Gain-Scheduled Control of a Tower Crane Considering Varying Load-Rope Length*

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This paper discusses control of a crane mounted on a tower-like structure in case of varying load-rope length. A fast transfer of the load causes the sway of the load rope and the vibration of the flexible structure. Our object is to control both the sway and the vibration by control of torque to the boom. If the length of the load rope varies, it is difficult to control by the fixed compensator, because not only the natural frequency of the load rope but also gain of the plant to the control input vary. We design the gain-scheduled $H_\infty$ controller based on the LMI for the length of the load rope and verify the efficiency of the controller by simulaitons and experiments.

**Key Words**: Positioning, Vibration Control, Robust Control, Tower Crane, Flexible Structure, $H_\infty$ Compensator, Gain-Scheduling, LMI

1. Introduction

This paper deals with the jib-type crane mounted on a tower-like flexible structure called 'tower crane'. A tower crane that is used for construction of buildings or dams is necessary to climb up by adding a part of the tower to itself. Because of this structure, vibration of the tower is caused by motion of the crane such as ups and downs of the boom, rotation of the boom, rolling-up and -down of the load rope, sudden change of the load by its landing or lifting-up, and the discharge of the concrete from the hose. This makes the efficiency of the operator worse and becomes the problem to be solved as well as the sway of the load rope. There were studies\(^{(1,2,3,4)}\) for these problems and the necessity of automation for the operation is growing up.

The orders of the operation for the crane are 1) rolling up the load rope, 2) lifting up and rotating the boom simultaneously and 3) rolling down the load rope. Therefore, the variation of the rope length is not negligible for the automated operation. One of the authors has designed the gain-scheduled controller for a moving crane with varying length of the load rope\(^{(5)}\) by the method proposed by Hyde and Glover\(^{(6)}\). The efficiency of the controller is verified by carrying out simulations.

Also, the gain-scheduled controller based on the linear matrix inequalities (LMI) that proposed by Gahinet\(^{(7)}\) has already been used for the time-varying or parameter-depending systems and some studies were reported on control of the rigid bodies\(^{(8,9,10)}\). The authors do not know the application for the vibration control including the unstructured dynamics.

This paper leads the equation of motion for the tower crane with variation of the load-rope length and derives the reduced-order model for the controller design. The reduced-order model that is a nonlinear equation depends on the rope length is formulated as the linear parameter-depending model. We design the gain-scheduled controller in accordance with the rope length, which the operator varies arbitrarily. In experiments the implementation of the compensator utilizes the Padé approximation for the digital computing. We perform the real-time
digitization of the gain-scheduled controller obtained in continuous time by the approximation. The gain-scheduled compensator is compared with the fixed compensator for a certain rope length by simulations and experiments and the efficiency of the gain-scheduled one is then verified.

2. Modeling

2.1 Full-order model and reduced-order model

Figure 1 shows the full-order model of the tower crane. Definition of symbols are shown as follows;

\[ x_i: \text{Displacement of tower in } x \text{ direction}, \]
\[ \nu: \text{Boom angle}, \]
\[ \phi_t: \text{Bending of tower}, \]
\[ \theta: \text{Swing angle of the load}, \]
\[ \alpha: \text{Rotational angle of roller}, \]
\[ \Omega_b: \text{Angle between gantry and } x-\text{axis}, \]
\[ L: \text{Length of boom}, \]
\[ R_b: \text{Radius of roller}, \]
\[ E: \text{Modulus of elasticity}, \]
\[ l_i: \text{Load-rop length}, \]
\[ l_b: \text{Length of rope supporting boom}, \]
\[ l_w: \text{Distance between top of gantry and roller}, \]
\[ a: \text{Length of tower part}, \]
\[ a_s: \text{Distance between root of boom and origin}, \]
\[ l_n: \text{Distance between equilibrium point and the origin}, \]
\[ \lambda: \text{Stretch of rope supporting boom}, \]
\[ k_s: \text{Spring coefficient of rope supporting boom}, \]
\[ c_s: \text{Damping coefficient of rope supporting boom}, \]
\[ A_s: \text{Cross-sectional area of boom support rope}, \]
\[ A_t: \text{Cross-sectional area of tower part}, \]
\[ \rho: \text{Density of tower part}, \]
\[ I_s: \text{Cross-sectional area moment of inertia of tower part}, \]
\[ W_b: \text{Weight of boom}, \]
\[ W_o: \text{Weight of load}, \]
\[ J_a: \text{Moment of inertia of top of crane}, \]
\[ J_n: \text{Moment of inertia of roller}, \]
\[ M_a: \text{Mass of top of tower}, \]
\[ M_b: \text{Equivalent mass of tower part}, \]
\[ k_i: \text{Equivalent spring coefficient of tower part}, \]
\[ c_i: \text{Equivalent damping coefficient of tower part}, \]
\[ u: \text{Control input}, \]
\[ g: \text{Gravitational acceleration}, \]

The origin of \( x-x \) axis is centered at the top of the tower. The followings are assumed;

1. Mass of the rope is neglected.
2. The boom, the load rope and the tower move in the \( x-x \) plane.
3. The tower is divided into four elements.
4. The boom is rigid.

Since the angle of the load rope \( \theta \) is small,

\[ \sin \theta = \theta, \quad \cos \theta = 1, \quad (1) \]

Assuming the angle of the boom varies from 20 to 70 deg, \( \sin \nu \) and \( \cos \nu \) can be transformed as follows;

\[ \sin \nu = \frac{\sin \nu}{\nu}, \quad \cos \nu = \frac{\cos \nu}{\nu}. \quad (2) \]

We linearized the governing equation by Eqs. (1) and (2) and derived the following state equation

\[
\ddot{x} = A_x(\nu, \dot{\nu}, \theta, \dot{\theta}, l, \dot{l}) x + B_x(\nu, \dot{\nu}, \theta, \dot{\theta}, l, \dot{l}) u + \Phi \quad (3)
\]

where \( \Phi \) is the nonlinear term, which is mainly due to the rope supporting the boom and cannot be linearized based on Eqs.(1) and (2), and the state vector \( x \) is

\[
x = [x_0, \nu, \theta, a, \dot{a}, \dot{\phi}_i, \dot{\theta}, \dot{a}_s]^T. \quad (22 \times 1)
\]

\[
i = 1, 2, 3, 4 \quad (4)
\]

The observation value \( y \) is written by the following output equation

\[
y = [x_0, \nu, \theta]^T = C_x x. \quad (5)
\]

The state equation (3) is not suitable for controller design, because the term \( \Phi \) exhibits strong nonlinearity and the order is too large. We employ the reduced-order model shown in Fig. 2 for controller design. The followings are assumptions of the model in addition to (1.1)-(1.4);

(2.1) The input force directly affects the edge of the boom without rope supporting the boom.

(2.2) The tower part is a one-degree-of-freedom system, which can move in \( x \) direction.

Derivation of equation of motion is shown as follows. The potential energy \( T_b \) and the kinetic energy \( V_b \) of the boom are

\[
T_b = \frac{W_b}{2L} \int_0^L (\dot{x}^2 + \dot{z}^2) \, dz \quad (6)
\]

\[
V_b = \frac{W_o}{L} \int_0^L z_s \, dz \quad (7)
\]

\[
W_b = \frac{L}{2} \int_0^L \left( \dot{x}^2 + \dot{z}^2 \right) \, dz
\]

\[
W_o = \frac{L}{2} \int_0^L z_s \, dz
\]

\[
\int_0^L \left( \dot{x}^2 + \dot{z}^2 \right) \, dz = \int_0^L \left( \dot{z}^2 + \dot{z}^2 \right) \, dz
\]

\[
\int_0^L z_s \, dz = \int_0^L \dot{z} \, dz
\]
where the displacement of the point $x_o$, $z_o$ from the root of the boom are

$$x_o = x_r + \xi \sin \nu, \quad z_o = \xi \cos \nu.$$  \hfill (7)

The potential energy $T_o$ and the kinetic energy of the load $V_o$ are

$$T_o = \frac{W_o}{2}(\dot{x}_r^2 + \dot{z}_o^2), \quad V_o = W_0 g z_o$$  \hfill (8)

where the position of the load $x_o$, $z_o$ are

$$x_o = x_r + L \sin \nu + l \sin \theta, \quad z_o = L \cos \nu - l \cos \theta.$$  \hfill (9)

The kinetic energy of the top of the tower $T_t$ is

$$T_t = \frac{1}{2} M_b \dot{x}_r^2.$$  \hfill (10)

From assumption (2.1) the force putting into the top of the boom is

$$Q_o = - \frac{1}{R_b} u.$$  \hfill (11)

The angle between the force $Q_b$ and the boom is

$$\theta_b = \sin \left(- \frac{\delta_o}{L_o} \cos (Q_b - \nu) \right).$$  \hfill (12)

From assumption (2.2) the equivalent forces $Q_i$ caused by the control input $u$ is

$$Q_i = \frac{3}{2L_i} u$$  \hfill (13)

and the force putting into the tower part is

$$Q_o = - k_i \dot{x}_r - c_i x_r,$$  \hfill (14)

where the equivalent spring coefficient $k_i$ and the equivalent damper coefficient $c_i$ adjust to the natural frequency of the first mode of the tower.

Using Lagrange's equation

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial \dot{q}_i} = Q_i,$$

$$T = T_o + T_t + V_t, \quad V = V_o + V_t.$$  \hfill (15)

$$q_i = x_r, \nu, \theta,$$

$$Q_i = Q_o + Q_{tb}, Q_o \sin \theta_b L_o, 0.$$  \hfill (16)

The governing equation of the reduced-order model is written as follows;

$$M_b \ddot{x}_r + W_b \left( \dot{x}_r + L \left( \frac{\dot{\nu} \cos (\nu)}{2} - \frac{\dot{\nu} \sin (\nu)}{2} \right) \right)$$

$$+ W_b \left( \ddot{x}_r + 2 \dot{\theta} \cos (\theta) + \dot{\theta} \cos (\theta) \right)$$

$$+ L (\dot{\nu} \cos (\nu) - \dot{\nu} \sin (\nu) + \dot{l} \sin (\theta) - \dot{\theta} \sin (\theta))$$

$$= Q_o + Q_{tb}.$$  \hfill (16)

| $\Omega_b$ | $\pi/2$ (rad) | $L$ | 0.71 (m) |
| $R_b$ | 0.01 (m) | $E$ | $2.1 \times 10^{11}$ (N/m$^2$) |
| $l$ | 0.8 (m) | $l_o$ | 0.2 (m) |
| $L_t$ | 1.2 (m) | $a$ | 0.05 (m) |
| $a_o$ | 0.19 (m) | $l_m$ | 0.085 (m) |
| $A_b$ | $3.1 \times 10^{-16}$ (m²) | $A_t$ | $6.4 \times 10^{-5}$ (m²) |
| $l$ | $1.1 \times 10^4$ (kg/m$^3$) | $e$ | 0.006 |
| $l_t$ | $7.34 \times 10^{-10}$ (m$^4$) | $W_t$ | 0.205 (kg) |
| $W_0$ | 0.15 (kg) | $J_d$ | 0.11 (kg · m$^2$) |
| $J_m$ | $1.6 \times 10^{-5}$ (kg · m$^2$) | $M_d$ | 1.3 (kg) |
| $M_b$ | 1.5 (kg) | $c_i$ | $1.96 \times 10^{-2}$ (N · s/m) |
| $k_b$ | $1.45 \times 10^2$ (N/m) | $g$ | $9.8 (N • s^2 /m)$ |

\[ W_b \left( \frac{\dot{\nu}^2}{3} + L \cos (\nu) \ddot{x}_r - \frac{\dot{\nu} \sin (\nu)}{2} \right) - W_o \cos (\nu) + W_o \left( L \dot{\nu} + \frac{\dot{\nu} \cos (\nu) x_r + 2 \dot{\theta} \cos (\nu + \theta) + \dot{\theta} \cos (\nu + \theta)}{1 + \dot{\nu} \sin (\nu + \theta)} - \frac{\dot{\nu} \sin (\nu + \theta)}{1 + \dot{\nu} \sin (\nu + \theta)} \right) \]

\[ W_b \left( \ddot{x}_r + 2 \dot{\theta} \cos (\theta) + \dot{\theta} \cos (\theta) \right) + L \left( \cos (\nu + \theta) - \dot{\nu} \sin (\nu + \theta) \right) = 0 \]

The state equation linearized by Eqs. (1) and (2) is as follows;

\[ \dot{x}_r = A \left( \nu, \dot{\nu}, \theta, \dot{\theta}, l, \dot{l}, \dot{l}_o, \dot{\dot{l}} \right) x_r \]

\[ + B \left( \nu, \dot{\nu}, \theta, \dot{\theta}, l, \dot{l}, \dot{l}_o \right) u, \]

\[ x_r = [x_r, \nu, \theta, \dot{\nu}, \dot{\theta}, \dot{l}]^T. \]

The observation value $y$ is written by the following output equation

\[ y = [x_r, \nu, \theta]^T = C \left( x_r, y, \theta \right) \]

Table 1 shows specifications of the tower-crane model.

### 2.2 Transformation to linear parameter-depending system

Because this study assumes that an operator of the tower crane varies the load-rope length arbitrarily, the matrices $A_i, B_i$ of Eq. (19) vary in accordance with the load-rope length. We derived the state equation depending on the varying load-rope length from the reduced-order model Eq. (19) for the design of the controller.

Although the matrices $A_i, B_i$ of Eq. (19) depend on $\nu, \dot{\nu}, \theta, \dot{\theta}, l, \dot{l}, \dot{l}_o$, assuming $\nu \approx \nu_0$ (some angle of $\nu$), $\nu, \theta, \dot{\theta}, \dot{l}, \dot{l}_o \approx 0$ leads the following parameter-depending state equation taking into account of only the varying load-rope length $l$,

\[ x_r = A_l \left( \nu_0, 0, 0, 0, 0 \right) x_r \]

\[ + B_l \left( \nu_0, 0, 0, 0, 0 \right) u \]

On the design of the LMI-based gain-scheduled...
controller, which will be shown in the next section, there is a constraint that the elements of the control input matrix $B_r$ must be constant. Since, in Eq.(22), $B_r$ contains the varying load-rope length $l$, the constraint is not satisfied. We added an actuator characteristic, which have an enough wide frequency range in comparison with that of the controlled plant, and let the parameter-dependent elements of the matrix $B_r$ in the system matrix $A$ of the augmented plant. 

The new state equation and the output equation of the reduced-order model are obtained as follows:

\begin{align*}
\dot{x} &= A(Z)x + Bu_e \\
y &= Cx
\end{align*}

where $u_e$ is an input voltage to the actuator and $Z$ is a new parameter depending on the load-rope length. With variation of the load-rope length $l$, the elements depend on $l$ in matrix $A$ vary. We especially remark 8 elements, which vary larger than others do. All varying elements can be represented as linear functions of $Z$ by making the element $(6, 1)$ $Z$ as follows:

\begin{equation}
A(M, Z) = 
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Z & aZ & \beta Z & \gamma Z & M & \eta Z \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\end{equation}

$a, \beta, \gamma, x, \lambda, \eta = \text{const.}$

and $M$ is represented as

\begin{equation}
M = 2.0 \cdot 10^{-4} - 7.0 \cdot 10^{-4} Z.
\end{equation}

In the sequel, the Eq.(23) is rewritten as follows

\begin{equation}
\dot{x} = (A_s + ZA_z)x + Bu_e
\end{equation}

where $A_s$ is a matrix, which is obtained by removing the varying elements from the matrix $A$, and $A_z$ is a matrix which has only coefficients of $Z$.

We fixed the boom angle of $\nu_0$ in

\begin{equation}
\nu_0 = 45(\text{deg})
\end{equation}

which is a center of movement of the boom angle, and the range of variation of the load-rope length $l$ is assumed to be

\begin{equation}
0.3 \text{m} \leq l \leq 1.5 \text{m}.
\end{equation}

The range of $Z$ becomes

\begin{equation}
Z_{\text{min}} \leq Z \leq Z_{\text{max}},
\end{equation}

\begin{equation}
Z_{\text{min}} = 3.75 \cdot 10^4, \quad Z_{\text{max}} = 1.88 \cdot 10^5.
\end{equation}

$Z$ can be computed by the load-rope length $l$ as follows,

\begin{equation}
Z = \frac{563}{l}.
\end{equation}

3. Control System Design

The parameter-depending model Eq.(27) for controller design does not contain the characteristics of the rope supporting the boom and the higher modes of the tower truncated from the full-order model.

Furthermore, if the length of the load rope varies, not only the natural frequency of the load rope vary, but also gain of the plant to the control input becomes sensitive. Then, a linear time-invariant (LTI) controller might fail to get a good control performance. We designed the gain-scheduled $H_\infty$ controller based on the LMI\textsuperscript{40} for the varying load-rope length.

The generalized plant is shown in Fig. 3 where $\nu$ is the control input, $w_1$ and $w_2$ are disturbances, $y$ is the observation value, $z_1$ and $z_2$ are the controlled values. Criterion function used for the controller design is

\begin{equation}
\|G_{zw}\|_\infty < \gamma = \begin{bmatrix} G_{zw1} & G_{zw2} \\ G_{zw1} & G_{zw2} \end{bmatrix}
\end{equation}

\begin{align*}
G_{zw1} &= W_1(I - PK)^{-1}KP \\
G_{zw2} &= W_2(I - PK)^{-1} \theta \\
G_{zw2} &= W_3(I - PK)^{-1}D_w
\end{align*}

where $W_1, W_2, D_w$ are shown as follows,

\begin{align*}
W_1 &= \frac{6.8 \cdot 10^4 \cdot (s^2 + 1.6(0.2 \pi s + (0.2 \pi s)^2) \\
& + 8 \cdot 10^5}{s^2 + (20 \pi s + (20 \pi s)^2)^2}, \\
W_2 &= \frac{1.2 \cdot 10^4}{s + \epsilon}, \quad \epsilon = 10^{-6}, \\
W_3 &= \frac{10^2}{s^2 + 2 \cdot 0.5(3 \pi)s + (3 \pi)^2}, \\
W_2 &= \text{diag}(W_{21}, W_{22}, W_{23}), \\
D_w &= \text{diag}(0.03, 0.05, 0.05).
\end{align*}

$W_1$ is a weighting function for stability robustness, and $W_2$ is $W_3$ are weighting functions for vibration of the tower parts, the boom angle, and the load-rope angle, respectively. $W_2$ includes the integrator for the positioning control of the boom angle. The gain of the frequency responses of weighting functions and the multiple error $A_b$ are shown in Fig. 4. Thus, LTI vertex controllers corresponding to the two vertexes are constructed as follows;

\begin{equation}
K_i = \begin{bmatrix} A_{K_1} & B_{K_1} \\ C_{K_1} & D_{K_1} \end{bmatrix} \quad (i = 1, 2)
\end{equation}

Gain-scheduled controller for the length of the load rope is obtained from convex interpolation using a parameter $Z$ corresponding to its length as follows;
The gain of the controller obtained is 14. Figure 5 shows the gain of the gain-scheduled controller obtained by Eq. (36) for the load rope length \( l = 0.8 \) m. In the frequency from 5 to 20 Hz, the experimental setup of the tower crane has vibration modes, such as transverse vibration of the rope supporting the boom, vibration of the rope supporting the load, and the rotary motion of the load. Even the full-order model does not include these modes. While the controller obtained can stabilize the full-order model in simulation, it causes spillover phenomena in experiment because it has a high gain in frequency range above 5 Hz. Then, we add a low-pass filter of third order shown in Fig. 6, which has a slight notch property in the frequency from 5 to 20 Hz. It also decreases the gain of the controller above 250 Hz of a sampling frequency in order to prevent the aliasing caused by digitizing the controller. Although the phase delay caused by the low-pass filter is about 30 deg around 1 Hz which is mainly the range of control, we verified that its effect to the control performance is very little by simulation. Figures 7 (a) and (b) show bode diagrams of the gain-scheduled controllers for \( l = 1.5 \) m and \( l = 0.3 \) m to which the low-pass filter is added, respectively.

4. Experiments

4.1 Experimental setup

Figure 8 shows the schematic diagram of the experimental setup. The displacement of the top of the tower is assumed to be equal to the strain multiplied by a certain constant value. The strain is
detected by the strain gauges attached on the lower part of the tower. The boom angle is detected by the potentiometer connected to the shaft that is a center of rotation of the boom. The edge of the boom equips with the potentiometer with the fork clipping the load rope and the potentiometer detects the angle of the load rope that is assumed to be straight. The ratio of the load rope in time is 0.15 m/s for rolling up and 0.23 m/s for rolling down.

The controller interpolated by Eq。(36) according to the load rope is digitized by the Padé approximation(40)

\[ A_{k+1} = I + \Delta t A_k \left( I - \frac{\Delta t}{2} A_k \right)^{-1} \]  

\[ B_{k+1} = (\Delta t \cdot B_k) \left( I - \frac{\Delta t}{2} A_k \right)^{-1} \]  

in every sampling time \( \Delta t = 4 \) ms. This results in realization of the gain-scheduled compensator digitized in real time as precisely as possible.

4.2 Experimental results

First, we performed the control experiments by giving the objective value for the boom angle in rolling up the load rope. The objective value was a ramp input whose change ratio was 5 deg/s.

Figure 9 shows the responses in case of \( \nu_0 = 45 \) deg \( \rightarrow \nu = 55 \) deg, \( l(0) = 1.4 \) m \( \rightarrow l = 0.3 \) m. The boom angle \( \nu \) is transferred to the angle \( \nu_0 \) that is counterclockwise from the \( x \)-axis.

The solid line indicates the response by the gain-scheduled compensator and the broken line indicates that by the fixed compensator for \( l = 0.8 \) m. From Fig. 9 it is seen that the gain-scheduled compensator suppresses the vibration of the tower, while the fixed compensator causes the spillover phenomenon in the modes from 5 Hz to 20 Hz that are not taken into consideration in the simulation model. As the load rope becomes short, the controlled plant is getting more sensitive against the control input and the gain diagram of the gain-scheduled compensator becomes

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Fig. 8 Schematic diagram of experimental setup

Fig. 9 Experimental results \( \nu_0 = 45 \rightarrow 55 \) deg, \( l = 1.4 \rightarrow 0.3 \) m

small. Since the fixed compensator cannot change the gain by itself, the spillover phenomenon is caused. The performance of following to the objective values is getting worse by the gain-scheduled compensator because of decrease in its gain, as the rope length is getting closer to 0.3 m. And it causes the sway of the load rope. But the boom angle is slowly settled to the objective value without spillover after 8 s. We also verified that the experimental results coincide with the simulation results.

Next, in order to verify the control performance against a wind input we performed the control experiments giving the impulse to the load with its rolling down from 0.3 m to 1.5 m. Figure 11 shows the results of the control experiment. The objective value for the boom angle is 45 deg. The solid line indicates the response by the gain-scheduled compensator and the broken line indicates that by the fixed compensator for \( l = 0.8 \) m. From Fig. 11 it is seen that the gain-scheduled compensator is superior to the fixed one from the viewpoints of suppression performance

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**Fig. 10** Simulation results \( \nu_a = 45 \rightarrow 55 \) (deg), \( l = 1.4 \rightarrow 0.3 \) (m)

**Fig. 11** Experimental result of impulse response \( \nu_a = 45 \) (deg), \( l = 1.5 \rightarrow 0.3 \) (m)
of the sway of the load rope and the settling performance of the boom angle.

5. Conclusions

This paper has led the equation of motion for the tower crane with variation of the load-rope length and has derived the reduced-order model for the controller design. The reduced-order model was formulated as the model depends on the parameter of the rope length. We designed the gain-scheduled controller in accordance with the load-rope length, which the operator varies arbitrarily. The gain-scheduled compensator was compared with the fixed one for a certain rope speed of length by simulations and experiments and its efficiency was verified. It was also shown that the real-time digitization of the gain-scheduled controller obtained in continuous time was performed by the Padé approximation.

References


