Adaptive Sliding Controller for Uncertain Second Order Systems*

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In this paper an adaptive sliding controller is presented for second order nonlinear systems with system parameter uncertainties. Using an adaptive technique to estimate uncertain system parameters, an adaptive sliding controller is then proposed to ensure that the sliding condition is maintained in finite time. The proposed adaptive sliding control scheme can be implemented without the requirement that the bounds of uncertain system parameters should be known in advance. Furthermore, the control chattering drawback occurred in the classical sliding mode control can be alleviated in the proposed control scheme. To show the effectiveness of the proposed scheme, some simulation results are presented.

Key Words: Adaptive Control, Sliding Mode Control, Uncertain System

1. Introduction

Sliding mode control, also called variable structure system control, is a nonlinear controller which provides an effective and robust means for controlling a system with uncertainties⁽¹⁾⁻⁽⁵⁾. However, traditional sliding mode control always results in a chattering control input owing to its discontinuous switching control (i.e. $k(t) \operatorname{sgn}(S)$). The control input chattering may damage system mechanisms or excite the high frequency unmodelled dynamics that may result in an unstable system. Hence, how to obtain a chattering-free sliding mode control becomes an open question for researchers^{(1),(6),(7)}. More recently, some attempts have been made to combine adaptive technique and sliding mode control approach⁽⁶⁾⁻⁽¹¹⁾.

In this paper, an adaptive law is presented to estimate uncertain system parameters recursively. According to estimated parameter values, we derive a sliding control input to satisfy the so-called sliding condition. In the adaptive sliding control scheme, there is a design flexibility to select some control values in the control input. We present the mathematical analysis for the proposed adaptive sliding control

system stability property. In mathematical analysis, it is proved that the sliding condition can be maintained in finite time. Finally, some simulation results are made to show the performance of the proposed adaptive sliding control scheme.

2. System Statement

Consider a second order system described in the form of

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x, \theta) + g(x, \theta)u\end{aligned}\quad (1)$$

where, $x = [x_1, x_2]^T$ and u are the system state and the control input respectively, and $\theta \in R^p$ is the system parameter vector which is unknown but constant. Here, some assumptions are made for the system (1).

Assumption 1: Functions $f(x, \theta)$ and $g(x, \theta)$ are linear in the system parameter vector $\theta \in R^p$ and have the following forms

$$\begin{aligned}f(x, \theta) &= f_1(x)\theta \\ g(x, \theta) &= g_1(x)\theta\end{aligned}$$

where $f_1(x)$ and $g_1(x)$ are known nonlinear functions of the system state x .

Function $g(x, \theta) \neq 0$ for all time.

Assumption 2: Nonlinear functions $f_1(x)$, $g_1(x)$ are continuous, the partial derivatives, $\frac{\partial f_1(x)}{\partial x_i}$, $\frac{\partial g_1(x)}{\partial x_i}$ exist for $i=1, 2$, and are bounded for x .

* Received 12th November, 1998

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3. Adaptive Sliding Controller

Following Assumption 1, the system (1) can be represented as follows

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1(x)\theta + g_1(x)\theta u \\ &= \phi(x, u)\theta\end{aligned}$$

where $\phi(x, u) = f_1(x) + g_1(x)u \in R^{1 \times p}$. In this work, The control objective is to design an adaptive sliding control scheme such that the system state x_1 can track the desired trajectory x_{m1} described by

$$\begin{aligned}\dot{x}_{m1} &= x_{m2} \\ \dot{x}_{m2} &= -a_m x_{m2} + b_m r\end{aligned}\quad (2)$$

where $a_m > 0$, $b_m > 0$, and r is the reference input which is a piecewise continuous and bounded function. To derive a sliding controller for the system (1) such that x_1 can track x_{m1} , a suitable sliding surface $S(e) = 0$ must be constructed firstly and then the controller is derived to satisfy the sliding condition, i.e.

$$S(e)\dot{S}(e) < 0$$

When the system parameter vector θ is known and the sliding surface is set as

$$S(e) = c_1 e_1 + e_2 \quad (3)$$

where $e_1 = x_1 - x_{m1}$, $e_2 = x_2 - x_{m2}$, $e = [e_1, e_2]^T$, and $c_1 > 0$, the system variable structure control can be derived directly as follows

$$\begin{aligned}u &= [-c_1 e_2 - f(x, \theta) - a_m x_{m2} + b_m r \\ &\quad - \rho \operatorname{sgn}(S)]/g(x, \theta) \\ &\quad ; \rho > 0\end{aligned}$$

In real system, the parameter vector is usually unknown and replaced by a so-called nominal parameter vector $\bar{\theta}$. Under the following bond conditions concerned with the system parameter,

$$\begin{aligned}|\theta_i - \bar{\theta}_i| &\leq \hat{\theta}_i, \quad i = 1, 2, \dots, p. \\ \hat{\theta} &= [\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_p]^T\end{aligned}$$

the control input is changed to be

$$\begin{aligned}u &= [-c_1 e_2 - f(x, \bar{\theta}) - a_m x_{m2} + b_m r \\ &\quad - \rho_1(x, x_{m2}, r, e_2, \bar{\theta}) \operatorname{sgn}(S)]/g(x, \bar{\theta}); \\ g(x, \bar{\theta}) &\neq 0\end{aligned}$$

where $\rho_1(x, x_{m2}, r, e_2, \bar{\theta})$ is a positive function which is obtained according to the bounds of uncertain system parameters such that the sliding condition can be ensured. To overcome the uncertainty of the system parameter vector, we usually set a larger $\rho_1(x, x_{m2}, r, e_2, \bar{\theta})$ value in short of information about system parameters. Unfortunately, there will be a larger oscillation in the control input when the sliding mode occurs. The control input chattering is quite unsuitable in the practical system control, because it may damage the system mechanism or excite the high frequency unmodelled dynamics so that it resulted in the system unstable. In this work, we will use adaptive approach to update system parameters recursively. It does not need the bounds of the system

parameters to design a sliding controller. Firstly, the sliding surface in Eq. (3) is still chosen to use and an adaptive law is proposed to estimate the system parameter vector as follows.

$$\dot{\hat{\theta}}^T = -\phi(x, u)\Gamma(S_0 e_2 + S) \quad (4)$$

where $S_0 = e^T e$, $\Gamma \in R^{p \times p}$ is a symmetric, positive definite gain matrix, and $\hat{\theta} = \theta - \bar{\theta}$. Here, $\hat{\theta} \in R^p$ is the estimated parameter vector of the system parameter vector θ . A variable structure control is then given in the following form

$$\begin{aligned}u &= [-c_1 e_2 - f(x, \hat{\theta}) - a_m x_{m2} + b_m r - \delta_1(e) \operatorname{sgn}(S) \\ &\quad - \delta_2(t)S]/g(x, \hat{\theta})\end{aligned}\quad (5)$$

where $g(x, \hat{\theta})$ is assumed to be bounded away from zero, $\delta_2(t) = \frac{k_3}{|S| + k_4}$ is a positive function with adjustable positive constants k_3 and k_4 , and $\delta_1(e)$ is a positive function determined by

$$\delta_1(e) = \begin{cases} k_1 |S_1| - \delta_2 |S|, & e_2 S < 0 \\ k_2 |S|, & e_2 S \geq 0 \end{cases} \quad (6)$$

where $0 < k_1 < 1$, $1 \leq k_2$, and $S_1 = c_1 e_2 - e_1$.

Remark: In $\delta_2(t)$, k_4 value is chosen sufficiently small. In conventional sliding mode control, k_4 value is set to be zero such that this setting results in the control chattering phenomena. Hence, k_4 value can not be chosen to be equal to zero in our work. To obtain a better output tracking ability, k_1 and k_2 values in Eq. (6) are set to be larger values usually. Unfortunately, too large k_1 and k_2 values may result in a too large control input. Hence, it make a compromise between the output tracking ability and control input value to choose more suitable k_1 and k_2 values. A large k_3 value is usually chosen. When k_3 value is set to be a larger value, it implies that the sliding condition $S(e)\dot{S}(e) < 0$ can be maintained in a shorter time (refer to Theorem 2). Similarly with k_1 and k_2 values, a large k_3 value will result in a large control input in the transient time. Without any restrictions, it is suggested that k_1 , k_2 , and k_3 values are set to be large values.

Definition 1^{(12),(13)}:

For any fixed $p \in [1, \infty)$, $f: R^+ \rightarrow R$ is said to belong to L^p if, and only if, f is locally integrable and

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p dt \right)^{1/p} < \infty$$

When $p = \infty$, $f \in L^\infty$ if, and only if,

$$\|f\|_\infty = \sup_{t \geq 0} |f(t)| < \infty$$

Lemma 1^{(12),(13)}:

If $f(t)$ is a uniformly continuous function, such that $\lim_{t \rightarrow \infty} \int_0^t f(\tau) d\tau$ exists and is finite,

Then $f(t) \rightarrow 0$ as $t \rightarrow \infty$.

Corollary 1^{(12),(13)}:

If $g(t)$, $\dot{g}(t) \in L^\infty$, and $g(t) \in L^p$, for some $p \in [1, \infty)$,

Then $g(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof: Direct from lemma, with $f = |g|^p$, since $g(t)$, $\dot{g}(t)$ bounded implies that $f(t)$ is uniformly continuous.

The property of the proposed adaptive sliding controller is presented in the following theorem.

Theorem 1: Consider the system (1) subjected to Assumption 1, the reference model (2), and the sliding controller given by (3), (4), (5), and (6). If suitable positive functions, $\delta_1(e)$, $\delta_2(t)$ are chosen, it is ensured that

(i) $S_0(e)$, $S(e)$, and $\hat{\theta}(t)$ are uniformly bounded with respect to t ;

(ii) Function $S(e)$ will approach to zero as time tends to infinity, i.e.

$$\lim_{t \rightarrow \infty} S(e) = 0$$

Proof: Choose a Lyapunov function candidate of the form

$$V(S_0, S, \hat{\theta}) = \frac{1}{2} \left(SS + \frac{1}{2} S_0 S_0 + \hat{\theta}^T \Gamma^{-1} \hat{\theta} \right)$$

It follows that the time derivative of function $V(S_0, S, \hat{\theta})$ is

$$\begin{aligned} \dot{V}(S_0, S, \hat{\theta}) &= S\dot{S} + \frac{1}{2} S_0 \dot{S}_0 + \hat{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \\ &= S[\phi(x, u)\hat{\theta} - \delta_1 \operatorname{sgn}(S) - \delta_2 S] + S_0\{e_1 e_2 + e_2 \\ &\quad \times [\phi(x, u)\hat{\theta} - c_1 e_2 - \delta_1 \operatorname{sgn}(S) - \delta_2 S]\} + \hat{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \\ &= \{[S + S_0 e_2]\phi(x, u) + \hat{\theta}^T \Gamma^{-1}\} \hat{\theta} + S_0\{e_2[e_1 - c_1 e_2 \\ &\quad - \delta_1 \operatorname{sgn}(S) - \delta_2 S]\} - S\delta_1 \operatorname{sgn}(S) - S\delta_2 S \end{aligned}$$

From Eq. (4), it is obtained that

$$\begin{aligned} \dot{V}(S_0, S, \hat{\theta}) &= S_0\{e_2[-S_1 - \delta_1 \operatorname{sgn}(S) - \delta_2 S]\} \\ &\quad - S\delta_1 \operatorname{sgn}(S) - S\delta_2 S \end{aligned}$$

When the positive function δ_1 is chosen from Eq. (6), it is ensured that

$$\begin{aligned} \dot{V}(S_0, S, \hat{\theta}) &\leq -S\delta_1 \operatorname{sgn}(S) - S\delta_2 S \\ &= -\delta_1 |S| - \delta_2 S^2 \leq 0 \end{aligned}$$

The above inequality indicates that function $V(S_0, S, \hat{\theta})$ is a nonincreasing function and then it follows that $V(S_0, S, \hat{\theta})$, S_0 , S , and $\hat{\theta}$ are uniformly bounded with respect to t , i.e. $S_0, S, \hat{\theta} \in L^\infty$. The above inequality also means

$$\delta_1 |S| + \delta_2 S^2 \leq -\dot{V}(S_0, S, \hat{\theta})$$

We then have

$$\int_0^\infty \delta_1 |S| dt + \int_0^\infty \delta_2 S^2 dt \leq V(0) - V(\infty) < \infty$$

or

$$\int_0^\infty \delta_2 S^2 dt < \infty \quad (7)$$

The above inequality (7) represents $S \in L^2$. From the definition of the sliding surface $S(e)$ in Eq. (3), its time derivative can be obtained in the form of

$$\dot{S} = \phi(x, u)\hat{\theta} - \delta_1 \operatorname{sgn}(S) - \delta_2 S$$

Since S_0 , S , and $\hat{\theta}$ are uniformly bounded, it implies that e_1 , e_2 , and $\hat{\theta}$ are also uniformly bounded. The fact that the reference model (2) is set to be a stable

system certainly means that the system state x is uniformly bounded. It follows that u , $\delta_1(e)$, and $\phi(x, u)$ are also uniformly bounded. It is concluded that $\dot{S}$ is uniformly bounded, i.e. $\dot{S} \in L^\infty$. Since $S, \dot{S} \in L^\infty$ and $S \in L^2$, according to Corollary 1, it is ensured that

$$\lim_{t \rightarrow \infty} S(e) = 0$$

From the definition of sliding surface $S = c_1 e_1 + e_2 = c_1 e_1 + \dot{e}_1$, it follows directly that $\lim_{t \rightarrow \infty} e_1 = 0$ and $\lim_{t \rightarrow \infty} e_2 = 0$.

We further to show that the sliding condition can be achieved in the following theorem.

Theorem 2: In the proposed adaptive sliding control scheme, the positive function $\delta_2(t)$ and the positive function $\delta_1(e)$ determined from Eq. (6) are chosen, then there exists t_1 such that for all $t \geq t_1$, the sliding condition

$$S(e)\dot{S}(e) < 0$$

is maintained.

Proof: The time derivative of function $S(e)$ is

$$\dot{S} = \phi(x, u)\hat{\theta} - \delta_1 \operatorname{sgn}(S) - \delta_2 S$$

Since $\lim_{t \rightarrow \infty} S = 0$ and $\delta_2 > 0$, it follows that

$$\lim_{t \rightarrow \infty} [\phi(x, u)\hat{\theta} - \delta_1 \operatorname{sgn}(S)] = 0$$

From Assumption 2 and the results of Theorem 1, it is concluded that

$$\frac{d}{dt} [\phi(x, u)\hat{\theta} - \delta_1 \operatorname{sgn}(S)] \leq k_6, \quad k_6 > 0$$

Hence, there exists t_1 such that for all $t \geq t_1$

$$|\phi(x, u)\hat{\theta} - \delta_1 \operatorname{sgn}(S)| < \frac{k_3 |S|}{|S| + k_4} < k_3$$

Then we have, for $t > t_1$

$$\begin{aligned} S\dot{S} &= S[\phi(x, u)\hat{\theta} - \delta_1 \operatorname{sgn}(S) - \delta_2 S] \\ &\leq S\phi(x, u)\hat{\theta} - \delta_1 |S| - \delta_2 S^2 \\ &= S\phi(x, u)\hat{\theta} - \delta_1 |S| - \delta_2 |S|^2 \\ &\leq |\phi(x, u)\hat{\theta} - \delta_1| |S| - \frac{k_3}{|S| + k_4} |S|^2 \\ &= -\left[\frac{k_3 |S|}{|S| + k_4} - |\phi(x, u)\hat{\theta} - \delta_1| \right] |S| < 0 \end{aligned}$$

In this work, it has stated that the proposed adaptive sliding controller can be implemented without the requirement that the bounds of uncertain system parameters should be known in advance. Hence, we can avoid the problem: to overcome system parameter uncertainties, more conservative switching gain values are set such that this setting usually results in a larger control chattering. Moreover, in our variable structure control (5), the switching gain $\delta_1(e)$ is function of error vector. When the sliding mode exists, the switching gain $\delta_1(e)$ will be decreased. Therefore, the control chattering drawback will be alleviated in the proposed adaptive sliding mode control system.

4. Simulation Examples

Simulation 1: Consider the system

$$\dot{x} = x_2$$

$$\dot{x}_2 = -ax_2^2 \cos(3x_1) + [b + c \sin(x_1)]u$$

where $a=1$, $b=6$, $c=2$, and the reference model

$$\dot{x}_{m1} = x_{m2}$$

$$\dot{x}_{m2} = -40x_{m1} - 12.5x_{m2} + 50r$$

where reference input r is a square wave with amplitude=1, period= 4π . In simulation, all constant values are set as follows:

$$c_1=2, \quad k_1=0.99, \quad k_2=3, \quad k_3=100, \quad k_4=0.01,$$

$$\text{and } \Gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, simulation is started with initial values $x(0)=[1, 1]^T$, $x_m(0)=[0, 0]^T$, and $\hat{\theta}(0)=[0.5, 4, 1]^T$. All the simulation results are shown in Fig. 1.

Simulation 2: Consider a nonlinear system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -ax_2^2 \cos(3x_1) + [b + c \sin(x_1)]u$$

(a) Phase trajectory in the (e_1, e_2) plane

(b) Time responses of x_1 and x_{m1}

The real values of the unknown system parameters a , b , and c are

$$a=1+|\sin(t)|, \quad b=6+\cos(2t), \quad c=2+sw(0.2t)$$

where $sw(0.2t)$ represents a square wave with period= 10π , and the desired state trajectory x_{m1} is given by

$$x_{m1} = \sin(\pi t/2)$$

In this simulation, the sliding surface is chosen as

$$S=2e_1+e_2$$

We let (1) $k_1=0.99$, $k_2=5$ and (2) $k_1=0.01$, $k_2=1.01$. The other values are chosen as $k_3=5$, $k_4=0.01$, $x(0)=[1, 0]^T$, $\hat{\theta}(0)=[0.5, 4, 1]^T$, $\Gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. From Fig. 2, it

is indicated that the proposed adaptive sliding control scheme is still stable to treat with time varying system. In the meantime, it is found that the proposed adaptive sliding mode control system has a better output tracking ability when large k_1 and k_2 values are chosen but it results in a larger control input.

Simulation 3: In this simulation, it is to show the

(c) Control input u

(d) Estimated system parameters $\hat{a}$, $\hat{b}$, and $\hat{c}$

Fig. 1 Simulation results of simulation 1

effects of the different k_3, k_4 value choosing. The system is given as follow:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -ax_2^2 \cos(x_1) + u$$

where $a=5$. The reference model is the same as that chosen in Simulation 2. For simulation, the following

values are given

$$k_1=0.99, \quad k_2=1.01, \quad k_4=0.01, \quad \Gamma=4,$$

$$\text{and } c_1=20.$$

In addition, the following initial values are chosen

$$x(0)=[1, 0]^T \text{ and } \hat{a}(0)=2$$

(a) Phase trajectory in the (e_1, e_2) plane

(b) Time responses of x_1 and x_{m1}

(c) Control input u

(d) Control input u

(e) Estimated system parameters $\hat{a}, \hat{b}$, and $\hat{c}$

(f) Estimated system parameters $\hat{a}, \hat{b}$, and $\hat{c}$

Fig. 2 Simulation results of simulation 2

Fig. 3 Simulation results of simulation 3. State error e_1 under different k_3 values

Figure 3 shows the state error $e_1(t)$ in the different k_3 value choosing. From simulation results, it is indicated that we have a better state error response when k_3 value is set to be large.

5. Conclusion

This paper presents a method to design an adaptive sliding controller for the second order nonlinear system with system parameter uncertainties. Unlike the traditional sliding mode control approach, the control input chattering can be alleviated in our adaptive sliding control scheme. To design the proposed control scheme, the requirement of the bond information of the uncertain system parameter is not needed. In this paper, the stability property of the proposed control scheme is analyzed. Finally, some simulation results are presented to show the performance of the adaptive sliding control scheme. From simulation results, it is illustrated that all signals are bounded, the system state x_1 can track the desired state x_{m1} asymptotically, and the sliding surface can be reached and maintained in finite time.

References

- (1) Utkin, V.I., Variable Structure Systems with Slid-

- ing Modes, IEEE Trans. Automat. Contr., Vol. 22, No. 2 (1977), p. 212-222.
- (2) Young, K.-K.D., Design of Variable Structure Model Following Control Systems, IEEE Trans. Automat. Contr., Vol. 23, No. 6 (1978), p. 1079-1085.
- (3) Baida, S.V., Unit Sliding Mode Control in Continuous and Discrete Time Systems. Int. J. Control, Vol. 57, No. 5 (1993), p. 1125-1132.
- (4) Wang, J.D., Lee, T.L. and Juang, Y.T., New Methods to Design an Integral Variable Structure Controller. IEEE Trans. Automat. Contr., Vol. 41, No. 1 (1996), p. 140-143.
- (5) Hsu, K.C., Variable Structure Control Design for Uncertain Dynamic Systems with Sector Nonlinearities. Automatica, Vol. 34, No. 4 (1998), p. 505-508.
- (6) Slotine, J.-J.E. and Coetsee, J.A., Adaptive Sliding Controller Synthesis for Nonlinear Systems. Int. J. Control, Vol. 43, No. 6 (1986), p. 1631-1651.
- (7) Chang, F., Twu, S. and Chang, S., Adaptive Chattering Alleviation of Variable Structure System Control, IEE Proc. D, Vol. 137, No. 1 (1990), p. 31-39.
- (8) Sira-ramirez, H. and Zribi, M., Adaptive Dynamical Variable Structure Feedback Regulation for Nonlinear Uncertain Systems. Int. Journal of Robust and Nonlinear Control, Vol. 3, No. 2 (1993), p. 183-195.
- (9) Shyu, K.K. and Yan, J.J., Variable Structure Model Following Adaptive Control for Systems with Time Varying Delay, Control Theory and Advanced Technology, Vol. 10, No. 3 (1994), p. 513-521.
- (10) Feng, C.B. and Wu, Y.Q., A Design Scheme of Variable Structure Adaptive Control for Uncertain Dynamic Systems. Automatica, Vol. 32, No. 4 (1996), p. 561-567.
- (11) Lee, T.H., Xu, J.X. and Wang, M., A Model-based Adaptive Sliding Controller, Int. J. of Systems Science, Vol. 27, No. 1 (1996), p. 129-140.
- (12) Narendra, K.S. and Annaswamy, A.M., Stable Adaptive Systems, (1989), Prentice-Hall, Englewood Cliffs, NJ.
- (13) Sastry, S. and Bodson, M., Adaptive Control: Stability, Convergence, and Robustness. (1989), Prentice-Hall, Englewood Cliffs, NJ.