A Method of Minimal-Time Control to a Quantized Linear Discrete System with Rational Coefficients*

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In quantized control systems, the control can take only discrete (e.g. integer) values. Optimization of such systems has been conventionally performed by the mathematical programming method such as dynamic programming or integer programming. In a quantized, linear, time-invariant, discrete system with rational coefficients, a minimal-time control problem is formulated by using the transition between two points in the state space. Considering the properties of the system expressed by the state equation employed in the present investigation and the relationship between a discrete-time control problem and a mathematical programming problem, a solving method of this optimal quantized linear discrete control problem is demonstrated with two numerical examples.

Key Words: Measurement and Control, Optimal Control, Control Method, Quantized Discrete System, Mathematical Programming Technique, Gauss-Jordan Method

1. Introduction

Control systems in which controls can take discrete-level values (e.g. integer values) are called quantized control systems. These systems were optimized by the mathematical programming method such as dynamic programming (DP)^{(1),(2)} or integer programming (IP)^{(3),(4)}.

Compared with the conventional continuous control systems, it can be stated that the control is not precise, but the control itself is simple and a predetermination of the control is expected. These quantized controls are thought essential in the problem of switching systems, ecological systems handling a number of individuals, investment problems, traffic control systems etc. which can be described by the state

equation (1). However, to the best of authors' knowledge, there are not so many researches on these quantized control systems from the stand point of control engineering except studies about a structure of these systems: reachability, controllability, realization, state estimation and discrete-time control etc. by Anzai^{(4),(5)}, Miyamichi⁽⁶⁾ and Munakata⁽⁷⁾.

In this paper, based on the results of Dr. Anzai and Dr. Miyamichi, the minimal-time control problem of a quantized linear discrete system with rational coefficients is investigated by using the reachability of this system, and a solving method of the control with mathematical programming technique (R. S. M.: Recursive Solving Method) is given with numerical examples.

2. Minimal-Time Control of a Quantized Linear Discrete Control System with Rational Coefficients

2.1 Minimal-time control problem

The state equation in this control problem is described as,

$$x(i+1) = Ax(i) + bu(i), \quad i=0, 1, 2, \dots \quad (1)$$

where,

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Fig. 1 A transition of state vectors in each time (step) based on state equation and a check of the terminal point in case of two-dimensional systems

x is a state vector, an element of the n -dimensional real space R^n ,

u is a scalar control (single input), an element of Z^1 , the subspace of real space R^1 ,

A is an n -dimensional square matrix with rational elements, and

b is an n -dimensional vector with rational elements.

Let x_1 and x_2 denote two elements of the state space R^n , where $x_1=0$ (origin) and x_2 is given as a vector that differs from x_1 . Let Gn denote the following matrix,

$$Gn = [A^{n-1}b \ A^{n-2}b \ \dots \ Ab \ b]. \quad (2)$$

It is assumed that

$$\text{rank } Gn = n.$$

Then, the minimal-time control problem for a quantized linear discrete system is given as follows.

Determine the minimal-time (the minimum number of step) i and the integer-valued controls $u(\hat{i})$, $0 \leq \hat{i} \leq i$ so that the state $x(i)$ can reach the other state x_2 under the condition that the state equation (1) is satisfied with $u(\hat{i}) \in Z^1$.

See Fig. 1.

The value i satisfying the foregoing control problem is called the optimal minimal-time (optimal minimal number of step), and the controls $u(\hat{i})$, $0 \leq \hat{i} \leq i$ are called the optimal integer control vectors.

2.2 Reachability in a quantized linear time-invariant discrete control system with rational coefficients

Define Gi by

$$Gi = [A^{i-1}b \ A^{i-2}b \ \dots \ Ab \ b].$$

In this system, with $x(0)=0$ as starting (initial) point, the foregoing state equation yields the set of reachable points (state vectors) in time (step) i as

$$Mi = \{x | x \in R^n, \exists u = (u(0), u(1), \dots, u(i))^T, \\ u(\hat{i}) \in Z^1, \forall \hat{i}, x = Gi \cdot u\}. \quad (3)$$

Here, a^T denotes the transpose of vector a .

The definition, lemmas, corollary and theorems on the reachability in this quantized linear discrete control system are given in the following.

Definition on the reachability:

If $x \in M_\infty$, then x is said reachable from the origin

($0 \in R^n$), where the set M_∞ is obtained by summing up the set Mi in time (step) i to infinity, i.e.

$$M_\infty = \bigcup_{i=0}^{\infty} Mi. \quad (4)$$

If the relation of $\bar{M}_\infty = R^n$ holds, then it is said that the complete reachability is achieved. Here, $\bar{M}_\infty$ denotes the enclosure of the set M_∞ .

Now, the reachable set M_∞ in a quantized linear discrete system contains only the countable state points (vectors), so it is not reachable in the sense of continuous state vectors and is called "almost reachable^{(5),(6)}".

Lemmas, corollary and theorems on the reachability:

Lemma 1

If the matrix A in a quantized linear discrete system has eigenvalues which are not algebraic integers, arbitrary $x \in M_\infty$ is an accumulation point in M_∞ . Here, algebraic integers are solutions of a monic polynomial with integer coefficients.

Proof)

See Ref. (6).

The next corollary will hold as the result of Lemma 1.

Corollary

If algebraic integers are not contained in eigenvalues of the matrix A , the system is completely reachable.

Lemma 2

Let A be the matrix with rational elements. Then, the next two conditions are equivalent.

(1) All eigenvalues of the matrix A are algebraic integers.

(2) The elements of A are integers.

Proof)

See Ref. (6).

Lemma 3

In a quantized linear discrete system given in Eq. (1), let A and b be the matrix and the vector with rational elements, respectively. Then, if all eigenvalues of the matrix A are algebraic integers, the set M_∞ consists of isolated points (vectors) only.

Proof)

By using transformation of coordinates by some nonsingular matrix T with rational elements based on Lemma 2, the system matrix A can be transformed to the matrix with integer elements. And, the relation of $M_\infty = Mn$ holds in a quantized linear discrete system in which elements (of the matrix) are integers. Therefore, the set M_∞ consists of isolated points (state vectors) only.

Q. E. D.

Theorem 1

In a quantized linear discrete system with

rational elements, the necessary and sufficient condition of the complete reachability is that algebraic integers are not contained in eigenvalues of the matrix A .

Proof)

Sufficiency: if algebraic integers are contained in eigenvalues of the matrix A , the matrix A can be decomposed into sub-matrices A_1 and A_2 with direct-sum forms by the nonsingular matrix; that is, A_1 is the sub-matrix of which eigenvalues are algebraic integers only and A_2 is the sub-matrix of which eigenvalues are not algebraic integers. And, by Lemma 3, a subspace spanned by vectors in the matrix A_1 is a discrete subspace. So, reachable sets of states in this system are discrete partially; that is, this system is not completely reachable. Here, a discrete space means the space with discrete elements.

Necessity: it is evident by the result on Corollary of Lemma 1.

Q. E. D.

According to the result of Theorem 1, an accumulation of the set M_∞ in a quantized linear discrete system with rational elements depends on the characteristics on eigenvalues of the matrix A only.

2.3 Relations between minimal-time control problem and reachability and solutions in this problem^{(8),(9)}

Based on Theorem 1, Theorem 2 given in 2. 3 is derived⁽¹⁰⁾ as follows.

Theorem 2

In a quantized linear discrete system given in Eq. (1), the next two facts hold.

(1) If algebraic integers are not contained in eigenvalues of the system matrix A , there exists $\hat{i}$ in $0 \leq i < \infty$ for arbitrary $x_2 \in R^n$.

(2) If algebraic integers are contained in eigenvalues of the system matrix A , and

- 1) in case of $x_2 \notin Mt$, $\hat{i}$ does not exist,
- 2) in case of $x_2 \in Mt$, there exists $\hat{i}$ in $0 \leq i \leq t-1$.

Here, $(t-1)$ th step means that time i is $t-1$ such that the relations of $Mt = Mt-1$ and $Mt = M_\infty$ hold.

Proof)

(1): based on Theorem 1, if algebraic integers are not contained in eigenvalues in the matrix A , arbitrary $x_2 (\in R^n)$ is represented as infinitely generated integers' submodule. Therefore, there exists $\hat{i}$ in $0 \leq i < \infty$.

(2): based on Theorem 1, if algebraic integers are contained in eigenvalues in the matrix A , the relation of $Mt = M_\infty$ holds and the set Mt does not increase by increasing t . And, Mt is the set of isolated points. Therefore, if x_2 is not contained in M_∞ (that is, $x_2 \notin M_\infty$), then x_2 can not be reached from x_1 ; so $\hat{i}$ does not exist. If x_2 is contained in the set M_∞ (that is, $x_2 \in M_\infty$),

then the relation of $M_\infty = Mt$ holds; so there exists $\hat{i}$ in $0 \leq i \leq t-1$.

Q. E. D.

Supposed that there exists $\hat{i}$ (optimal minimal-time) in this control problem, then as the result of Theorem 2,

$$x_2 = \sum_{j=0}^i A^{i-j} b u(j). \quad (5)$$

Because, from the system equation:

$$x(i+1) = Ax(i) + bu(i), \quad i=0, 1, 2, \dots$$

$$x(i+1) = A^{i+1}x(0) + \sum_{j=0}^i A^{i-j} b u(j), \quad (6)$$

and substituting $x_1 = x(0) = 0$, and $x_2 = x(i+1)$ in Eq. (6).

Let

$$\gamma_j = A^{i-j} b, \quad j=0, 1, 2, \dots, i$$

$$z_j = u(j), \quad j=0, 1, 2, \dots, i,$$

Eq. (5) becomes

$$x_2 = \sum_{j=0}^i \gamma_j \cdot z_j \quad (7)$$

This control problem is reduced to the next Mathematical Programming problem of an integer minimization:

minimize i

subject to

$$x_2 = \sum_{j=0}^i \gamma_j \cdot z_j$$

$$z_j \in Z^1.$$

By solving the above Mathematical Programming problem, the optimal $\hat{i}$ can be obtained. This problem can be solved by using the algorithms (the recursive solving method: R. S. M.) given in Section 2. 4.

2.4 Algorithm

If eigenvalues of A are algebraic integers, algorithm 1 can be used; otherwise, algorithm 2 is used.

Algorithm 1

Step 1 Set $i=0$.

Step 2 Calculate $A^{i-j}b$, for $j=0, 1, 2, \dots, i$.

Step 3 For i , solve the equation: $x_2 - \sum_{j=0}^i A^{i-j} b u(j) = 0$ by using the recursive solving method (R. S. M.). If $u(j) \in Z^1$ for all $j=0, 1, 2, \dots, i$, $\hat{i}=i$ and $u(\hat{j})=u(j)$, $j=0, 1, 2, \dots, \hat{i}$ and this algorithm terminates. If $u(j) \notin Z^1$ for some j , set $i=i+1$ and go to Step 4.

Step 4 If $i \leq n-1$, go to Step 2. If $i > n-1$, optimal solution $\hat{i}$ does not exist and this algorithm terminates.

Algorithm 2

Step 1 The same as Step 1 in Algorithm 1.

Step 2 The same as Step 2 in Algorithm 1.

Step 3 The same as Step 3 in Algorithm 1.

Step 4 If $i \leq N$ (N is a given integer value such that $N > n$), go to Step 2. If $i > N$, an optimal solution $\hat{i}$ can not be found and this

algorithm terminates.

Remarks :

1. When eigenvalues of A are algebraic integers (Algorithm 1), an increase of elements of the set M_i stops in time (step) $i=n-1$. Therefore, if the Eq.

(5): $x_2 - \sum_{j=0}^i A^{i-j} b u(j) = 0$ can not be solved in time (step) $i \leq n-1$, an optimal solution $\hat{i}$ can not be obtained. On the other hand, when eigenvalues are not algebraic integers (Algorithm 2), elements of the set M_i increase successively with time (step) $i > n-1$. Therefore, the check that integer-valued solutions of the above equation at time (step) $i > n$ are obtained or not has to be repeated infinitely. In this paper, the number N (the maximum number in this search) is set in 20 steps beforehand.

2. A judgement of integer-valued solutions $u(j)$, $j=0, 1, 2, \dots, i-1$, is done by the result that all solutions calculated as real-valued numbers in the Eq. (5) are within the limitation-bounds: $\pm 10^{-6}$ of some integer-value or not.

3. In Step 3 of both Algorithm 1 and Algorithm 2, if time (step) i is less than $n-1$ and when the Eq. (5): $\sum_{j=0}^i A^{i-j} b u(j) = x_2$ is solved, the size of coefficients' matrix in the left hand is $n \times i$, so this matrix is a rectangular and not square matrix. Therefore, it is needed that the $i \times i$ square matrix and the i dimensional vector are treated, which are given in the first i steps (that is, from 0 step to $i-1$ step) and are derived from the $n \times i$ coefficients' matrix and the vector x_2 , respectively; and the i dimensional equation is solved by Gauss-Jordan Method. By using obtained integer-valued solutions: $u(0), u(1), u(2), \dots, u(i-1)$, it is checked that an equality holds or not in the $(n-i) \times i$ matrix and the $(n-i)$ vector which are left of the coefficients' matrix and the vector x_2 , respectively⁽¹¹⁾. If time (step) i is $n-1$, the equation is given by n dimensional square matrix and it is solved by Gauss-Jordan Method. And, all solutions obtained are checked easily as integer-valued or not. If time (step) i is greater than $n-1$, the size of coefficients' matrix in the Eq. (5) is $n \times i$, so the undetermined simultaneous equation must be solved. In this paper, all searches within the range given beforehand: $-100 \sim u(j) \sim 100$ are done in each step $i > n-1$, by examining of the result in step $n-i$.

2.5 Numerical examples

Two numerical examples are given in this paper.

Example 1

This control system is given by the state equation,

$$x(i+1) = \begin{bmatrix} 2 & 0 \\ -1/2 & 1/3 \end{bmatrix} x(i) + \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} u(i), \quad i=0, 1, 2, \dots$$

$n=2$, and,

state variable: $x(i) \in R^2$, quantized control variable $u(i) \in Z^1$.

Let

$$x_1 = x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and, } x_2 = x(i+1) = \begin{bmatrix} 3 \\ 181/18 \end{bmatrix}.$$

Determine minimal-time i and integer-valued (quantized) controls $u(i)$ so that a state x_1 can reach the other state x_2 in this control system.

Result :

rank $G_2=2$, so the assumption is satisfied.

The equation of eigenvalues in this control system is the next :

$$f(\lambda) = \lambda^2 - (7/3)\lambda + (2/3) = 0.$$

So, eigenvalues are not algebraic integers.

Mathematical Programming problem :

Minimize i

subject to

$$\begin{bmatrix} 3 \\ 181/18 \end{bmatrix} = \sum_{j=0}^i \begin{bmatrix} 2 & 0 \\ -1/2 & 1/3 \end{bmatrix}^{i-j} \cdot \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} z_j$$

$z_j \in Z^1$.

Thus, by solving this Mathematical Programming problem, solutions of this control problem can be obtained. That is, by using R. S. M. (Algorithm 2), $i=3$, and then $u(0)=-3, u(1)=-12, u(2)=39$ and $u(3)=-4$.

Example 2

This control system is given by the state equation,

$$x(i+1) = \begin{bmatrix} 1 & -2/3 \\ 0 & 2 \end{bmatrix} x(i) + \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} u(i), \quad i=0, 1, 2, \dots$$

$n=2$, and

state variable: $x(i) \in R^2$, quantized control variable $u(i) \in Z^1$.

Let

$$x_1 = x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and}$$

$$x_2 = x(i+1) = \begin{bmatrix} -13/3 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1/2 \\ 7/4 \end{bmatrix}.$$

Determine minimal-time i and integer-valued quantized controls $u(i)$ so that a state x_1 can reach the other state x_2 in this control system.

Result :

rank $G_2=2$, so the assumption is satisfied.

The equation of eigenvalues in this control system is as follows,

$$f(\lambda) = \lambda^2 - 3\lambda + 2 = 0.$$

So, eigenvalues are algebraic integers.

$$\text{For } x_2 = \begin{bmatrix} -13/3 \\ 6 \end{bmatrix},$$

$x_2 \in M_2 (= M_\infty)$. By using Algorithm 1, the optimal solutions are obtained, $i=1$ and then $u(0)=7, u(1)=-2$.

$$\text{For } x_2 = \begin{bmatrix} 1/2 \\ 7/4 \end{bmatrix},$$

Fig. 2 A reachable set M_2 and the relation between two terminal states and minimal-time optimal controls in Example 2

$x_2 \notin M_2 (= M_\infty)$. And by using Algorithm 1, i does not exist.

3. Conclusion

The minimal-time control problem of a quantized, linear, time-invariant discrete system with rational coefficients is investigated. By considering reachability properties of these systems, the method (R. S. M.) is given which is based on an integer minimization by mathematical programming and two numerical examples are presented.

It remains as a problem that R. S. M. in this study is not necessarily an efficient method, and so a more efficient method must be proposed by the authors in the future.

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