Numerical Analysis of Nonlinear Responses to Uncertain Excitations Modeled by Convex Sets*

Cai-Ying HE**, Jing-Hui ZHANG** and Shi-Lin XIE**

In this paper, the nonlinear responses to uncertain excitations modeled by convex sets are discussed. First, the nonlinear differential inclusion problem is converted into a class of optimization problems to solve the worst response. Next, it is simplified to a two-point boundary value problem of nonlinear ordinary differential equations. Finally, as an example, the Duffing equation is considered to obtain the numerical results for the two-point boundary value problem using the shooting method. The worst response and the worst excitation at a given time are discussed. The results show that the worst excitation corresponding to the worst response is a rectangular wave, and not a sine or a cosine wave for maximum-bound convex excitations. The worst excitation of a linear system is a uniform rectangular wave with natural period, while that of a nonlinear system is not a uniform one. This nonuniformity is related to the nonlinearity, the initial conditions and the excitation bound of the system. Results of further analysis are consistent with the results of a classical nonlinear theory.

Key Words: Nonlinear Vibrations, Uncertainty, Convex Sets, Numerical Analysis, Differential Inclusions

1. Introduction

There exist many uncertain factors in mechanical and structural dynamics fields. The current method of processing uncertainty involves treating it a stochastic process, and the results under a certain probabilistic sense can be obtained by applying the random vibration theory. This method is feasible and practical under general situations, but not under the specific ones. For example, in some engineering fields, such as aerospace, nuclear power and military engineering, more reliable methods of describing uncertainty are required to provide important control parameters for structural strength and reliability design. A reliable uncertainty set model, which can be classified under the nonprobabilistic modeling of uncertainty, has been presented. In the control theory, this uncertainty set model has been developed extensively. For example, Schweppe⁽¹⁾, Rowland⁽²⁾ and Grantham and Vincent⁽³⁾

applied it to the optimization control problem very well, and a large number of papers and monographs have been published in interrelated fields. In structural dynamics, however, the uncertainty set model was used by Drenick⁽⁴⁾ and Shinozuka⁽⁵⁾ in the 60's and 70's, but not developed further. It was not until recently that it caught the attention of Ben-Haim and coworkers^{(6),(7)}, who researched structural fatigue life, thin bar dynamics, impact loading and buckling of thin shells, seismic excitation and vibration measurements. In the structural dynamics field, some new concepts, such as the worst environmental condition or loading, the worst responses, and their calculation methods and design criterion, which together comprise a new branch of structural dynamics, i.e., the vibration convex set theory, will be introduced using the uncertainty set model. In recent years, a general form of the convex set model that describes uncertainty has been established by us for use in the analysis of an extensive structural dynamic response. We have discussed the dynamic responses of a linear vibration system with uncertain excitations modeled by convex sets⁽⁸⁾⁻⁽¹¹⁾. The response sets have some good property for a linear time-invariant vibration

* Received 16th October, 1998

** Department of Engineering Mechanics, Xi'an Jiaotong University, 710049 Xi'an, Shaanxi, P.R. China. E-mail: icapv@xjtu.edu.cn

system but not for a nonlinear one. This will result in that the solving methods in a linear system can not be extended to solve the response set of nonlinear system. In the practical engineering, there exist uncertainty factors in many nonlinear systems. Hence, in this paper, the dynamic responses of a nonlinear vibration system will be considered, and the physical meaning of analysis results will be discussed.

2. Basic Concepts

A single degree of freedom (SDOF) nonlinear vibration system with uncertain convex-set loading is considered. Its vibration equation is

$$\ddot{x} + f(x, \dot{x}) = u(t), \quad (1)$$

where $u(t)$ is the uncertain excitation to satisfy $u(t) \in \mathcal{Q}$, in which $\mathcal{Q}$ is a convex set, and initial conditions are $x(0) = \dot{x}(0) = 0$.

To obtain the response of system (1), we define its response set as

$$R_{[0, T]} = \{x : x = x_u(t), x(0) = 0, \dot{x}(0) = 0, u(\cdot) \in \mathcal{Q}, t \in [0, T]\}, \quad (2)$$

where $x_u(t)$ is the displacement response to excitation $u(t)$ for all $t \in [0, T]$. Similarly, we define a response set at a given time T as follows:

$$R_T = \{x : x = x_u(T), x(0) = 0, \dot{x}(0) = 0, u(\cdot) \in \mathcal{Q}\}, \quad (3)$$

where $x_u(T)$ is the displacement at time T as the excitation is $u(t)$.

We define the total worst displacement response of a SDOF system as

$$X_w = \max_t \sup_{u \in \mathcal{Q}} |x(t)|, \quad (4)$$

where $x(t)$ is the response to loading u . It reflects the total maximum absolute displacement of the system with the excitation bounded in set $\mathcal{Q}$. Since the total worst response is sometimes difficult to obtain, it is essential to introduce the regional or local worst displacement response, which is defined as

$$X_{tw} = \max_{t \in [0, T]} \sup_{u \in \mathcal{Q}} |x(t)|. \quad (5)$$

Besides, we define the worst displacement response at a given time as follows:

$$X_w(T) = \sup_{u \in \mathcal{Q}} |x(T)|. \quad (6)$$

The excitation u^* corresponding to the worst response is called the worst excitation, and corresponding response $x^*(t)$ is called the worst response history sometimes also called the worst response.

Also, the worst velocity response and the worst acceleration response can be similarly defined.

3. Two-Point Boundary Value Problem for The Worst Response Solution

The response sets R_T have symmetry, convexity and inclusion property for a linear time-invariant

vibration system but not for a nonlinear one. This will result in inconvenience in solving the response set of nonlinear system. In the practical engineering, the worst response frequently attracts attention of engineers, so the worst responses of SDOF nonlinear systems are worthy to be studied. For convenience, only the response under zero initial condition is considered. To easily solve Eq. (1), it is changed into the state equation form as follows.

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, u), \quad (7)$$

where $\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$ is the state vector, and $\mathbf{F}(\mathbf{x}, u) = \begin{Bmatrix} x_2 \\ -f(x_1, x_2) + u \end{Bmatrix}$. Then, the equation used for solving the worst response of system (1) can be simplified into the following optimization problem.

$$\text{OP}(1) : \begin{aligned} &\max (J = |x_1(t_f)|) \\ &\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, u), \mathbf{x}(0) = \mathbf{0}, u \in \mathcal{Q} \end{aligned}$$

where t_f is a free or a fixed time. This problem can be further rewritten as two simple optimization problems, i.e., OP(2) and OP(3).

$$\text{OP}(2) : \begin{aligned} &\min (J_1 = x_1(t_f)) \\ &\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, u), \mathbf{x}(0) = \mathbf{0}, u \in \mathcal{Q} \end{aligned}$$

$$\text{OP}(3) : \begin{aligned} &\min (J_2 = -x_1(t_f)) \\ &\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, u), \mathbf{x}(0) = \mathbf{0}, u \in \mathcal{Q} \end{aligned}$$

In fact, only one problem needs to be discussed, as the other can be analyzed similarly. Let the Hamilton function H be

$$H = \mathbf{p}^T(t) \mathbf{F}(\mathbf{x}, u) = p_1(t) \dot{x}_1 + p_2(t) [-f(x_1, x_2) + u(t)], \quad (8)$$

where $\mathbf{p}(t) = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$ is the costate vector. By applying the Pontryagin maximum principle⁽¹²⁾, the necessary conditions for minimizing J_1 are that

$$(i) \quad \dot{\mathbf{p}}(t) = -\frac{\partial H}{\partial \mathbf{x}}, \text{ satisfying the boundary condition } \mathbf{p}(t_f) = \frac{\partial J_1}{\partial \mathbf{x}(t_f)},$$

$$(ii) \quad H(\mathbf{x}^*, \mathbf{p}, u^*) = \min_{u \in \mathcal{Q}} H(\mathbf{x}^*, \mathbf{p}, u), \text{ and}$$

$$(iii) \quad H(\mathbf{x}^*(t), \mathbf{p}(t), u^*(t)) = H(\mathbf{x}^*(t_f^*), \mathbf{p}(t_f^*), u^*(t_f^*)) = 0 \text{ as } t_f \text{ is free, and } H(\mathbf{x}^*(t), \mathbf{p}(t), u^*(t)) = H(\mathbf{x}^*(t_f), \mathbf{p}(t_f), u^*(t_f)) = \text{const as } t_f \text{ is fixed.}$$

In the above conditions, superscripts * indicate the optimum values for problem OP(2). Condition (ii) is a nonlinear convex programming problem; thus the convexity property of set $\mathcal{Q}$ is very essential.

We derive the following differential equations from condition (i):

$$\dot{\mathbf{p}}(t) = - \begin{bmatrix} 0 & 1 \\ -\frac{\partial f}{\partial x_1} & -\frac{\partial f}{\partial x_2} \end{bmatrix}^T \mathbf{p}(t), \quad \mathbf{p}(t_f) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}. \quad (9)$$

Combining Eq. (9) with Eq. (7), the two-point boundary value problem is obtained:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f(x_1, x_2) + u^* \\ \dot{p}_1 = \frac{\partial f}{\partial x_1} \cdot p_2 \\ \dot{p}_2 = -p_1 + \frac{\partial f}{\partial x_2} \cdot p_2 \end{cases} \quad (10)$$

with boundary conditions

$$x_1(0) = x_2(0) = 0, \quad p_1(t_f) = 1, \quad p_2(t_f) = 0. \quad (11)$$

u^* in Eq.(10) can be determined from condition (ii) using the given convex set Ω , and it is generally a function of p_1 and p_2 .

The optimal solution can be obtained from the two-point boundary value problem (10) when t_f is fixed. If the total worst response is required to be solved when t_f is free, the free boundary condition (iii) has to be added to Eq.(11). Substituting Eq.(8) into the condition (iii), we can obtain

$$p_1(t)x_2^* + p_2(t)[-f(x_1^*, x_2^*) + u^*] = 0. \quad (12)$$

Combining the boundary condition Eq.(11), we have

$$x_2(t_f) = 0, \quad p_2(0) = 0. \quad (13)$$

However, the optimal time t_f cannot be determined directly from Eq.(13), so we carry out a variable transformation of Eq.(10), i.e., $t = t_f\tau$ for $\tau \in [0, 1]$, and treat t_f as a state variable. Then, we add Eq.(12) to condition (11) to change the free-point boundary value problem to the fixed-point one. If the total worst response occurs at infinite time, or a unique t_f cannot be gained from Eq.(12), it will be difficult to obtain the numerical solutions of the transformed boundary value problem. Hence, the worst response at a given time t_f can be solved from problem (10); moreover the local worst response can be obtained from it.

For a given practical problem, the general steps for solving the worst response are summarized as follows. First, we write the Hamilton function H from Eq.(8) according to this given system. Next, we solve the worst excitation expression u^* from condition (ii) according to the given convex set model. Then, we judge if the total worst response exists, i.e., if there exists a unique and limited solution t_f . If it exists, we can apply the transformed two-point boundary value problem to solve it, or apply problem (10) to solve the worst response at a given time. Finally, if the local worst response is further required, we can estimate it by solving the worst responses at a series of given time.

The property of the solutions for the two-point boundary value problem (10) is determined by that of the corresponding ordinary differential equation. If there exists a unique solution, x^* and u^* obtained from it are the required worst response history and the worst excitation respectively. Otherwise, if there appears multivalued or instability, the obtained results

have to be discussed further, and determined whether or not to be the required solution. Here, we only consider the situation for the unique solution.

As an example, the worst response of a linear system from the above result is compared with a known one obtained from the theory of linear system to check the validity of the theoretical results in this section. Consider a SDOF linear vibration system

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2x = u(t), \quad (14)$$

where $u(t) \in \Omega_{MD}$. Initial conditions are $x(0) = \dot{x}(0) = 0$, and

$$\Omega_{MD} = \{u(t) : |u(t)| \leq \bar{u}\}. \quad (15)$$

The above set model is called the maximum-bound convex model, for which the following worst excitation can be derived from condition (ii).

$$u^*(t) = -\bar{u} \cdot \text{sign}(p_2(t)) \quad (16)$$

For the linear system, the last two equations are uncoupled with x_1 and x_2 , and written separately as

$$\dot{\mathbf{p}}(t) = - \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \mathbf{p}(t), \quad \mathbf{p}(t_f) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}$; then we have

$$\mathbf{p}(t) = e^{\mathbf{A}^T(t_f-t)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (17)$$

Then $p_2(t)$ can be derived as

$$p_2(t) = \frac{e^{-\zeta\omega(t_f-t)}}{\sqrt{1-\zeta^2}\omega} \sin \sqrt{1-\zeta^2}\omega(t_f-t).$$

Substituting $p_2(t)$ into Eq.(16), we can obtain the worst excitation corresponding to the worst response as follows:

$$u^*(t) = -\bar{u} \cdot \text{sign}(\sin \sqrt{1-\zeta^2}\omega(t_f-t)). \quad (18)$$

From the free final condition $p_2(0) = 0$, the final time t_f corresponding to the total worst response satisfies the following condition.

$$t_f = \frac{n\pi}{\sqrt{1-\zeta^2}\omega}, \quad n = 1, 2, 3, \dots \quad (19)$$

This shows that the final time is not unique.

In order to study the relationship between $x^*(t_f)$ and t_f , we do not consider condition (19) at first. Substituting Eq.(18) into Eq.(14), we derive $x^*(t_f)$ as

$$\begin{aligned} x^*(t_f) &= - \int_0^{t_f} \frac{\bar{u}}{\sqrt{1-\zeta^2}\omega} \text{sign}(\sin \sqrt{1-\zeta^2}\omega\tau) \\ &\quad \times \omega[t_f - (t_f - \tau)] e^{-\zeta\omega\tau} \sin \sqrt{1-\zeta^2}\omega\tau d\tau \\ &= - \frac{\bar{u}}{\sqrt{1-\zeta^2}\omega} \int_0^{t_f} e^{-\zeta\omega\tau} |\sin \sqrt{1-\zeta^2}\omega\tau| d\tau. \end{aligned} \quad (20)$$

Because the integrand is always positive, $x^*(t_f)$ decreases gradually as $t_f \rightarrow +\infty$. This shows that the condition (19), which t_f satisfies, is not a sufficient condition.

To integrate Eq.(20), let $t_f = \frac{k\pi}{\sqrt{1-\zeta^2}\omega} + q$. Then we have

$$x^*(t_f) = - \frac{\bar{u}}{\omega^2} \cdot \left\{ \frac{(1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}})(1 - e^{-k\pi\zeta/\sqrt{1-\zeta^2}})}{1 - e^{-\pi\zeta/\sqrt{1-\zeta^2}}} \right\}$$

$$\begin{aligned}
& + e^{-k\pi\zeta/\sqrt{1-\zeta^2}} \\
& + (-1)^{k+1} e^{-\zeta\omega t_f} \left[\frac{\zeta}{\sqrt{1-\zeta^2}} \sin \sqrt{1-\zeta^2} \omega t_f \right. \\
& \left. + \cos \sqrt{1-\zeta^2} \omega t_f \right] \}. \quad (21)
\end{aligned}$$

When $t_\infty \rightarrow +\infty$, $x^*(t_f)$ approaches the minimum, which is

$$x_{\min}^* = -\frac{\bar{u}}{\omega^2} \cdot \frac{1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}}}{1 - e^{-\pi\zeta/\sqrt{1-\zeta^2}}}. \quad (22)$$

As far as the maximum-bound convex model is concerned, the maximum response of a linear system with zero initial condition is equal to the absolute value of the minimum response, i.e., the total worst response is $X_w = |x^*(t_f)| = -x_{\min}^*$. This result is consistent with the analytical expression derived from the worst modal response by Ben-Haim et al.⁽⁷⁾

4. Numerical Results of Duffing Equation

In this section, we consider the following damped Duffing equation as an example to discuss the responses of a SDOF nonlinear system to maximum-bound convex loading.

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2x + \varepsilon x^3 = u(t), \quad (23)$$

where $u(t) \in \Omega_{MD}$, and $x(0) = \dot{x}(0) = 0$.

From the results of a linear system in the above section, we can see that the final time t_f derived from $p_2(0) = 0$ is not unique, and may in fact be infinite. Hence, problem (10) is still applied to solve the worst response $x^*(t_f)$ at a given time, and to gain the total or local worst response by analyzing the curve of $x^*(t_f)$ versus t_f .

For the Duffing Eq.(23), problem (10) can be rewritten as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(2\zeta\omega x_2 + \omega^2 x_1 + \varepsilon x_1^3) - \bar{u} \cdot \text{sign}(p_2) \\ \dot{p}_1 = (\omega^2 + 3\varepsilon x_1^2)p_2 \\ \dot{p}_2 = -p_1 + 2\zeta\omega p_2 \end{cases} \quad (24)$$

with boundary conditions

$$x_1(0) = x_2(0) = 0, \quad p_1(t_f) = 1, \quad p_2(t_f) = 0.$$

The shooting method is applied to solve numerically a two-point boundary value problem of nonlinear ordinary differential equations. The initial values for p_1 and p_2 are assumed to be the analytical solution of the linear system in Section 3 to improve iteration velocity. Assume $\omega = 18.23$ rad/s and $\zeta = 0.05$. Figure 1 indicates the worst displacement response for various ε , where the horizontal axis expresses the given final time, and the vertical axis expresses the worst responses (i.e., the absolute value of maximum and/or minimum responses). The curve of $\varepsilon = 0$ shows the analytical result of the linear system in Section 3. From Fig. 1, we can see that the worst displacement responses decrease with increasing nonlinear parameter ε when the natural frequency and the damping

ratio of the system are given. To illustrate this point clearly, Fig. 2 gives the curve of the worst displacement response at $t_f = 15$ s versus nonlinear parameter ε .

To further study the natural relationship between the worst excitation and the worst response time history corresponding to the worst displacement, we analyzed the worst response histories, the worst excitations and their Fourier spectra at $t_f = 15$ s when $\varepsilon = 100$, $\varepsilon = 1\,000$, $\varepsilon = 2\,000$ and $\varepsilon = 5\,000$. The results at $\varepsilon = 2\,000$ are shown in Fig. 3, where (a) shows the worst excitation curve and its Fourier spectrum, and (b) shows the worst displacement curve and its Fourier spectrum. By analyzing these results, we can see that the frequency content of the worst excitation is similar to that of the worst response, their fundamental frequencies are basically consistent with the natural frequency of the corresponding linear system, and there appear other frequency components. These results from the nonlinear part of the Duffing equation. From these results, an important conclusion can be drawn: For maximum-bound convex excitations,

Fig. 1 Responses of different ε versus final time

Fig. 2 Worst response versus nonlinear parameter ε

Fig. 3 Worst excitation and response at $t_f=15$ s and $\varepsilon=2\,000$

the worst excitation corresponding to the worst response is a rectangular wave, not a sine or a cosine wave. The worst excitation of a linear system is a uniform rectangular wave with natural period $2\pi/\omega_0$, and is not a uniform one for a nonlinear system. This nonuniformity is related to the nonlinearity, the initial conditions and the excitation bound of the system.

Figure 4 shows the influence of these factors on the semi-wave width of the worst excitation. In fact, this reflects the varying law of period for the worst excitation with time under different situations. From this figure, we can see that since width of semi-wave after an enough long time reflects the main period of the worst excitation, the main period of excitation corresponding to the worst response gradually

Fig. 4 Varying trends of semi-wave width of the worst excitations with different conditions

becomes short and the frequency becomes large as the nonlinear parameter ε increases from zero. As $\varepsilon > 0$, the main period of excitation corresponding to the worst response gradually becomes short, i.e., the frequency becomes large when the maximum magnitude increases. This reflects that when the frequency response curve for a rigid spring inclines along the side with increasing frequency, the larger the excitation magnitude is, the larger the response magnitude is, and the higher the corresponding frequency is. These are consistent with the results of classical nonlinear theory. In addition, the initial conditions also influence the initial period of the worst excitation.

5. Conclusions

(1) In this study, we applied the Pontryagin maximum principle to derive the conditions necessary for solving the total worst response and the worst excitation, and to simplify them into the two-point boundary value problem of nonlinear ordinary differential equations. For a linear system with maximum-bound convex set loading, the worst response expression obtained here is consistent with that from the worst modal response by Ben-Haim et al.⁽⁷⁾ Since analytical results could not be obtained for a nonlinear system, a numerical procedure was introduced.

(2) For the Duffing equation with the maximum-bound convex model, the analytical results show that the worst displacement response decreases with increasing nonlinear parameter ε , and the main frequency components of the worst excitation and the worst response are consistent with the natural frequency of the corresponding linear system. There

appear other components in their frequency components. This results from the nonlinear part of the Duffing equation.

(3) Whether linear or nonlinear system, the worst excitation in the maximum-bound convex model is a rectangular wave, and not a sine or a cosine one. However, the worst excitation of a linear system is a uniform rectangular wave, but not a uniform one for a nonlinear system. This nonuniformity is related to the nonlinearity, initial conditions and the excitation bound of system.

(4) The stability of a nonlinear system is not discussed, so bifurcation and chaos are not involved. In the other words, we assumed that the two-point boundary value problem has a unique solution, from which the obtained x^* and u^* are respectively defined as the worst response history and the corresponding worst excitation.

Acknowledgements

We wish to acknowledge the support and encouragement provided by the Shaanxi Natural Science Foundation of China, the Foundation of State Education Commission Opening Laboratory for Theory of Lubrication and Rotor-Bearing System and the Nuclear Industry Foundation of China.

References

- (1) Schweppe, F.C., *Uncertain Dynamic Systems*, Englewood Cliffs: Prentice, 1973.
- (2) Rowland, J.D.L. and Vinter R.B., *Dynamic Optimization Problems with Free-Time and Active State Constraints*, SIAM. J. Control and Optimization, Vol. 31, No. 3 (1993), p. 677.
- (3) Grantham, W.J. and Vincent, T.L., A Controllability Minimum Principle, *Journal of Optimization Theory and Applications*, Vol. 17, No. 1/2 (1975), p. 93.
- (4) Drenick, R.F., Model-Free Design of Seismic Structures, *Journal of Engineering Mechanics Division, Proceedings of the ASCE*, Vol. 96 (1970), p. 483.
- (5) Shinozuka, M., Maximum Structural Response to Seismic Excitations, *Journal of Engineering Mechanics Division, Proceedings of the ASCE*, Vol. 96 (1970), p. 729.
- (6) Ben-Haim, Y. and Elishakoff, I., *Convex Models of Uncertainty in Applied Mechanics*, Amsterdam: Elsevier, 1990.
- (7) Ben-Haim, Y., Chen, G.D. and Soong, T.T., Maximum Structural Response Using Convex Models, *Journal of Engineering Mechanics, ASCE*, Vol. 122 (1996), p. 325.
- (8) He, C.Y. and Zhang, J.H., The Convex Response Set of a Class of MDOF Vibration System, *ACTA Mechanica Sinica*, (in Chinese), Vol. 29, No. 5 (1997), p. 530.
- (9) He, C.Y. and Zhang, J.H., Convex-Model Analysis of the Seismic Response, *Applied Mathematics and Mechanics*, Vol. 18, No. 10 (1997), p. 941.
- (10) He, C.Y. and Zhang, J.H., The Response Analysis of Structures to Integral Energy-Bound Convex Loading, *Structure & Environment Engineering*, (in Chinese), Vol. 104, No. 2 (1997), p. 20.
- (11) He, C.Y. and Zhang, J.H., Support Hyperplane Method Finding Response Set of Vibration System with Uncertain Excitations, *Journal of Vibration Engineering*, (in Chinese), Vol. 10, No. 3 (1997), p. 300.
- (12) Xie, X.S., *Optimal Control: Theory and Application*, (in Chinese), 1986, Beijing: Tsinghua University Press.