Chaotic Behaviors of Nonlinear Vibration System with Stochastic Excitation*

(Probability Distribution of Local Expansion Rates)

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This paper reports on the study of chaotic behavior of nonlinear vibration systems with stochastic excitation. Previous studies examining mechanical chaotic vibrations have mainly focused on their deterministic properties even if the target system contains stochastic elements which are sometimes unavoidable in experimental situations. In view of this, this paper investigates random vibrations arising in nonlinear stochastic systems, which the authors call SCB (stochastic chaotic behavior). SCB has negative largest Liapunov exponents while it behaves similarly to deterministic chaos, exhibiting the expanding and folding process of chaos. This feature of SCB is in good agreement with that of a experimental chaotic behavior observed in our experiments and implies that the physical chaotic behavior may not have positive largest Liapunov exponents even though it behaves chaotically. These kinds of chaotic properties of SCB and experimental chaos can be characterized by the probability density function of local expansion rates.

Key Words: Chaos, Stochastic Excitation, Largest Liapunov Exponent, Local Expansion Rate, Probability Density Function

1. Introduction

The presence of chaos in mechanical vibration systems has been numerically and experimentally demonstrated⁽¹⁾⁻⁽⁶⁾. In these studies, purely deterministic models are used to confirm the existence of chaotic dynamics, in other words, a random response of deterministic systems. In this vein, we have already demonstrated in previous work⁽⁵⁾ that statistical mechanical properties of the experimental chaotic behavior of a physical pendulum under parametric excitation are well modeled by the deterministic model.

However, such physical mechanical systems may contain noisy elements due to mechanical fluctuations such as mechanical errors in the driving mechanism. Therefore, the chaotic behavior we observed⁽⁵⁾ may

not be a purely deterministic chaos but a chaotic behavior occurring in stochastic systems. The main interest of the present work is to investigate interaction between chaos and noise arising in such physical systems. A similar viewpoint can be found in former studies⁽⁷⁾⁻⁽⁹⁾. Importantly, Tanaka et al.⁽⁹⁾ implied problems possibly arising in practical situations. They showed that a purely random time series possibly produces positive largest Liapunov exponents.

In this paper, we model our apparatus using a stochastic model, i.e., a pendulum system with noisy sinusoidal parametric excitation. We first show that this stochastic model behaves similarly to deterministic chaos. Then, we numerically and experimentally demonstrate that such stochastic chaotic behavior exhibits negative largest Liapunov exponents which normally correspond to periodic behavior. To investigate whether such stochastic behavior has chaotic properties, we introduce the probability density function of local expansion rates of the behavior.

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2. Calculation of Local Expansion Rates and Liapunov Exponents

2.1 Definition

Consider a series of points produced by a Poincaré map $\mathbf{T}$ of a d -dimensional orbit $\mathbf{x}(t)$

$$\mathbf{x}_{n+1} = \mathbf{T}(\mathbf{x}_n) \quad (n=1, 2, \dots), \quad (1)$$

then the local expansion rate $\lambda(\mathbf{x}_n, \mathbf{e}_n)$ of the Poincaré map $\mathbf{T}$ is defined as⁽¹⁰⁾

$$\lambda(\mathbf{x}_n, \mathbf{e}_n) = \ln \|\mathbf{T}(\mathbf{e}_n)\| \quad (n=1, 2, \dots), \quad (2)$$

where $\mathbf{e}_n$ is the unit vector starting at $\mathbf{x}_n$ and $\|\cdot\|$ is a norm. Local expansion rates along unstable manifolds are then defined as

$$\lambda_1(\mathbf{x}_n) = \sup_{\mathbf{e}_n} \lambda(\mathbf{x}_n, \mathbf{e}_n) \quad (n=1, 2, \dots). \quad (3)$$

Practical calculation of $\lambda_1(\mathbf{x}_n)$ is performed as follows. Initially, $\lambda_1(\mathbf{x}_1)$ is calculated from any unit vector $\mathbf{e}_1$. Then, $\lambda_1(\mathbf{x}_2)$ is calculated from the unit vector $\mathbf{T}(\mathbf{e}_1)/\|\mathbf{T}(\mathbf{e}_1)\|$. Repeating this procedure, we obtain $\lambda_1(\mathbf{x}_n)$ ($n=1, 2, \dots$) with $\mathbf{e}_n$ expanding along unstable manifolds^{(5),(10),(11)}.

The long time average of $\lambda_1(\mathbf{x}_n)$ results in the largest Liapunov exponent Λ^∞

$$\Lambda^\infty = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \lambda_1(\mathbf{x}_n). \quad (4)$$

2.2 Calculation from time series

It is clear from Eqs. (2) and (3) that the solution $\mathbf{e}_n$ is required to calculate practical values of $\lambda_1(\mathbf{x}_n)$.

When equations of target systems are known, the solution $\mathbf{e}_n$ is easily obtained as a solution of variational equations, so that $\lambda_1(\mathbf{x}_n)$ and Λ^∞ can be calculated^{(5),(10),(11)}.

In contrast, when the equations of target systems are not known, the solution $\mathbf{e}_n$ is estimated in the following way. Consider the linearized system of the general system (1)

$$\mathbf{x}_{n+1} = \mathbf{A}_n \mathbf{x}_n \quad (n=1, 2, \dots), \quad (5)$$

where $\mathbf{A}_n$ is a d -dimensional square matrix. Then, the matrix $\mathbf{A}_n$ can be estimated by means of the least squares method as follows^{(5),(11),(12)}.

Consider the chaotic time series $\{\mathbf{x}_j\}$ ($j=1, 2, \dots$). Selecting points $\{\mathbf{x}_i\}_j$ ($i=1, 2, \dots, N$) within a ball of radius R centered at $\mathbf{x}_j$, we can define displacement vectors as

$$\{\mathbf{y}^i\}_j \equiv \{\mathbf{x}_i - \mathbf{x}_j \mid \|\mathbf{x}_i - \mathbf{x}_j\| < R\}. \quad (6)$$

The map $\mathbf{T}$ maps the vectors $\{\mathbf{y}^i\}_j$ onto other vectors as

$$\{\mathbf{z}^i\}_j \equiv \{\mathbf{x}_{i+1} - \mathbf{x}_{j+1} \mid \|\mathbf{x}_{i+1} - \mathbf{x}_{j+1}\| < R\}. \quad (7)$$

If R is sufficiently small, one can express the relationship between $\{\mathbf{y}^i\}_j$ and $\{\mathbf{z}^i\}_j$ as

$$\mathbf{z}^i = \mathbf{A}_j \mathbf{y}^i, \quad (8)$$

and the matrix $\mathbf{A}_j$ can be estimated by the least squares method of the condition

$$\min_{\mathbf{A}_j} \sum_{i=1}^N \|\mathbf{z}^i - \mathbf{A}_j \mathbf{y}^i\|^2. \quad (9)$$

The matrix $\mathbf{A}_j$ yields local expansion rates along unstable manifolds^{(5),(11)}

$$\lambda_1(\mathbf{x}_n) = \ln \|\mathbf{A}_n \mathbf{e}_n\| \quad (n=1, 2, \dots). \quad (10)$$

2.3 Calculation from one-dimensional time series

The original d -dimensional orbit $\mathbf{x}(t)$ can be reconstructed from a scalar observable $x(t)$ by the delay-coordinate

$$\mathbf{x}(t) = [x(t), x(t+\Delta t), \dots, x(t+(d-1)\Delta t)], \quad (11)$$

where Δt is delay time, which is empirically selected. For periodically forcing systems, Δt is often selected as several tenths of the forcing period⁽¹³⁾, for example, $\Delta t \approx 2/5$ in our case. It is demonstrated that the orbit reconstructed by the coordinate (11) is topologically equivalent to the original orbit⁽¹⁴⁾.

By using the orbit $\mathbf{x}(t)$ reconstructed by the coordinate (11), one can estimate $\lambda_1(\mathbf{x}_n)$ and Λ^∞ from physical chaotic behavior even when only a scalar observable is obtainable^{(5),(11),(12)}.

2.4 Probability density function of local expansion rates

We here introduce a probability density function of local expansion rates along unstable manifolds (Eq. (3)) as

$$P(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta(\lambda_1(\mathbf{x}_n) - \lambda), \quad (12)$$

where $\delta(\cdot)$ is the delta function.

3. Experimental Apparatus and Model

3.1 Experimental setup and model

We performed experiments on a mechanical pendulum system⁽⁵⁾ shown in Fig. 1. The pendulum oscillates in a vertical plane using a crank-slider mechanism driven by an AC servomotor. The angular displacement of the pendulum is measured using a rotary encoder. The damping coefficient around the point of suspension of the pendulum is controlled by an eddy-current brake with an input voltage E from 0 to 45 V.

The equation of motion of the experimental apparatus is given by⁽⁵⁾

$$\begin{cases} \ddot{\theta} + \frac{C}{ml^2 + I} \dot{\theta} \\ + \left(\frac{mgl}{ml^2 + I} + \frac{mlAf^2}{ml^2 + I} h(ft) \right) \sin \theta = 0 \\ h(t) = \cos t + \frac{1}{\kappa} (\sqrt{1 - \kappa^2 \sin^2 t} - 1), \end{cases} \quad (13)$$

where $\ddot{\cdot} = d^2/dt^2$, $\dot{\cdot} = d/dt$, and

g : Acceleration due to gravity m/s^2 ,

θ : Angular displacement rad,

m : Mass of pendulum kg,

I : Moment of inertia around axis of pendulum

Fig. 1 Experimental apparatus

kgm², l : Length of pendulum m, C : Damping coefficient around axis of pendulum Ns/m, A : Amplitude of excitation m, f : Frequency of excitation rad/s, κ : $A/(\text{Length of connecting rod})$.

Practical values of these parameters, except for the damping C , are listed in Table 1. Model (13) can be normalized by the natural angular frequency of the linearized system⁽⁶⁾ as

$$\ddot{\theta} + \gamma \dot{\theta} + (1 + 2Bh(\omega t)) \sin \theta = 0, \quad (14)$$

where $B = 1.047$ and $\omega = 2.047$ in our experiments. The damping γ is plotted as a function of the input voltage E V in Fig. 2. Therefore, γ can be varied from 0.086 to 0.267 2.

3.2 Stochastic model

In previous work⁽⁵⁾, the normalized model (14) was further simplified. That is, taking the limit $\kappa \rightarrow 0$, the motion of the crank-slider mechanism was approximated as $h(t) = \cos(t)$

$$\ddot{\phi} + c\dot{\phi} + (1 + 2a \cos(\omega t)) \sin \phi = 0. \quad (15)$$

By adding random noise to the excitation term of the deterministic model (15), we obtain a stochastic model

$$\begin{cases} \dot{\phi}_1 = \phi_2 \\ \dot{\phi}_2 = -c\phi_2 - (1 + 2a\{\cos(\omega t) + sW(t)\}) \sin \phi_1, \end{cases} \quad (16)$$

where $\phi_1 = \phi$, $\phi_2 = \dot{\phi}$, and $W(t)$ is the Gaussian noise

Table 1 Physical parameters

m	102.7×10^{-3} kg
I	1.643×10^{-3} kgm ²
l	160.0×10^{-3} m
A	130.0×10^{-3} m
f	12.57 rad/s
κ	0.348

Fig. 2 Damping (Experiment)

with zero mean and unit variance, and the parameter s represents the extent of the noise. The noisy term of the stochastic model (16) simulates fluctuations in the excitation due to, for example, rotational fluctuations of the servomotor and mechanical errors in ball bearings and belt drives, etc.

The variational equation of the model (16), which is used for calculation of Liapunov exponents, is given by

$$\begin{cases} \dot{\xi} = \eta \\ \dot{\eta} = -(1 + 2a\{\cos(\omega t) + sW(t)\}) \cos \phi_1 \cdot \xi - c \cdot \eta, \end{cases} \quad (17)$$

where ξ is a variation of θ_1 and η is a variation of θ_2 . Note that e_n in section 2.1 is obtained from the Poincaré map of the solutions (ξ, η) . Therefore, the local expansion rates defined in Eq.(3) and the Liapunov exponents defined in Eq.(4) can be calculated from the solutions (ξ, η) .

In the following numerical examples, we assume the fixed parameters $a = 1.04$ and $\omega = 2$, derived from the experimental condition of the apparatus in Fig. 1. The other parameters, the extent of noise s and the damping c , are assumed to be free.

Furthermore, numerical integrations of Eqs.(16) and (17) are performed by means of the RKG (Runge-Kutta-Gill) method with the step $2\pi/\omega/128$. The noise is simulated by substituting a series of normal random numbers with zero mean and unit variance into $W(t)$

at each step of the integrations. The series of normal random numbers is obtained by applying the Box-Muller method to uniform random numbers generated by the algorithm⁽¹⁵⁾ having a period more than 2.3×10^{18} . This period is long enough to regard the simulated noise as completely random⁽¹⁵⁾ because steps of all our calculations are less than 10^8 .

4. Numerical and Experimental Results

4.1 Negative largest Liapunov exponents (numerical results)

In this section, we only show the numerical results based on the model (16), because precise variational equations are sometimes barely obtainable for experimental systems.

The Liapunov exponents are calculated by substituting the numerical solutions of Eqs.(16) and (17) into Eq.(4). The limit $N \rightarrow \infty$ in Eq.(4) is approximated as $N \rightarrow 10\,000$.

We first show the results for the noise $s=0$. Figure 3 shows Poincaré plots of numerical solutions of the stochastic model (16) for $c=0.198$. The deterministic chaotic attractor having two islands is clearly observed (it is not purely deterministic due to discretization error but the amplitude of such error is fairly small, roughly 10^{-15}). The largest Liapunov exponent takes the positive value $\Lambda^\infty=0.1971$. The chaotic properties of the behavior are naturally evaluated by the positive largest Liapunov exponent, following the theory of the deterministic chaos.

We next show Poincaré plots for the noise $s=0.015$. The Poincaré plot for $c=0.218$ is shown in Fig. 4. The behavior resembles that in Fig. 3, while its largest Liapunov exponent takes a negative value -1.809 . Although negative largest Liapunov exponents normally correspond to the periodic behavior in the deterministic case, the behavior in Fig. 4 is not periodic. Figure 5 describes more detailed features of the behavior in Fig. 4 and consists of Poincaré plots at different phases of external excitation. The integer n indicates the n 'th phase obtained by dividing a period $T=4\pi/\omega$: two periods of external excitation, into thirty-two phases, i.e., each Poincaré plot is sampled at $(t - nT/32) \bmod T = 0$ ($0 \leq n \leq 32$). Therefore, we can alternatively say that the Poincaré plot in Fig. 4 was obtained by sampling points at $n=0$ and $n=15$.

It is observed in Fig. 5 that the attractors are expanding their length and folding their own shapes at the right edges, although this process is not so clearly seen as in the deterministic case. This means that the expanding and folding process, which is regarded as a typical feature of chaotic dynamics, is also found in this random behavior arising in the stochastic system.

In this paper, we refer to such stochastic behavior

Fig. 3 Poincaré plot of deterministic chaos (Simulation)

Fig. 4 Poincaré plot of stochastic chaotic behavior (Simulation)

Fig. 5 Expanding and folding process of stochastic chaotic behavior (Simulation)

resembling the deterministic chaos as SCB (stochastic chaotic behavior).

The above results are summarized as follows:

- The SCB has chaotic properties in the expanding and folding process of the attractors resembling deterministic chaos.
- Despite such chaotic properties, the SCB has negative largest Liapunov exponents.

The phenomena mentioned above is likely to be related to strange nonchaotic attractors⁽¹⁶⁾ arising in quasiperiodic forcing systems. The strange nonchaotic attractors exhibit negative largest Liapunov exponents while having fractal structures. Many researchers have focused on such nonchaotic attractors and have investigated their physical properties^{(16),(17)}. However, this paper does not discuss

whether or not the observed behavior (Fig. 4) is strange nonchaotic.

4.2 Chaotic behavior in the experiment (experimental results)

Figure 6 shows the global bifurcation diagram of the apparatus in Fig. 1. Figure 6 consists of eight Poincaré plots, labeled 1 to 8, corresponding to the input voltage E varied from 29.5 V ($\gamma=0.239$) to 33.5 V ($\gamma=0.250$) with steps of 0.5 V. These Poincaré plots of 500 periods are obtained from the reconstructed phase portraits by the delay coordinate (11) of the angular displacement with the delay time $\Delta t=0.198$ s. The horizontal axis represents the angular displacement of the pendulum $\theta(t)$ and the vertical axis represents the delay coordinate $\theta(t+\Delta t)$. The scale of the plots is identical among these figures.

It is clearly seen in Fig. 6 that parametric bifurcations occur as the damping γ is varied. First, the experimental behavior shown in 1 of Fig. 6 is strongly random, exhibiting fractal structures generated by the expanding and folding process of the chaotic attractors. As the damping γ is increased, mixed structures, consisting of islands of period 2 and wandering transient points, arise. A further increase in the damping results in isolated islands of period 2 in 4. This process repeats from 5 to 6. This indicates that the experimental apparatus is in a state of chaotic bursts (crisis-induced intermittency⁽¹⁸⁾ or noise-induced intermittency⁽⁷⁾) from 2 to 6. As the damping γ increases further from 7 to 8, the extent of attractors

Fig. 6 Chaotic bifurcation arising in the apparatus (Experiment)

becomes smaller and the periodicity of the behavior becomes stronger.

From the fractal structures observed in 1 and parametric bifurcations from chaos to nearly periodic behavior, from 1 to 8, it is clear that the responses of the apparatus are similar to deterministic chaos. In previous work⁽⁶⁾, we regarded such experimental behavior as chaos, and demonstrated that statistical mechanical properties of the experimental behavior are well modeled by the deterministic model.

We focus on the behavior for $E=31.0$ V (or $\gamma=0.2431$), which has an attractor resembling deterministic chaos (Fig. 3) and the SCB (Fig. 4). The 10 000 periods of the Poincaré plot of this behavior are shown in Fig. 7. The orbits are reconstructed by the delay coordinate (11) with the delay time $\Delta t=0.198$ s.

4.3 Largest Liapunov exponent from time series (numerical and experimental results)

In the previous section, three types of behavior resembling each other, shown in Figs. 3, 4, and 7, are exhibited.

Table 2 shows the largest Liapunov exponents Λ^∞ of these behaviors. The exponents are estimated by the method mentioned in section 2.2. The horizontal axis represents the radius R by which the Poincaré maps are locally linearized (see section 2.2). The value of R is denoted by the percentage of the horizontal extent of the corresponding attractor. The limit $N \rightarrow \infty$ in the definition (Eq. (4)) is approximated by $N \rightarrow 9\,999$.

It is clear from Fig. 8 that the deterministic chaos (Fig. 3) is exceptionally different from the others.

Table 2. Largest Liapunov exponents

Type	Variational Eq.	Estimated
DTM	0.1971	0.1915
SCB	-0.1809	-0.1900
EXP	(disable)	-0.1374

Fig. 7 Poincaré plot of chaos arising in the apparatus (Experiment)

Fig. 8 Estimated largest Liapunov exponents as a function of R

That is, the estimated values of Λ^∞ for the deterministic chaos, denoted by DTM, nearly maintain a constant value independent of R , while the values for SCB and the experimental behavior (denoted by DTM) tend to monotonically decrease as R is increased. Furthermore, it is clear from Fig. 8 that since Λ^∞ of the SCB and the experimental behavior (EXP) take both negative and positive values, the signs of Λ^∞ cannot be specified without additional conditions.

To solve this problem, we focus on saturated values of Λ^∞ found in the region $R \geq 2.3$. Averaged values of Λ^∞ in this region are listed in Table 2. For comparison, the values of Λ^∞ in Figs. 3 and 4, which have already been calculated from variational equations, are also listed.

First, it is clear based on the numerical results of the deterministic chaos (Fig. 3) and the SCB (Fig. 4) that the averaged values (denoted by DTM and SCB in the table) give adequately estimated values of the largest Liapunov exponents, i.e., their signs and values are in good agreement with the exact ones based on the variational equations.

In the case of the experiment (denoted by EXP), we cannot compare the estimated value with the exact value to confirm the estimation accuracies. However, we also regard the averaged value $\Lambda^\infty = -0.1374$ obtained for the experimental behavior as the adequately estimated value as in the case of the SCB because the experimental behavior and the SCB resemble each other in their dependency on the radius R . This leads to the observation that the experimental behavior (Fig. 7) has negative largest Liapunov exponents while it has chaotic properties (Fig. 6, Fig. 7) similar to deterministic chaos.

The above results are summarized as follows:

- The largest Liapunov exponent for SCB is greatly varied, depending on the condition of estimation R , and possibly takes both positive and negative values.
- The largest Liapunov exponents for the experi-

ment have properties similar to those of SCB but different from those of deterministic chaos.

- The chaotic behavior in the experiment has the negative largest Liapunov exponent as in the case of the SCB.

The first and second points quantitatively imply that the physical chaotic behavior (Fig. 7) occurring in the experimental apparatus is well modeled by the stochastic model (16) rather than by the deterministic model (15).

Furthermore, the above results imply that even if the experimental behavior resembles deterministic chaos, the value of the largest Liapunov exponent, especially its sign, may be incorrectly estimated without additional conditions such as the saturation of the values.

4.4 Probability distribution of local expansion rates (numerical and experimental results)

The results in the previous section are summarized as follows. That is, SCB (Fig. 4) and experimental chaotic behavior (Fig. 7) have the following properties:

- qualitative properties resembling deterministic chaos.
- quantitative properties evaluated by the negative largest Liapunov exponents.

The two properties above are mutually exclusive in the sense of deterministic chaos.

These exclusive properties lead to the question of whether or not the experimental behavior and the SCB contain nearby orbits which exponentially repel each other and are normally characterized by positive largest Liapunov exponents. To answer this question, we introduce the probability density function $P(\lambda)$ of local expansion rates along unstable manifolds $\lambda_1(\mathbf{x}_n)$. To calculate $P(\lambda)$, we reuse the data of $\lambda_1(\mathbf{x}_n)$ (data length 9 999) that have already been calculated for the largest Liapunov exponents of the attractors in Fig. 8. Applying Eq. (12) to the data, $P(\lambda)$ is obtained. The limit $N \rightarrow \infty$ is approximated by $N \rightarrow 9\,999$.

For convenience, we regard behavior as being chaotic in a narrow sense if it has the property of exponentially repelling orbits. Note that such chaotic properties are indicated as peaks in the right half of $P(\lambda)$ ($\lambda > 0$).

First, the probability density function $P(\lambda)$ for deterministic chaos (Fig. 3) is shown in Fig. 9. Each $P(\lambda)$ is calculated with R varied from 0.4 to 3.0 with steps of 0.2. Consequently, the 14 probability density functions are plotted in this figure. A large peak is found in the right half of $P(\lambda)$ near $\lambda = 1 > 0$. This peak indicates chaotic properties of the behavior. Even in this deterministic case, peaks in the left half of $P(\lambda)$ ($\lambda < 0$) do not vanish, and small but not

Fig. 9 Probability density of local expansion rates of deterministic chaos (Simulation)

negligible peaks are also found in this region. Therefore, it is considered that the sign of the largest Liapunov exponents strongly depends on this balance among peaks both in the left and right halves of $P(\lambda)$. It is clear from Fig. 9 that for deterministic chaos (Fig. 3), the peak near $\lambda=1$ in the right half of $P(\lambda)$ is dominant. This balance for deterministic chaos (Fig. 3) appears to make the largest Liapunov exponent positive. Additionally, it is worth noting that the shape of $P(\lambda)$ is nearly independent of the value of R .

Next, the results on $P(\lambda)$ for SCB (Fig. 4) are shown in Fig. 10. A considerable peak is found near $\lambda=1$ in the right half of $P(\lambda)$. This indicates that chaotic properties similar to deterministic chaos are also found in SCB while it has the negative largest Liapunov exponents (1). However, in contrast to the deterministic case, another peak is also found in the left-half region. Therefore, the negative largest exponent of SCB is obtained because of the balance so that the peak near $\lambda=1$ in the right-half region is not so dominant as in the deterministic case. Furthermore, independent of the value of R , the peak indicating the chaotic properties can be found near $\lambda=1$.

Lastly, the results of $P(\lambda)$ for the experimental behavior (Fig. 7) are shown in Fig. 11. Similar to the result for SCB (Fig. 10), the chaotic properties of the experimental behavior are indicated as a peak near $\lambda=1$, and the occurrence of the peak is independent of the value of R .

The above results are summarized as follows:

- The chaotic properties, in a narrow sense, are indicated by peaks in the right half of $P(\lambda)$ ($\lambda > 0$).
- Presence of the peaks does not depend on R , the condition of estimation.
- SCB and experimental chaos are considered to contain chaotic properties resembling deterministic chaos because they have large peaks in the right half of $P(\lambda)$.

The first result means that peaks in the right half of $P(\lambda)$ enable us to characterize the chaotic properties that cannot be evaluated by the positive largest

Fig. 10 Probability density of local expansion rates of stochastic chaotic behavior (Simulation)

Fig. 11 Probability density of local expansion rates of experimental chaos (Experiment)

Liapunov exponents. The second result means that the index based on $P(\lambda)$ is easily applicable to practical cases, i.e., in contrast to the largest Liapunov exponents, this index does not depend on R , and thus it directly gives us the desired information without careful optimization of the condition R . The third result quantitatively confirms our qualitative observation that SCB (shown in Figs. 4 and 5) and the experimental behavior (shown in Figs. 6 and 7) are chaotic.

The above results lead to the conclusion that the index based on the probability density function of the local expansion rates is effective to characterize the chaotic properties of the experimental behavior arising in physical systems such as our experimental apparatus.

5. Conclusions

In this paper, we theoretically and experimentally investigated the capability of the stochastic model to describe the physical chaotic behavior arising in our experiments, and compared our findings with the results based on a purely deterministic model.

First, we modeled the experimental apparatus by the pendulum system with noisy sinusoidal parametric excitation and showed that such a stochastic system produces random behavior resembling deterministic chaos, which we call SCB.

We then calculated the largest Liapunov

exponents of SCB based on the variational equations and obtained the following results:

- SCB qualitatively resembles deterministic chaos in its expanding and folding process of attractors.
- Despite such chaotic properties, SCB is qualitatively evaluated by negative largest Liapunov exponents.

The above results imply that in the stochastic case, chaotic properties may not be correctly evaluated by the positive largest Liapunov exponents normally used as the index of chaos.

We next estimated the largest Liapunov exponents of SCB and the experimental behavior and obtained the following results:

- The value of the largest Liapunov exponents depends considerably on R , the condition of estimation, so that the value may take both positive and negative values.
- This dependency is similar to that of SCB but different to that of deterministic chaos.
- The experimental chaotic behavior has the negative largest Liapunov exponent as in the case of SCB.

The above results quantitatively imply that the experimental chaotic behavior is well modeled by SCB rather than by deterministic chaos.

Furthermore, we introduced the probability density function of local expansion rates in order to evaluate chaotic properties of SCB and experimental chaotic behavior whose largest Liapunov exponents are negative; the following results were obtained:

- Chaotic properties of the behaviors are indicated by peaks in the right half of the probability density function.
- The peaks do not depend on R , the condition of estimation.
- SCB and experimental chaotic behavior have large peaks in the right half region, thus they contain chaotic properties resembling deterministic chaos.

The above results quantitatively imply that random behavior may contain chaotic properties resembling deterministic chaos even if the behavior does not have positive largest Liapunov exponents.

The above results give us a practical example that in experimental situations there is chaotic behavior that has negative largest Liapunov exponents while containing chaotic properties, and that the probability density function of local expansion rates evaluates such chaotic properties in detail.

In a future paper, we will show how detailed features of SCB can be described by stochastic process theory and how the strange nonchaotic attractors^{(16),(17)} are related to SCB and experimental

chaotic behavior.

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