Simultaneous Optimal Design of Structure and Control Systems Based on Youla Parameterization*

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This paper deals with simultaneous optimal design of structure and control systems. For structures with collocated sensors and actuators, a stabilizing state-feedback gain and an estimator gain can be obtained easily. Using these two gain matrices, we can parameterize all stabilizing controllers for the structure by Youla parameterization scheme. Since stability of the closed-loop system is guaranteed, we can optimize design parameters in the structure and the controller simultaneously without taking two-step design procedures which consist of the step for controller synthesis and that for structural parameter tuning. Hence, convergence to the local optima is achieved with standard descent-based nonlinear optimization techniques (steepest descent, Newton method, etc.). By a design example, we show the effectiveness of proposed design method.

Key Words: Vibration Control, Flexible Structure, Optimal Control, Structure and Control Simultaneous Optimization, Youla Parameterization, Non-linear Optimization Technique

1. Introduction

In control system design, there exist several cases in which the control object has adjustable design parameters in itself (such design parameters are referred as “passive parameter”). In such cases, we synthesize a control system (controller and the control object itself) using the following two step procedure:

Step 1 (Structural design): Passive parameters (defined as p) are determined for satisfying structural design requirements, e.g. stress, weight, etc.

Step 2 (Controller design): A controller (defined as K) is synthesized for the mathematical model of the control object which has been fixed in Step 1. Design parameters of the controller (e.g. gain constants in PID controller, weighting matrices in the

quadratic functional in LQ problem, etc.) are adjusted for satisfying requirements on the closed-loop system.

The above control system design scheme has been widely used in practical applications since the design theory is well established and several CAD tools are available for each step of the design.

However, in Step 1 all passive parameters are obtained without utilizing the controller information, and in Step 2 the controller is synthesized without considering the adjustable nature of each passive parameter. In other words, the control object is not designed evaluating the “ease of control” quantitatively and the model-based controller can be obtained only after all passive parameters are fixed at their nominal values. Compared to the above design methodology, the simultaneous tuning of passive parameters and controller is clearly more effective and can provide better results because the freedom of the original problem is not lost. We refer to this design methodology as the “simultaneous optimal design of structure and control systems”. In this paper we deal with the simultaneous optimal design problem of structure and control systems.

A lot of design strategies have been proposed for

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the simultaneous optimal design problem. These design methods are summarized in survey papers published recently^{(1)–(3)}. In most of the proposed design methods the simultaneous optimal design problem is formulated as an optimization problem of an objective function representing a performance of the closed-loop system and the specification on the control object. These design problems can be classified into the following three classes by observing the characteristic of the objective function:

(a) Multi objective design problem optimizing the closed-loop performance and the structural specification^{(4)–(6)}.

(b) Single objective design problem optimizing the closed-loop performance^{(7),(8)}.

(c) Single objective design problem optimizing an objective function representing both requirements on the closed-loop system and the structural system^{(9),(10)}.

The problem (a) is the most straightforward formulation in the sense of the simultaneous optimal design. Several gradient based optimal design algorithms are proposed for the problem (a). However, the convergence of the objective function is not generally guaranteed even in a local sense. In the problem (b), the formulated problem is decomposed into several LMIs (Linear Matrix Inequalities) and these LMIs are solved sequentially. In this case, the convergence to a local optimal solution is guaranteed. However, this design method can not be applied when mass, damping and stiffness matrices in the equation of motion of the control object are not affine functions of each passive parameter. Generally, the assumption on the affine nature of each matrix is not satisfied and these matrices become nonlinear functions of passive parameters⁽¹¹⁾. In problems (a) and (b), the desired passive parameter or the design parameter in the controller is obtained sequentially by optimizing one parameter while keeping the others fixed. This design strategy is generally applied for solving the simultaneous optimal design problem, since established theories on structural and controller design can be utilized in the design process. We refer this design method as “*p-K iteration*”. The detail of the *p-K iteration* is given in later. When one applies *p-K iteration* to the simultaneous optimal design problem, the passive parameter and the parameter in the controller must be selected satisfying a condition to decrease (in the case of the minimization problem) the objective function. However, since we have no means of satisfying the condition at present, we can point out a case in which difficulty occurs in performing *p-K iteration* (The details of this case is presented later.).

In the problem (c), a linear combination of the

quadratic performance index in LQ problem and a quadratic cost function on passive parameter is proposed as the objective function to be minimized. The quadratic cost on passive parameters evaluates the deviation of each value of the passive parameter from its initial value. A simultaneous tuning algorithm of passive parameters and the feedback controller is presented, based on the necessary condition of the optimality of the objective function. The algorithm guarantees the convergence to a local optimal solution. However, since the physical meaning of the terms of the objective function on passive parameters is unclear, the optimal design method is difficult to apply when an explicit specifications, such as weight and stress limit, etc. are given in the structural design.

In this paper, we propose a locally convergent design method for the simultaneous optimal design problem which is effective even in the case where the problem (a) is posed, and/or the case where matrices in the equation of motion are not affine functions of passive parameters. We assume mechanical systems with collocated sensor and actuator as the control object. For such systems, we can easily obtain a stabilizing state feedback gain matrix and an estimator gain matrix. Then we can parameterize all stabilizing controllers with these two gain matrices and a proper and stable transfer function matrix (free parameter) based on Youla parameterization scheme. The stability of the closed-loop system is guaranteed if all design parameters (passive parameters, parameters in feedback gain matrices and the free parameter in Youla parameterization) belong to specified sets respectively (in *p-K iteration*, the stability of the closed-loop system can not be guaranteed even when this condition is satisfied). Therefore, we do not need to employ *p-K iteration* mentioned above. Moreover, the convergence to the local optimal solution can be achieved by applying standard descent based nonlinear optimization techniques since all design parameters can be adjusted simultaneously with the closed-loop stability.

The rest of this paper is organized as follows: The formulation of the simultaneous optimal design problem is given in section 2. The parameterization of the controller and the mathematical description of the simultaneous optimal design problem are presented in section 3. The difficulty of *p-K iteration* process and the effectiveness of the problem formulation in this paper are also stated in this section. In section 4, a design example of a simply supported beam system is presented. The conclusion of this paper is given in section 5.

Notations

- t, τ : time constant
 M^T : transposition of a matrix M
 s : Laplace operator
 R^N : N dimensional real vector space
 $M(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$: a realization of $M(s)$, i.e.
 $M(s) = D + C(sI - A)^{-1}B$
 I : identity matrix
 $\| \cdot \|$: norm of a vector or a matrix in some sense
 $F_l\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, N\right)$: lower linear fractional transformation, i.e.
 $F_l\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, N\right) = M_{11} + M_{12}(I - M_{22}N)^{-1}M_{21}$
 with $\det(I - M_{22}N) \neq 0$
 RH_∞ : class of rational, proper and asymptotically stable transfer function matrices
 $E(a(t))$: expectation operator of a signal $a(t)$
 $\delta(\cdot)$: Dirac's delta function

2. Problem Formulation

2.1 Control object

Consider an equation of motion of degrees of freedom mechanical system given by

$$M(p)\ddot{q}(t) + D(p)\dot{q}(t) + K(p)q(t) = L_n(p)n(t) + L_a(p)u(t), \quad (1)$$

where $q(t) \in R^n$, $n(t) \in R^{n_n}$ and $u(t) \in R^{n_u}$ are the displacement, disturbance and control input vectors respectively. Matrices, $M(p) \in R^{n \times n}$, $D(p) \in R^{n \times n}$ and $K(p) \in R^{n \times n}$ are the mass, damping and stiffness matrices. Matrices $L_n(p) \in R^{n \times n_n}$ and $L_a(p) \in R^{n \times n_u}$ are determined depending upon the disturbance or actuator location. A vector $p \in R^{N_p}$ is denoted as the passive parameter vector. Each element of p becomes an adjustable structural design parameter such as mass, damping, stiffness, sensor/actuator placement, etc. For physical limitations on passive parameters, e.g. mass, damping and stiffness must be positive, the passive parameter vector p belongs to the following set given by:

$$P_m = \{p \in R^{N_p} | \underline{p} \leq p \leq \bar{p}, \bar{p} - \underline{p} > 0_{N_p}, \bar{p} > 0_{N_p}\}, \quad (2)$$

where $\bar{p}$ and $\underline{p}$ are the upper and the lower bounds of the vector p respectively. In this paper the following assumption is made on matrices, $M(p)$, $D(p)$ and $K(p)$.

Assumption Matrices $M(p)$, $D(p)$ and $K(p)$ satisfies

$$M(p) = M^T(p) > 0, \quad D(p) = D^T(p) \geq 0, \quad K(p) = K^T(p) \geq 0, \quad \forall p \in P_m. \quad (3)$$

Furthermore, all elements of each matrix in Eq. (1) are bounded functions of $p \in P_m$ and are not necessarily linear ones of p .

Equation (3) is a general assumption for many vibrating mechanical systems. The latter assumption is made to avoid any element of each matrix from becoming $\pm \infty$.

Let us assume that some displacement and velocity sensors are installed at the same location and the same direction as those of actuators, i.e. sensor/actuator collocation is realized. Then the measured displacement and velocity become

$$y_d = L_a(p)^T q(t), \quad y_v = L_a(p)^T \dot{q}(t). \quad (4)$$

Note that matrices $[K(p) \ L_a(p)]$ and $[D(p) \ L_a(p)]$ have full-row rank to satisfy the controllability of the rigid body mode⁽¹²⁾. Let the state vector $x_p(t) = [q(t)^T \ \dot{q}(t)^T]^T$ and the output vector $v(t) \in R^{N_v}$, given by

$$v(t) = C_v x_p(t) + D_v u(t), \quad (5)$$

where $C_v \in R^{N_v \times 2n}$ and $D_v \in R^{N_v \times n_u}$. Letting $y_p(t) = [y_d(t)^T \ y_v(t)^T]^T$, the transfer function matrix $P(s)$ from $[n^T \ u^T]^T$ to $[v^T \ y_p^T]^T$ is represented by

$$P(s) = \begin{bmatrix} P_{vn}(s) & P_{vu}(s) \\ P_{yn}(s) & P_{yu}(s) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_v & 0 & D_v \\ C_2 & D_{21} & 0 \end{bmatrix}, \quad (6)$$

where:

$$A = \begin{bmatrix} 0 & I \\ -M(p)^{-1}K(p) & -M(p)^{-1}D(p) \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ M(p)^{-1}L_n(p) \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ M(p)^{-1}L_a(p) \end{bmatrix}, \quad C_2 = \begin{bmatrix} L_a(p)^T & 0 \\ 0 & L_a(p)^T \end{bmatrix}.$$

Note that $M(p)^{-1}$ always exists by assumption.

2.2 Problem formulation

Define the feedback controller as $u = K(s)y_p$. For the closed-loop system with $P(s)$ and $K(s)$, define the following objective function given by

$$J_c = \|W_v(s)G_{vn}(s)W_n(s)\|, \quad (7)$$

where transfer function matrices $W_v(s)$ and $W_n(s)$ are frequency weighting functions respectively. The transfer function matrix $G_{vn}(s)$ is the closed-loop transfer function matrix with $P(s)$ and $K(s)$, i.e. $G_{vn}(s) = F_l(P(s), K(s))$. Each weighting function is specified by the designer considering the spectrum of the disturbance, a trade-off between the robustness and performance of the closed-loop system, etc. The block diagram of the closed-loop system is depicted in Fig. 1.

As the structural design specification, we formulate a minimization problem of an objective function $J_s(p)$ depending only on the passive parameter vector p . As an example of $J_s(p)$, we can mention the total weight of the structural system. With the above definition, the simultaneous optimal design problem is stated as the following:

[Simultaneous optimal design problem of structural and control systems]

Find the optimal design variable vector $p^* \in P_m$ and

Fig. 1 Closed-loop system

the controller $K^*(s)$ which satisfy $J_c < \gamma_c$ and $J_s < \gamma_s$ ($\gamma_c, \gamma_s > 0$).

Note that this problem is exactly corresponding to the problem (a) in section 1.

3. Simultaneous Optimal Design with Youla Parameterization

3.1 Parameterization of the controller

Let E_1 and E_2 be arbitrary positive definite $n_u \times n_u$ dimensional matrices. Define a state feedback gain matrix F as the following:

$$F(p) = -[E_1 L_a(p)^T \quad E_2 L_a(p)^T]. \quad (8)$$

Then the matrix $A + B_2 F$ becomes

$$\begin{aligned} A + B_2 F &= \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}L_a \end{bmatrix} [E_1 L_a^T \quad E_2 L_a^T] \\ &= \begin{bmatrix} 0 & I \\ -M^{-1}(K + L_a E_1 L_a^T) & -M^{-1}(D + L_a E_2 L_a^T) \end{bmatrix}. \end{aligned} \quad (9)$$

Note that the state feedback control law $u(t) = Fx_p(t)$ is a DVDFB (direct velocity and displacement feedback) control for the system in Eq. (6). By assumption, it is easy to verify that $K + L_a E_1 L_a^T > 0$ and $D + L_a E_2 L_a^T > 0$ are satisfied for all $E_1, E_2 > 0$. Therefore, the stability of the matrix $A + B_2 F$ is guaranteed for all positive definite E_1 and E_2 (12).

As a dual notion of the above statement, consider a full-order estimator represented by

$$\begin{cases} \dot{\hat{x}}_p(t) = A\hat{x}_p(t) + B_2 u(t) + H(\hat{y}_p(t) - y_p(t)) \\ \hat{y}_p(t) = C_2 \hat{x}_p(t) \end{cases}, \quad (10)$$

where $\hat{x}_p(t)$ and $\hat{y}_p(t)$ are estimates of $x_p(t)$ and $y_p(t)$, respectively. Construct the estimator gain matrix H as

$$H(p) = - \begin{bmatrix} 0 & 0 \\ M(p)^{-1}L_a(p)E_3 & M(p)^{-1}L_a(p)E_4 \end{bmatrix}, \quad (11)$$

Where E_3 and E_4 are arbitrary positive definite $n_u \times n_u$ dimensional matrices. Then, the system matrix of the output injected system becomes:

$$\begin{aligned} A + HC_2 &= \begin{bmatrix} 0 & I \\ -M(p)^{-1}K(p) & -M(p)^{-1}D(p) \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & 0 \\ M(p)^{-1}L_a(p)E_3 & M(p)^{-1}L_a(p)E_4 \end{bmatrix} \\ &\quad \times \begin{bmatrix} L_a^T(p) & 0 \\ 0 & L_a^T(p) \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ -M^{-1}(K + L_a E_3 L_a^T) & -M^{-1}(D + L_a E_4 L_a^T) \end{bmatrix}. \end{aligned} \quad (12)$$

By the above fact on the stability of state feedback control system, the stability of the matrix $A + HC_2$ is also guaranteed for all $E_3, E_4 > 0$.

Two matrices F and H are stabilizing gains for the state feedback system $A + B_2 F$ and the output injected system $A + HC_2$ respectively. With these gain matrices, a doubly coprime factorization of $P_{yu}(s)$ and that of a stabilizing controller for $P(s)$ can be derived (13). By Youla parameterization scheme (14), (15), all stabilizing controllers for $P(s)$ are parameterized as

$$K(s) = F_t(K_0(p), Q(s)), \quad (13)$$

where the transfer function matrix $Q(s)$ is an element of the set RH_∞ and $K_0(s)$ is given as

$$K_0(s) = \left[\begin{array}{c|cc} A + B_2 F + HC_2 & -H & B_2 \\ \hline F & 0 & I \\ C_2 & -I & 0 \end{array} \right].$$

Then, the closed-loop system $G_{vn}(s)$ with $P(s)$ and $K(s)$ in Eq. (13) is obtained as

$$G_{vn}(s) = F_t(P(s), K(s)) = T_1(p) - T_2(p)Q(s)T_3(p) \quad (14)$$

$$\begin{aligned} T_1(p) &= \left[\begin{array}{cc|c} A + B_2 F & -B_2 F & B_1 \\ \hline 0 & A + HC_2 & B_1 + HD_{21} \\ C_v + D_v F & -D_v F & 0 \end{array} \right], \\ T_2(p) &= \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_v + D_v F & D_v \end{array} \right], \\ T_3(p) &= \left[\begin{array}{c|c} A + HC_2 & B_1 \\ \hline C_2 & D_{21} \end{array} \right]. \end{aligned}$$

Note that the well-posedness condition of the closed-loop system with $P(s)$ and $K(s)$, i.e. $\det(I + P_{yu}(\infty)K(\infty)) \neq 0$ is always satisfied since $P_{yu}(s)$ is strictly proper ($P_{yu}(\infty) = 0$) from Eq. (6). For the closed-loop system $G_{vn}(s)$ in Eq. (14), the following proposition can be stated:

Proposition The closed-loop system $G_{vn}(s)$ is stable for all $p \in P_m$, $E_1, E_2, E_3, E_4 > 0$ and $Q(s) \in RH_\infty$. Moreover, $G_{vn}(s)$ represents all stable closed-loop systems for control objects given by Eq. (1) under sensor/actuator collocation, since $K(s)$ in Eq. (13) gives the parameterization of all stabilizing controllers for $P(s)$.

3.2 Simultaneous optimal design problem with Youla parameterization

From the discussion in the previous subsection, the simultaneous optimal design problem with Youla parameterization can be formulated as the following:

[Simultaneous optimal design problem with Youla parameterization]

For $G_{vn}(s)$ in Eq. (14), find $p^* \in P_m$, $E_1^*, E_2^*, E_3^*, E_4^* > 0$ and $Q^*(s) \in RH_\infty$ which satisfy $J_c < \gamma_c$ and $J_s < \gamma_s$.

Recall that this problem corresponds to the problem (a) in section 1. For problem (a), the following iterative design procedure has been applied:

[p-K iteration]

Step 0: Set iteration number $i=0$. Select the passive parameter vector $p(i) \in P_m$ and fix the model of the control object.

Step 1: For the control object with $p(i)$, obtain the "optimal" or "suboptimal" controller $K(i)$.

Step 2: For the fixed controller $K(i)$, update the passive parameter vector $p(i)$ as $p_{new} \in P_m$ while preserving closed-loop stability. If the considered objective function converges, then stop. Otherwise, let $p(i) \leftarrow p_{new}$, $i \leftarrow i+1$, and go to step 2.

In p - K iteration, design variable vectors p and the controller are sequentially obtained in each independent stage. This fact means that the designer can utilize various commercially available software (optimal structural design with FEM based software or optimal controller design with CAD software on control system design) in p - K iteration. However, even if such software is available, it can be shown that p - K iteration may fail to obtain the optimal design variable vectors in the structure and the controller. Without loss of generality, assume that the simultaneous optimal design problem is formulated as a minimization of a single objective function J_a (a multi objective optimization problem can be transformed to a single objective one by taking weighted sum of each objective function). Define $J_a^1(i)$ and $J_a^2(i)$ as the value of J_a with the pair $(p(i), K(i))$ (step 1) and the pair $(p_{new}, K(i))$ (step 2) of the i -th p - K iteration respectively. Then, J_a converges to a local minimum with p - K iteration if and only if $p(i) \in P_m$ and $K(i)$ satisfying

$$J_a(i+1) \leq J_a(i), \quad (15)$$

$$J_a^2(i) \leq J_a^1(i), \quad (16)$$

can be found for all i . In general, the above two conditions can not be satisfied with p - K iteration since $p_{new} \neq p(i)$ satisfying Eq. (16) can not always be obtained in step 2. The existence of such $p_{new} \in P_m$ clearly depends on the selection of $K(i)$ in step 1, however the method for obtaining $G_m(s)$ satisfying Eq. (16) is not established for the present. Note that even if $K(i)$ is synthesized such that $G_{vn}(s)$ has the robust performance property for a set of plants around the "nominal value" $p(i) \in P_m$ (Such $K(i)$ can be obtained with μ -synthesis procedure⁽¹⁶⁾), Eq. (16) is not necessarily satisfied for all i since such $K(i)$ only optimizes the worst case of the closed-loop performance in the

hyper ellipsoid around the center $p(i)$. Therefore, in the worst case p - K iteration may terminate without updating $p(i=0)$, since $p_{new} \in P_m$ satisfying Eq. (16) can not be found around $p(i)$ for the fixed $K(i)$.

On the other hand, the stability of the closed-loop system $G_{vn}(s)$ is guaranteed in the above problem formulation if $p \in P_m$, $E_1, E_2, E_3, E_4 > 0$ and $Q(s) \in RH_\infty$. This fact implies that p and the controller $K(s)$ can be adjusted simultaneously while maintaining the stability of the closed-loop system. Therefore, the above drawback of p - K iteration does not exist in this problem formulation. Moreover, since $G_{vn}(s)$ represents all stable closed-loop system with $P(s)$ in Eq. (6) and the stabilizing controller $K(s)$, the optimal design variable $p^* \in P_m$, $E_1^*, E_2^*, E_3^*, E_4^* > 0$ and $Q^*(s) \in RH_\infty$ can be obtained for the entire design space. Hence the convergence of the objective function to a local minimum is guaranteed with conventional descent-based optimization methods.

The simultaneous optimal design which guarantees the convergence to a local minimum is considered by Obinata et al.⁽⁹⁾ and Hiramoto et al.⁽¹⁰⁾ However, these formulations can not deal with the specification on the structural system explicitly. The formulation of this paper can directly incorporate the structural specification, e.g. weight, strength, etc.

In contrast to the formulation of Zou et al.⁽⁷⁾ and Tanaka et al.⁽⁸⁾, the formulated simultaneous optimal design problem does not assume that each matrix in the equation of motion in Eq. (1) is a linear function of elements of the passive parameter vector p . Hence, the above convergence property is not changed if matrices $M(p)$, $D(p)$ and $K(p)$ are nonlinear functions of elements of p .

Remark 1 For the formulated design problem, it is hard to obtain the optimal $Q(s)$ in Eq. (13) analytically, such as Francis⁽¹⁷⁾, from the function space RH_∞ . By assuming the application of a numerical optimization technique, it is practical to parameterize $Q(s)$ by a finite order transfer function matrix and optimize coefficients of each element of the transfer function matrix in RH_∞ . In this situation the space to be searched becomes smaller than the entire design space.

Remark 2 Even if only the velocity of the structure $y_v(t)$ (Eq. (4)) is available, all results in this subsection are valid by letting $C_2 = [0 \ L_a^T(p)]$ and $H = - \begin{bmatrix} 0 \\ M^{-1}(p)L_a(p)E_4 \end{bmatrix}$.

Remark 3 As a controller design adopting Youla parameterization, the convex optimization based scheme is established and is shown to be a powerful design method^{(18),(19)}. This design procedure may be applicable to simultaneous optimal design if p - K iteration is employed. Namely, the optimal $Q(s)$ is obtained first,

and p and $E_i (i=1, 2, 3, 4)$ are optimized next (and iterate this procedure to achieve convergence). However, the problem on the convergence of p -K iteration indicated above can not be avoided if we follow the design based on convex optimization.

4. Design Example

As a design example, consider a simply supported beam of length L as depicted in Fig. 2. Define the horizontal and the vertical coordinates as ξ and ψ respectively. A disturbance force $n(t)$ which satisfies $E(n(t))=0$ and $E(n(t)n(\tau))=\delta(t-\tau) \forall t, \tau>0$ is applied at $\xi=\xi_n$. At $\xi=\xi_a$ a sensor measuring $\dot{\psi}(\xi_a, t)$, and an actuator producing a lateral force $u(t)$ are installed. Assume that the cross-sectional area of the beam is distributed as

$$S(\xi)=\sum_{i=1}^{N_d} S_i(\xi), \quad (17)$$

where

$$S_i(\xi)=\begin{cases} a_i & ((i-1)L/N_d \leq \xi < iL/N_d) \\ 0 & (\text{otherwise}) \end{cases} \quad (18)$$

In Eq. (17), N_d denotes the number of elements whose cross sectional area can be adjusted independently in the ξ direction. The equation of motion of the system and the boundary conditions are described as a partial differential equation represented by

$$\frac{\partial^2}{\partial \xi^2} \left(EI(\xi) \frac{\partial^2 \psi(\xi, t)}{\partial \xi^2} \right) + \rho S(\xi) \frac{\partial^2 \psi(\xi, t)}{\partial t^2} + D_o \dot{\psi}(\xi, t) = n(t) \delta(\xi - \xi_n) + u(t) \delta(\xi - \xi_a), \quad (19)$$

$$\psi(0, t) = \psi(L, t) = 0, \quad \frac{\partial^2 \psi(0, t)}{\partial \xi^2} = \frac{\partial^2 \psi(L, t)}{\partial \xi^2} = 0, \quad (20)$$

where E and ρ are Young's modulus and density of the beam respectively. By assuming circular cross section with diameter d_i ($i=1, \dots, N_d$), a_i in Eq.(18) and moment of inertia of area $I(\xi)$ is given as

$$a_i = \frac{\pi d_i^2}{4}, \quad (21)$$

$$I(\xi) = \frac{1}{4\pi} S^2(\xi) = \frac{\pi}{64} \sum_{i=1}^{N_d} d_i^4. \quad (22)$$

The symbol D_o denotes the damping operator, i.e. $D_o \dot{\psi}(\xi, t)$ represents internal or external damping force in the beam. In this example, so called "proportional damping"⁽²⁰⁾ is assumed, that is the damping

Fig. 2 Simply supported beam

operator is assumed to have the form

$$D_o = \alpha \rho S(\xi) + \beta \frac{\partial^2}{\partial \xi^2} \left(EI(\xi) \frac{\partial^2}{\partial \xi^2} \right), \quad (23)$$

where $\alpha, \beta > 0$. In this example the distributed parameter system is approximated by an n degree of freedom system with Rayleigh-Ritz method. Assume that the displacement $\psi(\xi, t)$ is approximated by

$$\psi(\xi, t) = \sum_{k=1}^n \phi_k(\xi) q_k(t), \quad (24)$$

where $\phi_k(\xi) = \sqrt{2/L} \sin(k\pi\xi/L)$ and $q_k(t)$ are generalized coordinates. Substituting Eq. (24) into Eq.(19), multiplying by $\phi_j(\xi)$ and integrating over $0 \leq \xi \leq L$, the approximated system of order n is obtained in the form of Eq. (1) where

$$M = \begin{bmatrix} \rho \int_0^L S(\xi) \phi_1^2(\xi) d\xi & \dots \\ \vdots & \ddots \\ \rho \int_0^L S(\xi) \phi_1(\xi) \phi_n(\xi) d\xi & \dots \\ \vdots & \ddots \\ \rho \int_0^L S(\xi) \phi_n^2(\xi) d\xi \end{bmatrix},$$

$$K = \begin{bmatrix} \int_0^L \phi_1(\xi) \frac{\partial^2}{\partial \xi^2} \left(EI(\xi) \frac{\partial^2 \phi_1(\xi)}{\partial \xi^2} \right) d\xi & \dots \\ \vdots & \ddots \\ \int_0^L \phi_1(\xi) \frac{\partial^2}{\partial \xi^2} \left(EI(\xi) \frac{\partial^2 \phi_n(\xi)}{\partial \xi^2} \right) d\xi & \dots \\ \vdots & \ddots \\ \int_0^L \phi_n(\xi) \frac{\partial^2}{\partial \xi^2} \left(EI(\xi) \frac{\partial^2 \phi_n(\xi)}{\partial \xi^2} \right) d\xi \end{bmatrix},$$

$$D = \alpha M + \beta K, \quad L_n = [\phi_1(\xi_n) \dots \phi_n(\xi_n)]^T,$$

$$L_a = [\phi_1(\xi_a) \dots \phi_n(\xi_a)]^T.$$

Design specifications in this design example are summarized as follows:

Closed-loop design specification

1. Minimize an objective function defined as

$$J_c = \|G_{un}(s)\|_2^2, \quad (25)$$

where $G_{un}(s)$ is the transfer function from the disturbance n to the control effort u . By minimizing J_c the energy consumption for the active control can be suppressed.

2. Satisfy an inequality constraint given by

$$\varphi = \|G_{qn}(s)\|_2^2 \leq \gamma_c, \quad (26)$$

where $G_{qn}(s)$ is the closed-loop transfer function matrix from the disturbance n to the displacement vector $q = [q_1, q_2, \dots, q_n]^T$ in Eq.(1) and γ_c is a scalar specified by the designer. The inequality constraint assigns the closed-loop performance on vibration suppression in the H_2 norm sense.

Structural design specification

1. Minimize the weight of the structure given as

$$J_s = \sum_{i=1}^{N_d} \frac{\rho a_i L}{N_d} \quad (27)$$

Assume that $G_{un}(s)$ and $G_{qn}(s)$ have a state space form given as

$$\begin{bmatrix} G_{qn}(s) \\ G_{un}(s) \end{bmatrix} = \begin{bmatrix} A^c & B^c \\ C_1^c & 0 \\ C_2^c & 0 \end{bmatrix}$$

Then, J_c and φ can be obtained as

$$J_c = \text{trace}(C_2^c X^c C_2^{cT}) \quad (28)$$

$$\varphi = \text{trace}(C_1^c X^c C_1^{cT}) \quad (29)$$

where X^c is the symmetric and positive definite solution of the following Lyapunov equation:

$$A^c X^c + X^c A^{cT} + B^c B^{cT} = 0 \quad (30)$$

The design variable vector is defined as

$$p = [a_1, \dots, a_{N_d}, \xi_a]^T \quad (31)$$

where a_i ($i=1, \dots, N_d$) and ξ_a are the cross-sectional areas of the divided elements of the beam and the sensor/actuator placement respectively. With this definition, the feasible set P_m in Eq. (2) can be defined as the following:

$$P_m = \{p \in R^{N_d+1} | \underline{p} \leq p \leq \bar{p}\}, \quad (32)$$

In the above definition $\underline{p} = [\underline{a}, \dots, \underline{a}, 0]^T$ and $\bar{p} = [\bar{a}, \dots, \bar{a}, L]^T$, where $\bar{p}$ and $\underline{p}$ are upper and lower limits of the cross sectional area respectively. In this example, the free parameter $Q(s)$ in Eq. (13) is parameterized as

$$Q(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + c_4 s + c_5}, \quad (33)$$

where $c_1, c_2, c_3 \in R$ and $c_4, c_5 > 0$. As other design parameters in the controller, we define feedback gains $E_2 > 0$ and $E_4 > 0$ in Eqs. (8) and (11). Note that $E_1 = E_3 = 0$ in this example since the displacement at $\xi = \xi_a$ is not measured (see remark 2.). In this example the above multi objective optimal design problem is converted to a single objective optimization problem. Define an objective function J given as

$$J = cJ_c + J_s \quad (34)$$

where $c > 0$ is a weighting factor specified by the designer. All inequality constraints given in the above are incorporated in the optimization with penalty function method. Let J_p be the sum of J and penalty functions. Each value of the parameter employed in the optimization is summarized in Table 1. The objective function J_p is minimized using Quasi-Newton method. The gradient of J_p on each design variable is analytically obtained by the method of sensitivity analysis. The optimization is conducted several times from different initial design variables to obtain the "better" local optimal solution. The optimization history for J_p is shown in Fig. 3. Both values of J_c and J_s are reduced by 72.9% and 18.0% from their initial values respectively. The optimized cross-sectional area of the beam and the placement of the sensor and the actuator are depicted in Fig. 4. Note

Table 1 Initial values of design parameters and other variables

Parameters	Value
Length of the beam L	1
Young's modulus E	1
Initial cross-sectional area of the beam a_0	1
Number of considered mode n	10
Number of division N_d	10
Factors α, β (in Eq. (23))	0.005
Maximum cross-sectional area $\bar{a}$	1.5^2
Minimum cross-sectional area $\underline{a}$	0.5^2
Disturbance injected point ξ_n	$0.3L$
Weighting factor c (in Eq. (34))	10
γ_c (Upper bound of φ in Eq. (26))	0.2

Fig. 3 Optimization history of J_p

Fig. 4 Resulted $S(\xi)$ and ξ_a

that the superscript 0 and *opt* denote the initial and the optimized design variables. It is found that $S(\xi)$ becomes large around $\xi = \xi_n$ where the disturbance

Fig. 5 Gain plots of open-loop systems and closed-loop systems

$n(t)$ is injected. Figure 5 gives some gain plots of the open and the closed-loop system from n to $\psi(0.65L, t)$. In this figure, the thin dotted line (open-loop (Initial configuration)) denotes the gain plot of the open-loop system with initial design variable. The thin line (open-loop (Init. conf. with LQG)) gives the closed-loop version of the above. The controller is synthesized to minimize J_c with the inequality constraint $\psi \leq \gamma_c$ using the method proposed by Hsieh et.al.⁽²¹⁾ The thick line and dotted line plots are of the closed and the open-loop system with optimal design variables. In the case of initial design variables the resultant J_c becomes 0.42988 while on the other hand $J_c=0.2889$ with optimized design variables. This result implies that the optimized closed-loop achieves the same performance on vibration suppression (in the sense of Eq. (26)) with $0.2889/0.429 \times 100 \cong 67.2\%$ energy consumption.

5. Conclusion

The simultaneous optimal design with Youla parameterization is considered in this paper and the main points are as follows:

1. All stabilizing controllers for collocated n degree of freedom systems can be obtained easily with Youla parameterization. Considering this fact the simultaneous optimal design problem with Youla parameterization is formulated.
2. This formulation enables one to adjust passive parameter vector p simultaneously while maintaining the stability of the closed-loop system. Since p - K iteration technique does not need to be employed the convergence to a local optimal solution is guaranteed

with standard descent-based optimization techniques.

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