Stability Robustness Analysis of Linear Time-Variant Systems with Both Time-Varying Structured and Unstructured Perturbations*

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The stability robustness of linear time-variant systems with both time-varying structured and unstructured perturbations is investigated by using the Lyapunov approach and some essential properties of matrix measures. Some new sufficient conditions are proposed for ensuring the linear time-variant systems with both time-varying structured and unstructured perturbations to be uniformly asymptotically stable. For the case that the linear time-variant system only subjects to time-varying structured perturbations, by mathematical analysis, the presented sufficient conditions are proved to be less conservative than the existing ones reported in the literature. The result only for structured independent time-varying perturbations case is also extended to the stability analysis of time-variant interval systems.

Key Words: Time-Variant Systems, Stability Robustness, Time-Varying Uncertainties, Structured and Unstructured Perturbations

1. Introduction

Stability is a very important and necessary requirement for any workable system. The robust stability analysis for linear time-invariant systems with time-varying structured/unstructured perturbations has attracted much attention in the literature (see, for example, Zhou and Khargonekar, 1987; Sezer and Siljak, 1989; Weinmann, 1991; Gao and Antsaklis, 1993; Chou, 1994; Luo et al., 1995; Chen and Yang, 1997; Schreier et al., 1998; and references therein). On the other hand, the robust stability problem of linear time-variant systems with time-varying structured/unstructured perturbations has

also been investigated by Xi (1994) by using the Lyapunov approach. To the authors' best knowledge, only the article of Xi (1994) investigated the robust stability problem of linear time-variant systems with time-varying structured/unstructured perturbations. That is, the research on the stability robustness of linear time-variant systems with time-varying structured/unstructured perturbations is considerably rare and almost embryonic.

It is well known that an approximate system model is always used in practice and sometimes the approximation error should be covered by introducing both structured and unstructured perturbations in control system analysis and design. Therefore, the purpose of this paper is to propose another different approach for investigating the stability robustness problem of linear time-variant systems with both time-varying structured and unstructured perturbations by using the Lyapunov approach and some essential properties of matrix measures. The main results are presented in Section 2. For the case that the linear time-variant system only subjects to time-varying structured perturbations, by mathematical analysis, some conservatism comparisons between the

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criteria, which are, respectively, proposed in this paper and by Xi (1994), are also given in this section. In Section 3, the result only for structured independent time-varying perturbations case is extended to the stability analysis of time-variant interval systems. Three illustrated examples are given in Section 4. Finally, Section 5 offers some conclusions.

2. Stability Robustness of Linear Time-Variant Uncertain Systems

Consider a linear time-variant uncertain system described by

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + E(t)x(t) + F(t)x(t), \\ x(0) &= x_0,\end{aligned}\quad (1)$$

where $x(t)$ is the $n \times 1$ state vector, and $A(t) \in R^{n \times n}$ is the u.a.s. (uniformly asymptotically stable) time-variant matrix and its elements are formed by piecewise-continuous and uniformly bounded functions defined on $[0, +\infty)$; $E(t) \in R^{n \times n}$ denotes time-varying structured perturbation matrix. The time-varying unstructured perturbation matrix $F(t)$ is assumed to be uniformly bounded, i.e.,

$$\bar{\sigma}_{\max}(F(t)) \leq \alpha, \quad (2)$$

for every $t \geq 0$, where $\bar{\sigma}_{\max}(F(t)) = \sup_{t \geq 0} \sigma_{\max}(F(t))$, $\sigma_{\max}(F(t))$ denotes the maximum singular value of $F(t)$, and α is a real non-negative constant.

Before we analyze the stability robustness of the linear time-variant uncertain system (1), the following lemmas need to be introduced first.

Lemma 1 (Kalman and Bertram, 1960; Yushizawa, 1996):

If $A(t)$ is a uniformly bounded matrix, the following statements are equivalent:

- (i) $A(t)$ is u.a.s.;
- (ii) There exists a continuous and differentiable Lyapunov function $V(x(t), t) = x^T(t)P(t)x(t) > 0$, which has the following properties: $a_1\|x(t)\|^2 \leq V(x(t), t) \leq a_2\|x(t)\|^2$ and $\dot{V}(x(t), t) \leq -b\|x(t)\|^2$, where $a_1 > 0$, $a_2 > 0$, $b > 0$, and $\|\bullet\|$ denotes the vector norm on R^n ;
- (iii) For any uniformly bounded and uniformly positive definite symmetric matrix $Q(t)$, the matrix differential equation

$$-\frac{dP(t)}{dt} = P(t)A(t) + A^T(t)P(t) + 2Q(t) \quad (3)$$

gives the uniformly bounded and uniformly positive definite symmetric solution $P(t)$.

Lemma 2 (Desoer and Vidyasagar, 1975; Vidyasagar, 1978; Fang and Kincaid, 1996):

The matrix measures of the piecewise-continuous matrices $W(t)$ and $Y(t)$, $\mu(W(t))$ and $\mu(Y(t))$, are well defined for any norm and have the following

properties:

- (i) $-\|W(t)\| \leq -\mu(-W(t)) \leq \text{Re}(\lambda(W(t))) \leq \mu(W(t)) \leq \|W(t)\|$, for every $t \geq 0$, any matrix norm $\|\bullet\|$ and any matrix $W(t) \in R^{n \times n}$;
- (ii) $\mu(W(t) + Y(t)) \leq \mu(W(t)) + \mu(Y(t))$, for every $t \geq 0$ and any two matrices $W(t), Y(t) \in R^{n \times n}$;
- (iii) $\mu(\gamma W(t)) = \gamma \mu(W(t))$, for every $t \geq 0$, any matrix $W(t) \in R^{n \times n}$ and any non-negative number γ ;

where $\lambda(W(t))$ denotes any eigenvalue of $W(t)$, and $\text{Re}(\lambda(W(t)))$ denotes the real part of $\lambda(W(t))$.

Remark 1: From the property (iii) in Lemma 2, we can get that $\mu(\gamma W(t)) = -\gamma \mu(-W(t))$, for $\gamma \in R$ and $\gamma < 0$.

Lemma 3 (Chen and Yang, 1997):

For any two piecewise-continuous matrices $W(t), Y(t) \in R^{n \times n}$, and $|W(t)| \leq Y(t)$, the matrix measures of the matrices $W(t)$, $|W(t)|$ and $Y(t)$ are well defined and have the property:

$$\mu(W(t)) \leq \mu(|W(t)|) \leq \mu(Y(t)),$$

for every $t \geq 0$, where $|W(t)|$ denotes the modulus matrix of the matrix $W(t)$, i.e., $|W(t)| = (|W_{ij}(t)|)_{n \times n}$ which means to take absolute value for each element.

Proof: This lemma can be immediately obtained from the definition of matrix measure given by Desoer and Vidyasagar (1975). **Q.E.D.**

2.1 Structured independent perturbation case

Suppose that the time-varying structured perturbation matrix $E(t)$ is bounded by the inequality

$$|E(t)| \leq \varepsilon U, \quad (4)$$

where ε is the real non-negative uncertain parameter, and U is the given real non-negative constant matrix.

In what follows, we present a new sufficient condition for ensuring that the linear time-variant uncertain system (1) with both $|E(t)| \leq \varepsilon U$ and $\bar{\sigma}_{\max}(F(t)) \leq \alpha$ is still u.a.s..

Theorem 1:

The linear time-variant uncertain system (1) with both $|E(t)| \leq \varepsilon U$ and $\bar{\sigma}_{\max}(F(t)) \leq \alpha$ is u.a.s., if the following inequality holds

$$\varepsilon \cdot \bar{\mu}(U^T|P(t)| + |P(t)|U) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t)) < 2\sigma_{\min}(Q(t)), \quad (5)$$

for every $t \geq 0$, where $\bar{\mu}(U^T|P(t)| + |P(t)|U) = \sup_{t \geq 0} \mu(U^T|P(t)| + |P(t)|U)$, and $P(t)$ is the solution of the matrix differential equation (3), $Q(t)$ is any uniformly bounded and uniformly positive definite symmetric matrix, and $\sigma_{\min}(Q(t)) = \inf_{t \geq 0} \sigma_{\min}(Q(t))$, in which $\sigma_{\min}(Q(t))$ denotes the minimum singular value of $Q(t)$.

Proof:

Let $V(x(t), t) = x^T(t)P(t)x(t)$ be the Lyapunov

function of the linear time-variant uncertain system (1), then, from Lemma 1 and (2), we have

$$\begin{aligned}\dot{V}(x(t), t) &= x^T(t)[A^T(t)P(t) + E^T(t)P(t) \\ &\quad + \dot{P}(t) + P(t)A(t) + P(t)E(t) \\ &\quad + F^T(t)P(t) + P(t)F(t)]x(t) \\ &\leq x^T(t)[-2Q(t) + E^T(t)P(t) + P(t)E(t)]x(t) \\ &\quad + 2\|F(t)\|_2\|P(t)\|_2\|x(t)\|_2^2 \\ &= x^T(t)[-2Q(t) + E^T(t)P(t) + P(t)E(t)]x(t) \\ &\quad + 2\sigma_{\max}(F(t))\sigma_{\max}(P(t))\|x(t)\|_2^2 \\ &\leq x^T(t)[-2Q(t) + E^T(t)P(t) + P(t)E(t)]x(t) \\ &\quad + 2\bar{\sigma}_{\max}(F(t))\bar{\sigma}_{\max}(P(t))\|x(t)\|_2^2 \\ &\leq x^T(t)[-2Q(t) + E^T(t)P(t) + P(t)E(t) \\ &\quad + 2\alpha\bar{\sigma}_{\max}(P(t))I]x(t),\end{aligned}\quad (6)$$

where $\|F(t)\|_2$ and $\|P(t)\|_2$ denote the spectral norms of $F(t)$ and $P(t)$, $\|x(t)\|_2$ denotes the 2-norm of $x(t)$, and I is the identity matrix.

Since $E^T(t)P(t) + P(t)E(t) + 2\alpha\bar{\sigma}_{\max}(P(t))I$ is symmetric, the eigenvalues of $E^T(t)P(t) + P(t)E(t) + 2\alpha\bar{\sigma}_{\max}(P(t))I$ are all real. That is,

$$\begin{aligned}Re(\lambda(E^T(t)P(t) + P(t)E(t) + 2\alpha\bar{\sigma}_{\max}(P(t))I)) \\ = \lambda(E^T(t)P(t) + P(t)E(t) + 2\alpha\bar{\sigma}_{\max}(P(t))I).\end{aligned}$$

Thus, using the properties in Lemmas 2 and 3, and from (4) and (5), we can get

$$\begin{aligned}\lambda_{\max}(E^T(t)P(t) + P(t)E(t) + 2\alpha\bar{\sigma}_{\max}(P(t))I) \\ \leq \mu(E^T(t)P(t) + P(t)E(t) + 2\alpha\bar{\sigma}_{\max}(P(t))I) \\ \leq \mu(E^T(t)P(t) + P(t)E(t)) \\ \quad + \mu(2\alpha\bar{\sigma}_{\max}(P(t))I) \\ = \mu(E^T(t)P(t) + P(t)E(t)) \\ \quad + 2\alpha\bar{\sigma}_{\max}(P(t))\mu(I) \\ \leq \mu(|E^T(t)P(t)| + |P(t)E(t)|) \\ \quad + 2\alpha\bar{\sigma}_{\max}(P(t))\mu(I) \\ \leq \mu(\varepsilon U^T|P(t)| + \varepsilon|P(t)|U) \\ \quad + 2\alpha\bar{\sigma}_{\max}(P(t))\mu(I) \\ = \varepsilon\mu(U^T|P(t)| + |P(t)|U) + 2\alpha\bar{\sigma}_{\max}(P(t)) \\ \leq \varepsilon\bar{\mu}(U^T|P(t)| + |P(t)|U) + 2\alpha\bar{\sigma}_{\max}(P(t)) \\ < 2\sigma_{\min}(Q(t)) \\ \leq 2\lambda_{\min}(Q(t)),\end{aligned}\quad (7)$$

for every $t \geq 0$, where $\lambda_{\min}(\bullet)$ denotes the minimum eigenvalue. Now, Eq.(7) implies that

$$\begin{aligned}\lambda_{\max}(E^T(t)P(t) + P(t)E(t) \\ + 2\alpha\bar{\sigma}_{\max}(P(t))I - 2Q(t)) < 0,\end{aligned}\quad (8)$$

for every $t \geq 0$. This implies that, for every $t \geq 0$,

$$E^T(t)P(t) + P(t)E(t) + 2\alpha\bar{\sigma}_{\max}(P(t))I - 2Q(t)$$

is uniformly negative definite. That is, $\dot{V}(x(t), t) < 0$, uniformly. So, we have the stated results. **Q.E.D.**

Remark 2: Consider the linear time-invariant system with both time-varying structured and unstructured perturbations described by

$$\dot{x}(t) = Ax(t) + E(t)x(t) + F(t)x(t), \quad x(0) = x_0,$$

where A is an $n \times n$ real Hurwitz matrix. Then, the sufficient condition (5) becomes

$$\varepsilon\mu(U^T|P| + |P|U) + 2\alpha\sigma_{\max}(P) < 2\sigma_{\min}(Q),$$

in which the symmetric positive definite matrix P is the unique solution of the Lyapunov equation

$$A^TP + PA = -2Q$$

for some symmetric positive definite matrix Q . That is, the result for time-invariant uncertain system case may be viewed as a special case of the sufficient condition (5).

Remark 3: If the linear time-variant uncertain system (1) only subjects to time-varying unstructured perturbations, the linear time-variant uncertain system (1) becomes

$$\dot{x}(t) = A(t)x(t) + F(t)x(t). \quad (9)$$

Thus, the sufficient condition (5) becomes

$$\alpha\bar{\sigma}_{\max}(P(t)) < \sigma_{\min}(Q(t)), \quad (10)$$

which coincides with that proposed by Xi (1994). Therefore, the result given by Xi (1994) may be viewed as a special case of ours. On the other hand, if the linear time-variant uncertain system (1) only subjects to time-varying structured perturbations, the linear time-variant uncertain system (1) becomes

$$\dot{x}(t) = A(t)x(t) + E(t)x(t). \quad (11)$$

Thus, the sufficient condition (5) becomes

$$\varepsilon\bar{\mu}(U^T|P(t)| + |P(t)|U) < 2\sigma_{\min}(Q(t)). \quad (12)$$

Xi (1994) has shown that the linear time-variant uncertain system (11) with $|E(t)| \leq \varepsilon U$ is u.a.s., if the following inequality holds

$$\varepsilon\bar{\sigma}_{\max}(|P(t)|U) < \sigma_{\min}(Q(t)), \quad (13)$$

for every $t \geq 0$, where $\bar{\sigma}_{\max}(|P(t)|U) = \sup_{t \geq 0} \sigma_{\max}(|P(t)|U)$, and $\sigma_{\max}(|P(t)|U)$ denotes the maximum singular value of $|P(t)|U$. If we adopt the 2-norm-based matrix measure, the proposed sufficient condition (12) becomes

$$\varepsilon\bar{\mu}_2(|P(t)|U) < \sigma_{\min}(Q(t)), \quad (14)$$

where $\bar{\mu}_2(|P(t)|U) = \sup_{t \geq 0} \mu_2(|P(t)|U)$. Now we compare the proposed sufficient condition (14) with the sufficient condition (13) of Xi (1994). From the property (i) in Lemma 2, we have

$$\begin{aligned}\sup_{t \geq 0} \mu_2(|P(t)|U) &\leq \sup_{t \geq 0} \| |P(t)|U \|_2 \\ &= \sup_{t \geq 0} \sigma_{\max}(|P(t)|U),\end{aligned}\quad (15)$$

where $\|\bullet\|_2$ denotes the spectral norm. This mathematically proves that the sufficient condition (14) proposed in this paper is less conservative than the sufficient condition (13) proposed by Xi (1994).

2.2 Structured dependent perturbation case

In many interesting problems, we have only a small number of uncertain parameters, but those uncertain parameters may enter into many entries of the system matrix (Xi, 1994). In particular, we assume that the time-varying structured perturbation matrix $E(t)$ is of the form

$$E(t) = \sum_{i=1}^m k_i(t)E_i(t), \quad (16)$$

where $E_i(t)$ ($i=1, 2, \dots, m$) are the given constant or uniformly bounded certain matrices, and $k_i(t)$ ($i=1, 2,$

..., m) are time-varying real uncertain parameters.

In the following, we present two sufficient conditions for ensuring that the linear time-variant uncertain system (1) with both $E(t) = \sum_{i=1}^m k_i(t)E_i(t)$ and $\bar{\sigma}_{\max}(F(t)) \leq \alpha$ is still u.a.s..

Theorem 2:

The linear time-variant uncertain system (1) with both $E(t) = \sum_{i=1}^m k_i(t)E_i(t)$ and $\bar{\sigma}_{\max}(F(t)) \leq \alpha$ is u.a.s., if either of the following inequalities holds

$$(i) \quad \sum_{i=1}^m k_i(t) \bar{\phi}_i(t) + \alpha \cdot \bar{\sigma}_{\max}(P(t)) < \underline{\sigma}_{\min}(Q(t)); \quad (7)$$

$$(ii) \quad |k_j(t)| \cdot \bar{\mu} \left(\sum_{i=1}^m |P_i(t)| \right) + \alpha \cdot \bar{\sigma}_{\max}(P(t)) < \underline{\sigma}_{\min}(Q(t)), \text{ for } j=1, 2, \dots, m; \quad (18)$$

for every $t \geq 0$, where $\bar{\mu} \left(\sum_{i=1}^m |P_i(t)| \right) = \sup_{t \geq 0} \mu \left(\sum_{i=1}^m |P_i(t)| \right)$, $P_i(t) = (E_i^T(t)P(t) + P(t)E_i(t))/2$, and

$$\bar{\phi}_i(t) = \begin{cases} \sup_{t \geq 0} \mu(P_i(t)), & \text{for } k_i(t) \geq 0. \\ -\sup_{t \geq 0} \mu(-P_i(t)), & \text{for } k_i(t) < 0. \end{cases} \quad (19)$$

Proof:

Using the properties of Lemma 2 and Remark 1, and from (16) and (17), we can obtain

$$\begin{aligned} \lambda_{\max}(E^T(t)P(t) + P(t)E(t) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t))I) \\ \leq \mu(E^T(t)P(t) + P(t)E(t) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t))I) \\ = \mu \left(\sum_{i=1}^m k_i(t)E_i^T(t)P(t) + \sum_{i=1}^m k_i(t)P(t)E_i(t) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t))I \right) \\ = 2\mu \left(\sum_{i=1}^m k_i(t)P_i(t) + \alpha \cdot \bar{\sigma}_{\max}(P(t))I \right) \\ \leq 2 \sum_{i=1}^m \mu(k_i(t)P_i(t)) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t))\mu(I) \\ \leq 2 \sum_{i=1}^m k_i(t) \bar{\phi}_i(t) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t)) \\ < 2\underline{\sigma}_{\min}(Q(t)) \\ \leq 2\lambda_{\min}(Q(t)), \end{aligned} \quad (20)$$

for every $t \geq 0$. This implies that, for every $t \geq 0$, $E^T(t)P(t) + P(t)E(t) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t))I - 2Q(t)$ is uniformly negative definite. That is, $\dot{V}(x(t), t) < 0$ in (6), uniformly.

On the other hand, using the properties of Lemmas 2 and 3, and from (16) and (18), we have

$$\begin{aligned} \lambda_{\max}(E^T(t)P(t) + P(t)E(t) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t))I) \\ \leq 2\mu \left(\sum_{i=1}^m k_i(t)P_i(t) + \alpha \cdot \bar{\sigma}_{\max}(P(t))I \right) \\ \leq 2\mu \left(\sum_{i=1}^m |k_i(t)| \cdot |P_i(t)| \right) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t)) \\ \leq 2 \max_j |k_j(t)| \cdot \bar{\mu} \left(\sum_{i=1}^m |P_i(t)| \right) + 2\alpha \cdot \bar{\sigma}_{\max}(P(t)) \\ < 2\underline{\sigma}_{\min}(Q(t)) \\ \leq 2\lambda_{\min}(Q(t)), \end{aligned} \quad (21)$$

for $j=1, 2, \dots, m$ and every $t \geq 0$. This also implies that $\dot{V}(x(t), t) < 0$ in (6), uniformly. Thus, the proof

is completed.

Q.E.D.

Remark 4: Consider the linear time-variant uncertain system (11) with $E(t) = \sum_{i=1}^m k_i(t)E_i(t)$ in Remark 3. Then, the sufficient conditions (17) and (18) become

$$(i) \quad \sum_{i=1}^m k_i(t) \bar{\phi}_i(t) < \underline{\sigma}_{\min}(Q(t)); \quad (22)$$

$$(ii) \quad |k_j(t)| \cdot \bar{\mu} \left(\sum_{i=1}^m |P_i(t)| \right) < \underline{\sigma}_{\min}(Q(t)), \text{ for } j=1, 2, \dots, m. \quad (23)$$

Xi (1994) has also shown that the linear time-variant uncertain system (11) with $E(t) = \sum_{i=1}^m k_i(t)E_i(t)$ is u.a.s., if either of the following inequalities holds

$$(i) \quad \sum_{i=1}^m |k_i(t)| \cdot \bar{\sigma}_{\max}(P_i(t)) < \underline{\sigma}_{\min}(Q(t)); \quad (24)$$

$$(ii) \quad |k_j(t)| \cdot \bar{\sigma}_{\max} \left(\sum_{i=1}^m |P_i(t)| \right) < \underline{\sigma}_{\min}(Q(t)), \text{ for } j=1, 2, \dots, m; \quad (25)$$

where $\bar{\sigma}_{\max}(P_i(t)) = \sup_{t \geq 0} \sigma_{\max}(P_i(t))$ and $\bar{\sigma}_{\max} \left(\sum_{i=1}^m |P_i(t)| \right) = \sup_{t \geq 0} \sigma_{\max} \left(\sum_{i=1}^m |P_i(t)| \right)$. Now, we compare the proposed sufficient conditions (22) and (23) with the sufficient conditions (24) and (25) of Xi (1994). From the property (i) in Lemma 2 and adopting the 2-norm-based matrix measure, we can get

$$\begin{aligned} \sum_{i=1}^m k_i(t) \bar{\phi}_i(t) &\leq \sum_{i=1}^m |k_i(t)| \cdot \sup_{t \geq 0} \|P_i(t)\|_2 \\ &= \sum_{i=1}^m |k_i(t)| \cdot \bar{\sigma}_{\max}(P_i(t)), \end{aligned}$$

and

$$\bar{\mu} \left(\sum_{i=1}^m |P_i(t)| \right) \leq \sup_{t \geq 0} \left\| \sum_{i=1}^m |P_i(t)| \right\|_2 = \bar{\sigma}_{\max} \left(\sum_{i=1}^m |P_i(t)| \right).$$

Thus, these also mathematically prove that the sufficient conditions (22) and (23) presented in this paper are less conservative than the sufficient conditions (24) and (25) given by Xi (1994).

3. Stability Analysis of Time-Variant Interval Systems

Let $B(t)$ and $C(t)$ be piecewise continuous and uniformly bounded matrices defined on $[0, +\infty)$, and let an $n \times n$ time-variant interval matrix $A(t)$, denoted $A(t) = [B(t), C(t)]$, be a set of time-variant matrices defined by $B_{ij}(t) \leq A_{ij}(t) \leq C_{ij}(t)$.

Now, consider the time-variant interval system $\dot{x}(t) = [B(t), C(t)]x(t)$. (26)

From the sufficient condition (12) in Remark 3 and by assuming $(B(t) + C(t))/2$, $B(t)$ or $C(t)$ to be u.a.s., we can get the following corollaries to analyze the stability of the time-variant interval system (26).

Corollary 1:

The time-variant interval system (26) is u.a.s., if $A(t) = (B(t) + C(t))/2$ is u.a.s. and if

$$\varepsilon \cdot \bar{\mu}(U^T|P(t)|+|P(t)|U) < 2\sigma_{\min}(Q(t)), \quad (27)$$

for every $t \geq 0$, where $|E(t)| = (C(t) - B(t))/2 \leq \varepsilon U$, and $P(t)$ is the solution of the matrix differential equation (3).

Corollary 2:

The time-variant interval system (26) is u.a.s., if $A(t) = B(t)$ (or $A(t) = C(t)$) is u.a.s., and if

$$\varepsilon \cdot \bar{\mu}(U^T|P(t)|+|P(t)|U) < 2\sigma_{\min}(Q(t)), \quad (28)$$

for every $t \geq 0$, where $|E(t)| = C(t) - B(t) \leq \varepsilon U$, and $P(t)$ is the solution of the matrix differential equation (3).

4. Examples

In this section, three examples are given for illustration.

Example 1:

Consider the following output feedback system

$$\dot{x}(t) = [A(t) + B(t)K(t)C(t)]x(t), \quad (29)$$

in which

$$A(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 + \frac{1}{(1+t)} & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C(t) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

and

$$K(t) = \begin{bmatrix} -1 + k_1(t) & 0 \\ 0 & -1 + k_2(t) \end{bmatrix}.$$

In this example, we would like to find the upper and lower bounds on the uncertain functions $k_1(t)$ and $k_2(t)$ such that the system (29) remains u.a.s.. We have

$$A(t) + B(t)K(t)C(t) = A_0(t) + k_1(t)E_1(t) + k_2(t)E_2(t),$$

where

$$A_0(t) = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -3 + \frac{1}{(1+t)} & 0 \\ -1 & 0 & -4 \end{bmatrix},$$

$$E_1(t) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \text{ and } E_2(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solving the following matrix differential equation

$$-\frac{dP(t)}{dt} = P(t)A_0(t) + A_0^T(t)P(t) + 2I,$$

in which I is the identity matrix, we can obtain

$$P(t) = \begin{bmatrix} \frac{13}{27} & 0 & \frac{1}{27} \\ 0 & \frac{1}{3} + \frac{1}{9(t+1)} + \frac{1}{54(t+1)^2} & 0 \\ \frac{1}{27} & 0 & \frac{7}{27} \end{bmatrix}.$$

By applying the sufficient condition (24) of Xi (1994), we have that the system (29) remains u.a.s. if

$$0.5334|k_1(t)| + 0.4630|k_2(t)| < 1,$$

for every $t \geq 0$.

(30)

By using the proposed sufficient condition (22) with the 2-norm-based matrix measure, we can get that the system (29) remains u.a.s. if the following inequalities hold

$$0.5334k_1(t) + 0.4630k_2(t) < 1,$$

for $k_1(t) \geq 0$ and $k_2(t) \geq 0$;

$$0.5334k_1(t) < 1,$$

for $k_1(t) \geq 0$ and $k_2(t) < 0$;

$$-0.3112k_1(t) + 0.4630k_2(t) < 1,$$

(31)

for $k_1(t) < 0$ and $k_2(t) \geq 0$;

$$-0.3112k_1(t) < 1,$$

for $k_1(t) < 0$ and $k_2(t) < 0$;

for every $t \geq 0$. It is obvious that the bounds on $k_1(t)$ and $k_2(t)$ in (31) are better than those in (30). This illustrates that the proposed sufficient condition (22) can overcome the conservatism of the sufficient condition (24) given by Xi (1994). Note that this fact has been proved by mathematical analysis in Remark 4.

Example 2:

Consider a linear time-variant system with both time-varying structured and unstructured perturbations described by

$$\dot{x}(t) = A(t)x(t) + k_1(t)E_1(t)x(t) + k_2(t)E_2(t)x(t) + F(t)x(t) \quad (32)$$

with $\bar{\sigma}_{\max}(F(t)) \leq \alpha$, where the matrices $A(t) = A_0(t)$, $E_1(t)$ and $E_2(t)$ are the same as those given in Example 1.

In this example, we still adopt the 2-norm-based matrix measure and we still choose the same matrix $Q(t)$ as that given in Example 1 (i.e., $Q(t) = I$). Applying the sufficient condition (17), we know that the linear time-variant uncertain system (32) is u.a.s., if $k_1(t)$, $k_2(t)$ and α satisfy the following conditions

$$0.5334k_1(t) + 0.4630k_2(t) + 0.4875\alpha < 1,$$

for $k_1(t) \geq 0$ and $k_2(t) \geq 0$;

$$0.5334k_1(t) + 0.4875\alpha < 1,$$

for $k_1(t) \geq 0$ and $k_2(t) < 0$;

$$-0.3112k_1(t) + 0.4630k_2(t) + 0.4875\alpha < 1,$$

for $k_1(t) < 0$ and $k_2(t) \geq 0$;

$$-0.3112k_1(t) + 0.4875\alpha < 1,$$

for $k_1(t) < 0$ and $k_2(t) < 0$;

for every $t \geq 0$. If we use the sufficient condition (18), we know that, if the following conditions hold

$$0.5645|k_j(t)| + 0.4875\alpha < 1,$$

for $j=1$ and 2 , and for every $t \geq 0$,

the linear time-variant uncertain system (32) is u.a.s..

Example 3:

Consider the following linear time-variant system with both time-varying structured and unstructured perturbations

$$\dot{x}(t) = A(t)x(t) + E(t)x(t) + F(t)x(t) \quad (33)$$

with $|E(t)| \leq \varepsilon U$ and $\bar{\sigma}_{\max}(F(t)) \leq \alpha$, where $A(t) =$

$A_0(t)$ and

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

As Example 2, we still adopt the 2-norm-based matrix measure and we still choose $Q(t)=I$. Using the sufficient condition (5), we know that, if ε and α satisfy the following condition

$$1.2933\varepsilon + 0.4875\alpha < 1,$$

the linear time-variant uncertain system (33) is u.a.s..

5. Conclusions

Based on the Lyapunov approach and some essential properties of matrix measures, the stability robustness of linear time-variant systems with both time-varying structured and unstructured perturbations is investigated. Some new sufficient conditions are proposed for ensuring the linear time-variant systems with both time-varying structured and unstructured perturbations to be uniformly asymptotically stable. In Remarks 3 and 4, for the case that the linear time-variant system only subjects to time-varying structured perturbations, by mathematical analysis, the presented sufficient conditions are proved to be less conservative than those proposed by Xi (1994). The result only for structured independent time-varying perturbations case is also extended to the stability analysis of time-variant interval systems.

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