrix with each entry $a_{0ij} = (\underline{a}_{ij} + \bar{a}_{ij})/2$. Thus, the inversion of the constant real matrix, A_0 , can be found by any matrix inversion algorithm. Also, let $E^I \in \mathcal{IR}^{n \times n}$ be an interval error matrix given by

$$E^I = I_n - A^I A_0^{-1} = [e_{ij}] \in \mathcal{IR}^{n \times n} \quad (1)$$

Define a norm of an interval matrix F^I as

$$\|F^I\| = \max_i \sum_{j=1}^n \max(|\underline{f}_{ij}|, |\bar{f}_{ij}|) \quad (2)$$

Notice that $\|F^I\|$ is an upper bound to the "maximum row sum" norm of any degenerate (real) matrix $F_0 = \{f_{0ij}\} \in F^I$, i.e.,

$$\|F^I\| \geq \|F_0\| = \max_{1 \leq i \leq n} \sum_{j=1}^n |f_{0ij}|$$

In (1), if $\|E^I\| < 1$ and by the use of Theorem 2. 1, the desired $(A^I)^{-1}$ satisfies

$$(A^I)^{-1} \subseteq A_0^{-1} [S_m^I + P_m^I] \in \mathcal{IR}^{n \times n}, \quad (3)$$

where

$$S_m^I = I_n + E^I (I_n + E^I (I_n + E^I (I_n + E^I (\dots)))) \quad (4)$$

to m sums, and P_m^I is an $n \times n$ interval matrix with identical elements, each of which is the interval

$$\left[\frac{-\|E^I\|^{m+1}}{1 - \|E^I\|}, \frac{\|E^I\|^{m+1}}{1 - \|E^I\|} \right] \quad (5)$$

The use of Hansen's method gives a less conservative estimate of $(A^I)^{-1}$ than the direct use of any matrix inversion method with interval matrix operations. To reduce the inherent conservativeness of interval arithmetic operations, various improvements⁽¹⁰⁾⁻⁽¹³⁾ such as the use of nested form, centered form and the Moore's partition approach etc., are proposed.

3. Digital Interval Model Conversion

Consider a continuous-time interval system

$$\dot{x}_c(t) = A^I x_c(t) + B^I u_c(t), \quad x_c(0) = x_{c0}, \quad (6)$$

where the interval system matrices $A^I \in \mathcal{I} \mathcal{R}^{n \times n}$ and $B^I \in \mathcal{I} \mathcal{R}^{n \times m}$, $x_c(t) \in \mathcal{R}^{n \times 1}$ is the state, $u_c(t) \in \mathcal{R}^{m \times 1}$ is the input. The degenerate real matrices A_r and B_r are enclosed by A^I and B^I , respectively, where $A^I = [\underline{A}, \bar{A}] = [A_0 - \Delta A, A_0 + \Delta A]$, $B^I = [\underline{B}, \bar{B}] = [B_0 - \Delta B, B_0 + \Delta B]$, $A_0 = 0.5(\underline{A} + \bar{A})$ and $B_0 = 0.5(\underline{B} + \bar{B})$. (A_0, B_0) is the pair of nominal system matrices. ($\Delta A, \Delta B$) is the pair of perturbed system matrices.

The associated discrete-time interval model of (6) for a piecewise constant input $u_c(t)$ is

$$x_d(kT + T) = G^I x_d(kT) + H^I u_d(kT), \quad x_d(0) = x_{c0}, \quad (7.a)$$

where

$$G^I = e^{A^I T}, \quad (7.b)$$

$$H^I = (G^I - I_n)(A^I)^{-1} B^I, \quad (7.c)$$

$$u_d(t) = u_c(kT) \quad \text{for } kT \leq t < (k+1)T \quad (7.d)$$

The interval matrices $G^I \in \mathcal{I} \mathcal{R}^{n \times n}$ and $H^I \in \mathcal{I} \mathcal{R}^{n \times m}$ contain corresponding degenerate real matrices $G_r \in \mathcal{R}^{n \times n}$ and $H_r \in \mathcal{R}^{n \times m}$, such that

$$G_r = e^{A_r T} \subseteq G^I = e^{A^I T}, \quad (8.a)$$

$$H_r = (G_r - I_n)(A_r)^{-1} B_r \subseteq H^I = (G^I - I_n)(A^I)^{-1} B^I, \quad (8.b)$$

where T is the sampling period. Shieh et al.⁽¹⁾ have proposed the Pade and the second-order bilinear approximation methods to solve G_r and H_r shown below:

$$G_{pr} = I_n + \left[I_n - \frac{1}{2} A_r T + \frac{1}{12} (A_r T)^2 \right]^{-1} A_r T, \quad (9.a)$$

$$H_{pr} = \left[I_n - \frac{1}{2} A_r T + \frac{1}{12} (A_r T)^2 \right]^{-1} B_r T, \quad (9.b)$$

$$G_{b2r} = I_n + \left[I_n - \frac{1}{2} A_r T + \frac{1}{16} (A_r T)^2 \right]^{-1} A_r T, \quad (9.c)$$

$$H_{b2r} = \left[I_n - \frac{1}{2} A_r T + \frac{1}{16} (A_r T)^2 \right]^{-1} B_r T \quad (9.d)$$

If we consider the worst-case approximation, the interval Pade system matrix G_p^I is

$$\begin{aligned} G_p^I &= I_n + \left[I_n - \frac{1}{2} A^I T + \frac{1}{12} (A^I T)^2 \right]^{-1} A^I T + P_{pg}^I \\ &= [\underline{G}_p, \bar{G}_p] = [G_{p0} - \Delta G_{pg}, G_{p0} + \Delta G_{pg}] \supseteq G^I, \end{aligned} \quad (10.a)$$

where $P_{pg}^I \in \mathcal{I} \mathcal{R}^{n \times n}$ is an interval matrix with identical elements, each of which is the interval $[-e_{pg}, e_{pg}]$ obtained from Shieh et al.⁽¹⁾, that is

$$\begin{aligned} e_{pg} &= \frac{2 \left(1 + \frac{1}{2} \|A^I T\| + \frac{1}{16} \|A^I T\|^2 \right)}{1 - \frac{1}{2} \|A^I T\| + \frac{1}{16} \|A^I T\|^2} \\ &\quad - \frac{1 + \frac{1}{2} \|A^I T\| + \frac{1}{12} \|A^I T\|^2}{1 - \frac{1}{2} \|A^I T\| + \frac{1}{12} \|A^I T\|^2} - e^{\|A^I T\|}, \end{aligned} \quad (10.b)$$

$G_{p0} = \frac{1}{2} [\underline{G}_p + \bar{G}_p]$ and $\Delta G_{pg} = \frac{1}{2} [\bar{G}_p - \underline{G}_p]$. The associated interval Pade input matrix H_p^I is

$$\begin{aligned} H_p^I &= \left\{ \left[I_n - \frac{1}{2} A^I T + \frac{1}{12} (A^I T)^2 \right]^{-1} + P_{ph}^I \right\} B^I T \\ &= [\underline{H}_p, \bar{H}_p] = [H_{p0} - \Delta H_{ph}, H_{p0} + \Delta H_{ph}] \supseteq H^I, \end{aligned} \quad (11.a)$$

where $P_{ph}^I \in \mathcal{I} \mathcal{R}^{n \times n}$ is an interval matrix with identical elements, each of which is the interval $[-\nu_{ph}, \nu_{ph}]$ obtained from Shieh et al.⁽¹⁾, that is

$$\begin{aligned} \nu_{ph} &= \frac{2}{1 - \frac{1}{2} \|A^I T\| + \frac{1}{16} \|A^I T\|^2} \\ &\quad - \frac{1}{1 - \frac{1}{2} \|A^I T\| + \frac{1}{12} \|A^I T\|^2} - (e^{\|A^I T\|} - 1) \|A^I T\|^{-1}, \end{aligned} \quad (11.b)$$

$$H_{p0} = \frac{1}{2} [\underline{H}_p + \bar{H}_p] \text{ and } \Delta H_{ph} = \frac{1}{2} [\bar{H}_p - \underline{H}_p].$$

In this paper, we propose an interval tuning bilinear method to find G^I and H^I (designed as G_b^I and H_b^I in (7) whose corresponding interval matrix bounds will be narrower than those obtained via the Pade method. Consider a power series of the form

$$\begin{aligned} G_r &= e^{A_r T} = I_n + A_r T + \frac{1}{2!} (A_r T)^2 + \dots \\ &\quad + \frac{1}{k!} (A_r T)^k + \dots \\ &= I_n + A_r T + \frac{1}{2!} (A_r T)^2 + \frac{1}{2^2(1.5)} (A_r T)^3 \\ &\quad + \frac{1}{2^3(3)} (A_r T)^4 + \dots \end{aligned} \quad (12.a)$$

$$\cong I_n + A_r T + \frac{(A_r T)^2}{2} + \frac{(A_r T)^3}{2^2 \xi} + \frac{(A_r T)^4}{2^3 \xi^2} + \dots \quad (12.b)$$

$$= I_n + A_r T + \frac{(A_r T)^2}{2} + \left(I_n - \frac{A_r T}{2\xi} \right)^{-1} \frac{(A_r T)^3}{2^2 \xi} \quad (12.c)$$

$$\begin{aligned} &= \left[I_n - \frac{1}{2\xi} A_r T \right]^{-1} \left[I_n + \frac{2\xi - 1}{2\xi} A_r T \right. \\ &\quad \left. + \frac{\xi - 1}{2\xi} (A_r T)^2 \right], \text{ for } T < \frac{2\xi}{\|A_r\|} \\ &\triangleq G_{br} \end{aligned} \quad (12.d)$$

Obviously, from (12.a) and (12.b), one has $\|G_{br}\| > \|G_r\|$ for parameter $0 < \xi \leq 1.5$ and $G_{br} = G_{b1r}$ for $\xi = 1$, where G_{b1r} and G_{b2r} are the first-order and second-order bilinear approximate system matrices, respectively, and $\|G_{b1r}\| > \|G_{b2r}\| > \|G_r\|$ is proved by Shieh et al.⁽¹⁾. So, under the condition $1 < \xi \leq 1.5$, we can tune the parameter ξ to satisfy $\|G_r\| < \|G_{br}\| \leq \|G_{b2r}\|$ which will produce the desired result $\|G_{br} - G_r\| < \|G_{pr} - G_r\|$ (see Appendix A), for a suitably small sampling (see Appendix B). The best criterion for the selection of the parameter ξ is to choose the one such that the $\|G_{b2r}\| - \|G_{br}\|$ is as large as possible. So, $\xi = 1.5$ is the best choice for (12.d) which is equivalent to second-

order geometrical formula⁽¹⁸⁾, where Shieh et al.⁽¹⁸⁾ considered the degenerate matrix only, not interval ones,

$$G_{g2r} = \left[I_n - \frac{1}{3} A_r T \right]^{-1} \left[I_n + \frac{2}{3} A_r T + \frac{1}{6} (A_r T)^2 \right] \quad (13)$$

Whenever the sampling time is determined from physical restrictions, if $T \geq \frac{2\xi}{\|A_r\|}$, the matrix-valued function of $G_r = e^{A_r T}$ can be approximated as $G_r = e^{A_r T} = (e^{A_r \frac{T}{m}})^m$ for some appropriate integer $m^{(18)}$. Without loss of generality, let the absolute value of the largest eigenvalue of A_r be designed as ρ to find which parameter $\xi \in [1, 1.5]$ is the critical point. From Appendix (A6) we know if $\|G_{br}\| \leq \|G_{b2r}\|$, then $\|G_{br} - G_r\| < \|G_{pr} - G_r\|$ is guaranteed. So, from (9.c) and (12.c), let

$$1 + \rho T + \frac{(\rho T)^2}{2} + \left(1 - \frac{\rho T}{2\xi}\right)^{-1} \frac{(\rho T)^3}{4\xi} \leq 1 + \left(1 - \frac{1}{2}\rho T + \frac{1}{16}(\rho T)^2\right)^{-1} \rho T, \quad (14)$$

then one has $\xi \geq 4/3$ for $\rho T/(2\xi) \ll 1$. According to above discussion, The error-bound relationship between the bilinear, Pade, tuning bilinear and geometric series approximations in terms of ξ can be expressed as

$$\begin{aligned} \|G_{br} - G_r\|_{\xi=1} &= \|G_{b1r} - G_r\| > \|G_{br} - G_r\|_{\xi \in (1, 1.33)} \\ &\geq \|G_{pr} - G_r\| > \|G_{br} - G_r\|_{\xi \in [1.33, 1.5]} \\ &> \|G_{br} - G_r\|_{\xi=1.5} = \|G_{g2r} - G_r\| \end{aligned} \quad (15)$$

The associated tuning bilinear approximate input model is

$$\begin{aligned} H_{br}^- &= (G_{br}^- - I_n) A_r^{-1} B_r \\ &= \left(I_n - \frac{A_r T}{2\xi} \right)^{-1} \left(I_n + \frac{(\xi-1)A_r T}{2\xi} \right) B_r T \end{aligned} \quad (16)$$

The modeling error matrix between G_r and G_{br}^- in (12) can be defined as

$$E_{bgr}^- = G_r - G_{br}^- = \sum_{i=3}^{\infty} \left(\frac{1}{i!} - \frac{1}{2^{i-1}\xi^{i-2}} \right) (A_r T)^i \quad (17)$$

Since

$$\left(\frac{1}{2^{i-1}\xi^{i-2}} - \frac{1}{i!} \right) \geq 0 \quad \text{for } i \geq 3 \text{ and } 1 < \xi \leq 1.5,$$

we have

$$\begin{aligned} \|E_{bgr}^-\| &\leq \sum_{i=3}^{\infty} \left(\frac{1}{2^{i-1}\xi^{i-2}} - \frac{1}{i!} \right) \|A_r T\|^i \\ &= \left[1 - \frac{\|A_r T\|}{2\xi} \right]^{-1} \left[1 + \frac{2\xi-1}{2\xi} \|A_r T\| \right. \\ &\quad \left. + \frac{\xi-1}{2\xi} \|A_r T\|^2 \right] - e^{\|A_r T\|} \end{aligned} \quad (18)$$

Based on Theorem 2.1, we have

$$\begin{aligned} G_r &= e^{A_r T} = \left(I_n - \frac{A_r T}{2\xi} \right)^{-1} \left(I_n + \frac{2\xi-1}{2\xi} A_r T \right. \\ &\quad \left. + \frac{\xi-1}{2\xi} (A_r T)^2 \right) + E_{bgr}^- \\ &\subseteq \left(I_n - \frac{A_r T}{2\xi} \right)^{-1} \left(I_n + \frac{2\xi-1}{2\xi} A_r T \right. \end{aligned}$$

$$\begin{aligned} &\left. + \frac{\xi-1}{2\xi} (A_r T)^2 \right) + P_{bh}^I \\ &\triangleq G_{bh}^I \supseteq G^I = e^{A^I T}, \text{ for } T \leq \frac{2\xi}{\|A^I\|}, \end{aligned} \quad (19.a)$$

where $P_{bh}^I \in \mathcal{I} \mathcal{R}^{n \times n}$ is an interval matrix with identical elements each of which is the interval $[-e_{bh}^-, e_{bh}^-]$ obtained from (18) as below

$$\begin{aligned} e_{bh}^- &= \left(1 - \frac{\|A_r T\|}{2\xi} \right)^{-1} \left(1 + \frac{2\xi-1}{2\xi} \|A_r T\| \right. \\ &\quad \left. + \frac{\xi-1}{2\xi} \|A_r T\|^2 \right) - e^{\|A_r T\|} \end{aligned} \quad (19.b)$$

Also, the degenerate modeling input error matrix is defined as

$$\begin{aligned} E_{bhr}^- &= H_r - H_{br}^- = (G_r - G_{br}^-) A_r^{-1} B_r \\ &= \left[\sum_{i=3}^{\infty} \left(\frac{1}{i!} - \frac{1}{2^{i-1}\xi^{i-2}} \right) (A_r T)^{i-1} \right] B_r T \\ &\triangleq V_{bhr}^- B_r T \end{aligned} \quad (20.a)$$

Hence

$$\begin{aligned} \|V_{bhr}^-\| &\leq \sum_{i=3}^{\infty} \left(\frac{1}{2^{i-1}\xi^{i-2}} - \frac{1}{i!} \right) \|A_r T\|^{i-1} \\ &= \left(1 - \frac{\|A_r T\|}{2\xi} \right)^{-1} \left(1 + \frac{\xi-1}{2\xi} \|A_r T\| \right) \\ &\quad - (e^{\|A_r T\|} - 1) \|A_r T\|^{-1} \end{aligned} \quad (20.b)$$

Based on the approximate model (16), we obtain the desired H^I (designated as H_{bh}^I) in (7.a) as

$$\begin{aligned} H_r &= (e^{A_r T} - I_n) A_r^{-1} B_r = \left(I_n - \frac{A_r T}{2\xi} \right)^{-1} \left(I_n \right. \\ &\quad \left. + \frac{\xi-1}{2\xi} A_r T \right) B_r T + E_{bhr}^- \\ &\subseteq \left[\left(I_n - \frac{A_r T}{2\xi} \right)^{-1} \left(I_n + \frac{\xi-1}{2\xi} A_r T \right) + P_{bh}^I \right] B^I T \\ &\triangleq H_{bh}^I \supseteq H^I = (e^{A^I T} - I_n) (A^I)^{-1} B^I, \text{ for } T < \frac{2\xi}{\|A^I\|}, \end{aligned} \quad (21.a)$$

where $P_{bh}^I \in \mathcal{I} \mathcal{R}^{n \times n}$ is an interval matrix with identical elements each of which is interval $[-\nu_{bh}^-, \nu_{bh}^-]$ obtained from (20.b) as

$$\begin{aligned} \nu_{bh}^- &= \left(1 - \frac{\|A_r T\|}{2\xi} \right)^{-1} \left(1 + \frac{\xi-1}{2\xi} \|A_r T\| \right) - (e^{\|A_r T\|} \\ &\quad - 1) \|A_r T\|^{-1} \end{aligned} \quad (21.b)$$

Considering the discrete-time model (7) and the tuning enclosing one based on (19) to (21), we have $x_b^-(kT) \supseteq x_a(kT)$ for the same initial condition, according to Theorem 2.1. To find the solution bound of (6) is very difficult, but we can use the enclosing discrete-time model based on (19) to (21) to solve the approximately enclosed interval solution by recursive way.

Comparing the Pade interval system model (G_{bh}^I in (10) and H_{bh}^I in (11)) with the tuning bilinear interval system model (G_{bh}^I in (19) and H_{bh}^I in (21)), It has been shown by Shieh et al.⁽¹⁾ that the Pade approximate matrix exactly matches the first five moments of the power series (12.a), but the tuning bilinear approximate matrix just matches the first three moments of

the power series (12.a). However, one can adjust the parameter ξ of the tuning bilinear approximate matrix in (19.a) to get less conservative modeling error bound in (19.b) than that in (10.b) (see Appendix A). The result can be extended to the case of H_b^l and H_b^r .

4. Analog Interval Model Conversion

Consider the discrete-time uncertain system (7)

$$x_d(kT+T) = G^l x_d(kT) + H^l u_d(kT), \quad x_d(t) = x_{co}, \quad (22)$$

where $G^l \in \mathcal{I}\mathcal{R}^{n \times n}$ and $H^l \in \mathcal{I}\mathcal{R}^{n \times m}$ are interval matrices containing, respectively, degenerate interval matrices $G_r \in \mathcal{R}^{n \times n}$ with $\det(G_r - I_n) \neq 0$ and $H_r \in \mathcal{R}^{n \times m}$ such that $G_r \subseteq G^l$, $G^l = [\underline{G}, \bar{G}] = [G_o - \Delta G, G_o + \Delta G]$, $G_o = 0.5(\underline{G} + \bar{G})$, $\Delta G = 0.5(\bar{G} - \underline{G})$, $H_r \subseteq H^l$, $H^l = [\underline{H}, \bar{H}] = [H_o - \Delta H, H_o + \Delta H]$, $H_o = 0.5(\underline{H} + \bar{H})$ and $\Delta H = 0.5(\bar{H} - \underline{H})$. It is desired to find the equivalent continuous-time uncertain model of the form

$$\dot{x}_c(t) = A^l x_c(t) + B^l u_c(t), \quad (23)$$

where $A^l \in \mathcal{I}\mathcal{R}^{n \times n}$ and $B^l \in \mathcal{I}\mathcal{R}^{n \times m}$ are interval matrices which enclose corresponding degenerate matrices $A_r \in \mathcal{R}^{n \times n}$ and $B_r \in \mathcal{R}^{n \times m}$ such that

$$A_r = \frac{1}{T} \ln(G_r) \subseteq A^l = \frac{1}{T} \ln(G^l) \in \mathcal{I}\mathcal{R}^{n \times n} \quad (24.a)$$

and

$$B_r = A_r(G_r - I_n)^{-1} H_r \subseteq B^l = A^l(G^l - I_n)^{-1} H^l \in \mathcal{I}\mathcal{R}^{n \times m} \quad (24.b)$$

Shieh et al.⁽¹⁴⁾ have proposed a matrix continued fraction method for calculating the approximants of A_r and B_r . Based on the matrix continued fraction method⁽¹⁴⁾, we also induct one parameter $\bar{\xi}$ to obtain less conservative modeling error matrices between (A_r and A_{br}) and between (B_r and B_{br}), respectively. The tuning bilinear approximation is shown below:

$$\begin{aligned} A_r &= \frac{1}{T} \ln(G_r) \\ &= \frac{2}{T} \left[R_r + \frac{1}{3} R_r^3 + \dots + \frac{1}{n} R_r^n + \frac{1}{n+2} R_r^{n+2} + \dots \right] \\ &\quad \text{for } \|R_r\| < 1 \\ &\cong \frac{2}{T} R_r \left(I_n - \frac{1}{3} R_r^2 \right)^{-1} \triangleq A_{pr} \\ &\cong \frac{2}{T} R_r \left[I_n + \frac{1}{3} R_r^2 + \frac{\bar{\xi}}{3^2} R_r^4 \left(I_n + \frac{R_r^2}{3} + \frac{R_r^4}{3^2} + \dots \right) \right] \\ &= \frac{2}{T} R_r \left[I_n + \frac{\bar{\xi}-1}{9} R_r^4 \right] \left[I_n - \frac{1}{3} R_r^2 \right]^{-1} \\ &\triangleq A_{br}, \text{ for } \|R_r\|^2/3 < 1, \end{aligned} \quad (25.a)$$

$$\begin{aligned} B_r &= A_r(G_r - I_n)^{-1} H_r \\ &\cong A_r(G_{br} - I_n)^{-1} H_r \\ &= \frac{1}{T} \left(I_n + \frac{\bar{\xi}-1}{2\bar{\xi}} A_r T \right)^{-1} \left(I_n - \frac{1}{2\bar{\xi}} A_r T \right) H_r \\ &\triangleq B_{br}, \end{aligned} \quad (25.b)$$

where $R_r = (G_r - I_n)(G_r + I_n)^{-1}$. The associated Pade approximation method's formula⁽¹⁾ are

$$A_{pr} \triangleq \frac{2}{T} R_r \left(I_n - \frac{1}{3} R_r^2 \right)^{-1}, \quad (26.a)$$

$$B_{pr} \triangleq \frac{1}{T} \left(I_n - \frac{1}{2} A_r T + \frac{1}{12} (A_r T)^2 \right) H_r \quad (26.b)$$

Whenever a relatively large sampling time is determined from physical restrictions, the matrix-valued function of $A_r = \frac{1}{T} \ln(G_r)$ can be approximated as $A_r = \frac{1}{T} \ln(G_r) = \frac{1}{T} \ln({}^q\sqrt{G_r})^q = \frac{q}{T} \ln({}^q\sqrt{G_r})$, where

matrix ${}^q\sqrt{G_r}$ is the principle q th root of the matrix G_r , and R_r is replaced by $R_r = ({}^q\sqrt{G_r} - I_n)({}^q\sqrt{G_r} + I_n)^{-1(14)}$.

Here, it is desired to adjust $\bar{\xi}$ in (25.a) so that tuning bilinear approximate model error is less than that obtained via the Pade approximation method. For this case, if we choose $1 \leq \bar{\xi} \leq 1.8$, then $\|A_{pr}\| \leq \|A_{br}\| \leq \|A_r\|$. Meanwhile, the best choice for $\bar{\xi}$ is 1.8 to obtain the minimal error bound $\|A_r\| - \|A_{br}\| \leq \|A_r\| - \|A_{pr}\|$ (see Appendix C). In order to find the error matrix between A_r and A_{br} , we define a degenerate error matrix $E_{br} \in \mathcal{R}^{n \times n}$ as

$$E_{br} \triangleq A_{br} - A_r = \frac{2}{T} \sum_{j=3}^{\infty} \left(\frac{\bar{\xi}}{3^{j-1}} - \frac{1}{2j-1} \right) R_r^{2j-1} \quad (27.a)$$

Because

$$\frac{1}{2j-1} \geq \frac{\bar{\xi}}{3^{j-1}} \quad \text{for } 1 \leq \bar{\xi} \leq 1.8 \text{ and } j \geq 3,$$

hence

$$\begin{aligned} \|E_{br}\| &\leq \frac{2}{T} \sum_{j=3}^{\infty} \left(\frac{1}{2j-1} - \frac{\bar{\xi}}{3^{j-1}} \right) \|R_r\|^{2j-1} \\ &= \frac{1}{T} \ln \left(\frac{1+\|R_r\|}{1-\|R_r\|} \right) - \frac{2\|R_r\|}{T} \left(1 + \frac{\bar{\xi}-1}{9} \|R_r\|^4 \right) \\ &\quad \left(1 - \frac{\|R_r\|^2}{3} \right)^{-1} \end{aligned} \quad (27.b)$$

$$= \|A_r\| - \|A_{br}\| \leq \|A_r\| - \|A_{pr}\|,$$

$$\text{for } \|A_r\| \geq \|A_{br}\| \geq \|A_{pr}\| \quad (27.c)$$

Thus, tuning the parameter $\bar{\xi}$, one can determine the desired tuning bilinear uncertain system matrix A^l (designated an A_b^l) in (23) with less error bound than that obtained via the Pade approximation as follows:

$$\begin{aligned} A_r &= \frac{1}{T} \ln(G_r) = \frac{2}{T} R_r \left(I_n + \left(\frac{\bar{\xi}-1}{9} \right) R_r^4 \right) \left(I_n - \frac{1}{3} R_r^2 \right)^{-1} + E_{br} \\ &\subseteq \frac{2}{T} R_r^l \left(I_n + \left(\frac{\bar{\xi}-1}{9} \right) (R_r^l)^4 \right) \left(I_n - \frac{1}{3} (R_r^l)^2 \right)^{-1} + P_{ba}^l \\ &\triangleq A_b^l \supseteq A^l, \end{aligned} \quad (28.a)$$

where $R^l = (G^l - I_n)(G^l + I_n)^{-1}$ and $P_{ba}^l \in \mathcal{I}\mathcal{R}^{n \times n}$ is an interval matrix with identical elements, each of which is the interval obtained from (27.b), that is

$$\begin{aligned} e_{ba} &= \frac{1}{T} \ln \left(\frac{1+\|R^l\|}{1-\|R^l\|} \right) - \frac{2\|R^l\|}{T} \left(1 + \frac{\bar{\xi}-1}{9} \|R^l\|^4 \right) \\ &\quad \left(1 - \frac{\|R^l\|^2}{3} \right)^{-1} \end{aligned} \quad (28.b)$$

As a result, we obtain the interval tuning bilinear uncertain system matrix A_b^l as

$$A_b^l = \frac{2}{T} R^l \left(I_n + \frac{\xi-1}{9} (R^l)^4 \right) \left(I_n - \frac{1}{3} (R^l)^2 \right)^{-1} + P_{bb}^l$$

$$= [\underline{A}_b, \bar{A}_b] = [A_{bo} - \Delta A_b, A_{bo} + \Delta A_b] \supseteq A^l, \quad (29)$$

where $A_{bo} = 0.5(\bar{A}_b + \underline{A}_b)$, $\Delta A_b = 0.5(\bar{A}_b - \underline{A}_b)$. In short, we can adjust the scalar parameter ξ in (29) for $1 \leq \xi \leq 1.8$ to satisfy condition (27.c), and to yield (29) $A_b^l \supseteq A^l$.

To find the associated tuning bilinear uncertain input matrix B^l (designated an B_b^l) in (23), we define a degenerate error matrix $V_{bbr} \in \mathcal{R}^{n \times n}$ as

$$V_{bbr} = [(G_r - I_n)A_r^{-1}]^{-1} \left[I_n + \frac{\xi-1}{2\xi} A_r T \right]^{-1} \left[I_n - \frac{1}{2\xi} A_r T \right] \frac{1}{T}$$

$$= [(G_r - I_n)A_r^{-1}]^{-1} \left[I_n - (G_r - I_n)(A_r T)^{-1} \right. \\ \left. \left[I_n + \frac{\xi-1}{2\xi} A_r T \right]^{-1} \left[I_n - \frac{1}{2\xi} A_r T \right] \right] \\ = [(G_r - I_n)A_r^{-1}]^{-1} \left[\left(\frac{a+1}{2^2} - \frac{1}{3!} \right) (A_r T)^2 + \left(\frac{1}{3!2} + \frac{a(a+1)}{2^3} \right) \right. \\ \left. - \frac{1}{4!} (A_r T)^3 + \left(\frac{1}{4!2} + \frac{1}{3!2^2} + \frac{a^2(a+1)}{2^4} - \frac{1}{5!} \right) (A_r T)^4 + \dots \right], \quad (30.a)$$

where

$$a = \frac{1-\xi}{\xi}, \quad 1 < \xi \leq 1.5, \quad a < 0 \text{ and } |a| < 1$$

Hence, each coefficient of (30.a) is positive and the norm of V_{bbr} becomes

$$\|V_{bbr}\| \leq \left\| \frac{1}{T} [(G_r - I_n)(A_r T)^{-1}]^{-1} \left(1 - (e^{\|A_r T\|} - 1) \|A_r T\|^{-1} \left(1 + \frac{\xi-1}{2\xi} \|A_r T\| \right)^{-1} \left(1 - \frac{\|A_r T\|}{2\xi} \right) \right) \right\| \quad (30.b)$$

According to the inclusion theorem, the desired continuous-time tuning uncertain input matrix B_b^l is

$$B_{br} = \left[\frac{1}{T} \left(I_n + \frac{\xi-1}{2\xi} A_r T \right)^{-1} \left(I_n - \frac{A_r T}{2\xi} \right) + V_{bbr} \right] H_r$$

$$\subseteq B_b^l = \left[\frac{1}{T} (I_n + \frac{\xi-1}{2\xi} A^l T)^{-1} \left(I_n - \frac{A^l T}{2\xi} \right) + P_{bb}^l \right] H^l$$

$$= [\underline{B}_b, \bar{B}_b] = [B_{bo} - \Delta B_b, B_{bo} + \Delta B_b] \supseteq B^l, \quad (31)$$

where $B_{bo} = \frac{1}{2}[\underline{B}_b + \bar{B}_b]$, $\Delta B_b = \frac{1}{2}[\bar{B}_b - \underline{B}_b]$ and $P_{bb}^l \in \mathcal{R}^{n \times n}$ is an interval matrix with identical elements, each of which is the interval $[-\nu_{bb}, \nu_{bb}]$ obtained from (30.b), that is

$$\nu_{bb} = \left\| \frac{1}{T} [(e^{A^l T} - I_n)(A^l T)^{-1}]^{-1} \right\|$$

$$\times \left(1 - (e^{\|A^l T\|} - 1) \|A^l T\|^{-1} \left(1 + \frac{\xi-1}{2\xi} \|A^l T\| \right)^{-1} \left(1 - \frac{\|A^l T\|}{2\xi} \right) \right) \quad (32)$$

Similarly the interval Pade approximate model proposed by Shieh et al.⁽¹⁾ is:

$$A_b^l = \frac{2}{T} R^l \left[I_n - \frac{1}{3} (R^l)^2 \right]^{-1} + P_{pa}^l = [\underline{A}_p, \bar{A}_p], \quad (33.a)$$

where $R^l = (G^l - I_n)(G^l + I_n)^{-1}$, $P_{pa}^l \in \mathcal{R}^{n \times n}$ is an interval matrix with identical elements each of which is the interval $[-e_{pa}, e_{pa}]$ obtained as

$$e_{pa} = \frac{1}{T} \ln \left(\frac{1 + \|R^l\|}{1 - \|R^l\|} \right) - \frac{2}{T} \|R^l\| \left(1 - \frac{1}{3} \|R^l\|^2 \right)^{-1} \quad (33.b)$$

The associated input matrix obtained via the Pade method is

$$B_p^l = \left\{ \frac{1}{T} \left[I_n - \frac{1}{2} A^l T + \frac{1}{12} (A^l T)^2 \right] + P_{pb}^l \right\} H^l, \quad (34.a)$$

where $P_{pb}^l \in \mathcal{R}^{n \times n}$ is an interval matrix with identical elements each of which is the interval $[-\nu_{pb}, \nu_{pb}]$ obtained as

$$\nu_{pb} = \frac{1}{T} \left\| \left(\sum_{i=1}^{\infty} \frac{1}{i!} (A^l T)^{i-1} \right)^{-1} [1 - (e^{\|A^l T\|} - 1) \|A^l T\|^{-1} \left(1 - \frac{1}{2} \|A^l T\| + \frac{1}{12} \|A^l T\|^2 \right)] \right\| \quad (34.b)$$

We use the Hansen's inverse algorithm and interval arithmetic operation to compute and compare A_b^l , B_b^l , G_b^l and H_b^l proposed in this paper with A_p^l , B_p^l , G_p^l and H_p^l proposed by Shieh et al.⁽¹⁾, respectively. We will show that interval bounds of the system matrices via the tuning bilinear model are narrower than those obtained via the Pade approximation method in the next section.

5. Illustrative Examples

Consider an asymptotically stable linear R-L-C circuit⁽¹³⁾ described by an uncertain state Eq.(6). The nominal and perturbed system matrices of the dynamic system (6) are

$$A_o = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad B_o = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (35)$$

The initial vector $x_c(0) = 0_{2 \times 1}$ and the input vector $u_c(t)$ is a unit-step function. The continuous-time uncertain model of (35) is desired to be converted to its discrete-time counterpart in (7) for a sampling time $T = 0.1$ sec.

Based the tuning bilinear approximation method, one determines the interval system matrices in (19) and (21) by choosing the parameter $\xi = 1.5$ as

Fig. 1 Unit-step response of the uncertain system, Pade state $x_p(t)$, tuning bilinear state $x_b(t)$ and the nominal state $x_o(t)$.

$$G_b^I = \begin{bmatrix} [0.8082 & 0.8292] & [-0.0005 & 0.0005] \\ [0.0673 & 0.0883] & [0.7404 & 0.7414] \end{bmatrix},$$

$$H_b^I = \begin{bmatrix} [0.1798 & 0.1827] \\ [0.1798 & 0.1827] \end{bmatrix}, \quad (36.a)$$

and the enclosing interval solution $x_b^-(t)$ at $t = t_f = 6$ sec. obtained by recursive way is

$$x_b^-(t) = \begin{bmatrix} [0.9153 & 1.0948] \\ [0.9155 & 1.0949] \end{bmatrix} \quad (36.b)$$

Also, the Pade enclosing interval system matrices (10) and (11) and the enclosing interval solution $x_p(t)$ at $t = t_f = 6$ sec. are shown below, respectively,

$$G_p^I = \begin{bmatrix} [0.8056 & 0.8319] & [-0.0044 & 0.0044] \\ [0.0645 & 0.0914] & [0.7364 & 0.7452] \end{bmatrix},$$

$$H_p^I = \begin{bmatrix} [0.1761 & 0.1864] \\ [0.1761 & 0.1865] \end{bmatrix}, \quad (37.a)$$

$$x_p(t) = \begin{bmatrix} [0.8801 & 1.1390] \\ [0.8834 & 1.1403] \end{bmatrix} \quad (37.b)$$

It is observed that each entry of the tuning bilinear enclosing system matrices (36.a) is enclosed by those of the Pade interval system (37.a) (i. e. $G_b^I \subset G_p^I$ and $H_b^I \subset H_p^I$). From (36.b) and (37.b), one has the interval enclosing steady-state solution $x_b^-(t_f) \subset x_p(t_f)$. The transient enclosing state responses of (35) are shown in Fig. 1, and $x_b^-(t) \subset x_p(t)$ is observed from the simulation result which is identical to the one shown by Shieh et al.⁽⁷⁾. From their work⁽⁷⁾ we can also see that the enclosing first-order bilinear interval system

matrix bound is wider than that of the second-order bilinear method, $G_{b1} \supset G_{b2}$ and $H_{b1} \supset H_{b2}$, but the proposed tuning first-order bilinear interval system matrices bounds (36.a) are highly decreased, that is $G_{b1} \supset G_{b2} \supset G_b^l$ and $H_{b1} \supset H_{b2} \supset H_b^l$. In the meantime, those system matrices (36.a) obtained via the proposed method are also enclosed by the Pade enclosing interval system matrices (37.a). Hence, the proposed tuning bilinear approximate interval model yields less conservative enclosing bounds for the exact interval solution than those of the Pade and the second-order bilinear enclosing interval models, respectively.

For the discrete-time to continuous-time uncertain model conversion, we consider the discrete-time interval model (22) with the interval system matrices (36.a) and (37.a), respectively. Here, the corresponding continuous-time interval model (6) is desirable.

By setting $\bar{\xi}=1.8$, the proposed tuning bilinear approximate interval system matrices (29) and (31) can be computed as

$$A_b^l = \begin{bmatrix} [-2.1296 & -1.8726] & [-0.0070 & 0.0070] \\ [0.8592 & 1.1394] & [-3.0064 & -2.9911] \end{bmatrix},$$

$$B_b^l = \begin{bmatrix} [1.9683 & 2.0322] \\ [1.9649 & 2.0348] \end{bmatrix} \quad (38.a)$$

Also, the Pade approximate interval system matrices⁽¹⁾ (33) and (34) can be computed as

$$A_p^l = \begin{bmatrix} [-2.1664 & -1.8355] & [-0.0579 & 0.0579] \\ [0.8184 & 1.1855] & [-3.0632 & -2.9368] \end{bmatrix},$$

$$B_p^l = \begin{bmatrix} [1.9219 & 2.0793] \\ [1.9120 & 2.0868] \end{bmatrix} \quad (38.b)$$

From the results shown in (38.a) and (38.b), we observe that each entry of the tuning bilinear and the Pade's uncertain system matrices encloses the associated entry of the uncertain matrices (35), respectively. Also, each interval element of the Pade approximate model (38.b) encloses that of the corresponding tuning system matrices (38.a). In other words, $A_p^l \supset A_b^l \supset A^l$, $B_p^l \supset B_b^l \supset B^l$. Hence the proposed tuning approximate model (38.a) presents a narrower interval than that obtained from the Pade approximate model (38.b).

6. Conclusions

A tuning bilinear approximation method together with the interval arithmetic operation is proposed to convert a continuous-time (discrete-time) interval system to an equivalent discrete-time (continuous-time) interval model. We present a tuning bilinear approximation method which induces one parameter to adjust the error bounds of the system matrices such that interval model bound is narrower than that obtained via the Pade approximation method. So, the

modified tuning bilinear approximate uncertain model encloses the original exact uncertain system. Hence, the analysis and design of an original uncertain system can be carried out using the converted uncertain model in either the continuous-time or the discrete-time domain.

Appendix A

Let $x \triangleq \|A^l T\| \in \mathcal{R}$, $x > 0$ and $g \triangleq e^x$. Also, let

$$g \cong g_{pr} = \left(1 - \frac{x}{2} + \frac{x^2}{12}\right)^{-1} \left(1 + \frac{x}{2} + \frac{x^2}{12}\right) \quad \text{in (9.a),} \quad (A1)$$

$$g \cong g_{b2r} = \left(1 - \frac{x}{2} + \frac{x^2}{16}\right)^{-1} \left(1 + \frac{x}{2} + \frac{x^2}{16}\right) \quad \text{in (9.c),} \quad (A2)$$

$$g \cong g_{br} = \left(1 - \frac{x}{2\xi}\right)^{-1} \left(1 + \frac{2\xi-1}{2\xi}x + \frac{\xi-1}{2\xi}x^2\right) \quad \text{in (12.a),} \quad (A3)$$

where Shieh et al.⁽¹⁾ have shown that $g_{b2r} \geq g$, $g_{b2r} \geq g_{pr}$ and the absolute error between g and g_{pr} can be written as

$$e_{pgr} \triangleq |g - g_{pr}| = |g - g_{b2r} + g_{b2r} - g_{pr}|$$

$$\leq |g - g_{b2r}| + |g_{b2r} - g_{pr}|$$

$$= 2g_{b2r} - g - g_{pr} \quad (A4)$$

We have to choose the scalar parameter ξ , where $1 < \xi \leq 1.5$, to satisfy $g \leq g_{br} \leq g_{b2r}$. Let the error between g and g_{br} be written as

$$e_{br} \triangleq g_{br} - g \quad (A5)$$

Based on the results shown in (A4) and (A5), we can calculate the difference between (A4) and (A5) as

$$e_{pgr} - e_{br} = 2g_{b2r} - g_{br} - g_{pr}$$

$$= (g_{b2r} - g_{pr}) + (g_{b2r} - g_{br}) > 0 \quad (A6)$$

Hence, the proposed modeling errors is less conservative than the Pade modeling error.

Appendix B

Let $x \triangleq \|A^l T\| \in \mathcal{R}$, $x > 0$. The tuning bilinear approximant of g (defined as g_{br}) can be written as

$$g_{br} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2^2\xi} + \frac{x^4}{2^3\xi^2} + \frac{x^5}{2^4\xi^3} + \frac{x^6}{2^5\xi^4} + \dots \quad (B1)$$

The second-order bilinear approximant of g is

$$g_{b2r} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2^2 \frac{4}{3}} + \frac{x^4}{2^3 \frac{8}{4}} + \frac{x^5}{2^4 \frac{16}{5}} + \frac{x^6}{2^5 \frac{32}{6}} + \dots \quad (B2)$$

From (B1) and (B2), we observe that if $1 < \xi \leq 1.5$, then one can tune the scalar parameter ξ in the bound (1, 1.5] to let the sum of the first six terms of (B1) be smaller than that of the first six terms of (B2), but for the remain higher-order terms of (B1) no ξ exists in the bound (1, 1.5] such that the coefficients of the higher-order terms of (B1) are smaller than those of (B2), respectively. Although the coefficients of the higher-order terms of (B1) are greater than those of

(B2) for $(x/2\xi) < 1$, respectively, a more higher-order term of (B1) has a smaller value. So, if T is suitably small, then we still can find the desired ξ in the bound $(1, 1.5]$ to have $g_{\bar{b}r} \leq g_{b2r}$.

Appendix C

The approximate models⁽¹⁾ A_r are shown as follows:

$$A_r = \frac{2}{T} \left[R_r + \frac{1}{3} R_r^3 + \frac{1}{5} R_r^5 + \frac{1}{7} R_r^7 + \dots \right] \text{ for } \|R_r\| < 1 \quad (\text{C1.a})$$

$$\cong \frac{2}{T} \left[R_r + \frac{1}{3} R_r^3 + \frac{1}{3^2} R_r^5 + \frac{1}{3^3} R_r^7 + \dots \right] = \frac{2}{T} R_r \left[I_n - \frac{1}{3} R_r^2 \right]^{-1} \triangleq A_{pr} \quad (\text{C1.b})$$

$$\cong \frac{2}{T} \left[R_r + \frac{1}{3} R_r^3 + \frac{\bar{\xi}}{3^2} R_r^5 \left(I_n + \frac{1}{3} R_r^2 + \frac{1}{3^2} R_r^4 + \dots \right) \right] \triangleq A_{\bar{b}r} \quad (\text{C1.c})$$

From (C1.b) and (C1.c), we observe that if $\bar{\xi} \geq 1$, then $A_{\bar{b}r} \geq A_{pr}$. Comparing all the coefficients of (C1.a) with those of (C1.c), one has

$$\text{coefficient of } R_r^5: \frac{1}{5} > \frac{\bar{\xi}}{9} \Rightarrow \frac{9}{5} > \bar{\xi},$$

$$\text{coefficient of } R_r^7: \frac{1}{7} > \frac{\bar{\xi}}{3^3} \Rightarrow \frac{27}{7} > \bar{\xi},$$

$$\text{coefficient of } R_r^9: \frac{1}{9} > \frac{\bar{\xi}}{3^4} \Rightarrow \frac{81}{9} > \bar{\xi},$$

So, if we choose $1 \leq \bar{\xi} \leq 1.8$, then $\|A_{pr}\| \leq \|A_{\bar{b}r}\| \leq \|A_r\|$ is guaranteed. From (C1.c), we observe that if $\bar{\xi} = 1$, then $A_{pr} = A_{\bar{b}r}$; meanwhile, if $\bar{\xi} = 1.8$, then the $\|A_{\bar{b}r}\| - \|A_{pr}\|$ is maximal. So, $\bar{\xi} = 1.8$ is the best choice for getting the minimal error bound $\|A_r\| - \|A_{\bar{b}r}\|$.

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