Driveline Load Analysis with Multiple Cardan Joints*

Vedam KRISHNA**, Saravanan M. PEELAMEDU***, Rahul PHADNIS**, Nagi G. NAGANATHAN**** and Rao V. DUKKIPATI*****

A general equilibrium analysis of automotive drivelines with multiple universal joints and numerous supporting conditions is presented. For the analysis, components of the driveline in terms of their angular rotation, angular velocity and acceleration are defined. Using kinematic conditions and a known torque and speed, the angular speed of the axle output shaft and all intermediate shafts and forces at contact points is computed using a software developed to perform the analysis. To perform a time domain analysis, the quasi-static equilibrium conditions are imposed on the system at an instant. The instantaneous behavior of the system is determined as a function of the transmission output shaft rotation. Finally numerical example is provided to illustrate the analysis. Results of the numerical example are plotted with respect to time. The general behavior of the forces is found to be oscillatory and harmonic in nature.

Key Words: Driveline, Power Train, Cardan Joints, Automotive, Highway Vehicles, Off-highway Vehicles, Dynamics

1. Introduction

The power transmission unit (PTU) in road vehicles is an assembly of a large number of mechanical elements. The design of a PTU is constrained by the overall system requirements of compactness wherein the driveline must continue to supply uniform power even in the presence of large load variations, road undulations or any vibrations of the transmission and suspension units. The major requirement of the overall system is to suppress all possible vibrations and to transmit a uniform speed and torque.

Current design procedures of PTU involve some empirical formulations. These estimates are sufficiently conservative not to cause gross mechanical failures and may not be very different from theoretically predicted values unless the vehicle experiences a drastic, unfavorable condition. Some of the important parameters in a driveline such as the torsional fluctuations or the secondary couples are given in SAE AE711.

For larger joint angularities or for long drivelines with multiple couplings, the loads as well as kinematic variables computed by conventional SAE design practices show considerable variation from those calculated using analytical methods. The result of such differences is seen as a different loading profile on driveline elements such as the crosses and the trunion bearings among others. As a result, some of the failures of drivelines can be more clearly understood if the complete loading profile is known. For example, high tangential loads on the crosses can be as dangerous as even small amplitude and high frequency bearing loads leading to fatigue failure of bearings, since a bearing failure can instigate a very large loading on the joint causing a failure. Since the bearings suffer from large stress concentrations, this aspect cannot be neglected.

1.1 Basic components

The automobile power train consists of the
transmission followed by the driveline, the differential, and the wheel axles. The driveline as in Fig. 1 transfers the power from the transmission output to the differential by a combination of coupling and propeller shafts. Also, the driveline must accommodate the vertical motion of the rear axle as the rear wheels encounter road undulations and flexion of the rear springs. The most commonly used coupling is the universal joint or the cardan joint. The other specialized couplings are the Constant Velocity joints and the ball and trunnion joint.

The Cardan joint, as in Fig. 2, is a double hinged joint consisting of two yokes, one each on the driving shaft and the driven shaft. Connecting the two yokes is a cross member with four arms called the trunnions. Each trunnion is provided with the bearings so that it can execute rotary motion smoothly about the yoke axis. The advantage of this joint is that since the cross member is free to move about the yoke, the shaft angularities are easily accommodated by a simple reorientation of the cross member. The following section deals with common driveline configurations.

1.2 Common driveline configurations

A typical driveline consists of propeller shafts, Cardan joints and one spline to provide the freedom to accommodate the suspension movement. Historically, there have been two types of designs in drivelines. One is the torque tube drive where the drive had to resist the torsional reaction and transmit the driving and braking thrust forces from the wheel to the vehicle. In the currently used design, only the driveline torque is transmitted while the driving and braking thrust forces are transmitted through the suspension unit. Generally, the driveline consists of two or more jointed units. The use of trucks with more than one driving axle as in military applications led to a commonly used system of designating drive train configurations. An $N \times M$ truck refers to an $N$-wheeled truck with an $M$-wheeled drive. In addition, some vehicles have universal jointed axle driveshfts or halfshafts in their power train.

The two jointed design is most commonly used in cars and trucks. Three joint drivelines are used in some passenger cars and are frequently required in trucks. Different combinations of support bearings are possible. A propeller shaft containing a joint, tube, support bearing and a slip or fixed spline is called a single joint coupling shaft. The active spline in the driveline can be used in any of the driveshfts and there is no restriction. When a passenger car has independent rear suspension, the driveline is not required to accommodate axial movements and a slip spline may be needed only to provide build tolerances.

1.3 Role of computer aided analysis

As seen in the previous sections, a number of combinations of driveline elements are possible and a complete analysis requires each of the elements to be in stable equilibrium condition. Since the degree of freedom of the state of a driveline is very large, no closed-form solutions are possible, and hence computer analysis is resorted to. Analysis of a typical driveline with a single driveshaft, as will be seen later, involves 31 variables and hence, a closed-form solution is not feasible.

In a practical driveline, in addition to the heavy shaft and joints, the angular acceleration is also very high and so the inertial effect may become quite significant. This limits the size of the operating joint angles. If the effect of the inertia is included in the driveline analysis, then it is possible to predict the mechanical behavior of the power train more accurately.

The analysis of a power train involves accurate descriptions of the kinematics of the cross member. The motion of the cross member is such that the center of the trunnions moves in a circular path along with the respective yoke, while the trunnion itself rotates about its axis of symmetry. The rotation of the cross member about the three axes will be determined. The first step in the analysis is to deduce the
kinematics of the Cardan joint. Following this, a quasi-static analysis as well as an extension to
dynamic analysis will be carried out.

Nomenclature

\( \vec{e} \): Unit vector in 1 direction
\( F_{\text{net}} \): Bearing force at point 1 in Y direction
\( F_{\text{net}} \): Bearing force at point 1 in Z direction
\( F_C \): Force at point C in the X direction
\( F_E \): Force at point E in the tangential direction
\( F_T \): Force on frame 1 in radial direction
\( I \): Moment of inertia
\( \vec{r}_C \): Position vector of a particular point C (The
subscript indicate the point)
\( R \): Radius of journal cross
\( T_1 \): Transformation matrix for rotation about
the shaft axis
\( T_2 \): Transformation matrix for rotation about
a driven yoke axis
\( T_3 \): Transformation matrix for rotation about
a driven yoke axis
\( T_{\text{Input}} \): Input torque
\( T_{\text{Output}} \): Output torque
\( \alpha \): Angular acceleration
\( \beta \): Angular displacement of a driven shaft
\( \delta \): Phase index angle
\( \phi \): Rotation about driven yoke axis
\( \theta \): Angular displacement about a shaft axis
\( \omega \): Angular velocity about a shaft axis
\( \xi \): Rotation about driven yoke axis

2. Computer-Aided Driveline Analysis

This section deals with the analysis of a driveline
to determine the different loads on the driveline
elements. As a first step in such an analysis, the 3D
kinematics of the joint must be completely derived.
Then the equilibrium equations must be derived and
solved.

2.1 Overview of the analysis

The motion of the cross member in a universal
coupling is described in terms of two types of
rotational motion. These are:

1. Primary rotation along the respective shaft
axis.
2. Secondary rotations along the two yoke axes.

Expressions to quantify the primary rotation of
the cross were first developed by Poncelet, the creator
of projective geometry, in 1824 with the aid of spherical
geometry. Here, efforts to estimate the secondary
rotation will be made. Quasi-static equilibrium conditions
are then developed for each element and the
constraints on the possible configurations for a
statically indeterminate system (SDS) are imposed. The
solution to this formulation will give the loads on the
driveline elements. Finally, the extension to dynamic
equilibrium is achieved by including the shaft inertia
in the analysis. The analysis is expected to assist a
design engineer in two possible ways:

1. Determine loads on driveline elements such as
the yokes, cross members, bearings, and shafts.
2. Determine time independence of these forces;
thus, apply in fatigue life estimates.

2.2 Kinematics of the Cardan joint

The objective is to determine the kinematic state of
every driveshaft and cross following the input
shaft, given the geometric layout of the driveshafts,
and the kinematic state of the transmission output
shaft \((\theta, \omega_0, \text{ and } a_0)\). The kinematic state of any
component will be assumed to consist of \( \theta, \omega \) and \( a \).

To determine the kinematic states of all driveshafts,
consider the spatial motion of any two shafts
coupled by a Cardan joint. For an input shaft angular
displacement of \( \theta \), the driven shaft undergoes an
angular displacement of \( \beta \). The relationship between
them can be obtained by using the condition of
orthogonality of two adjacent trunnions of the cross,
assuming a rigid coupling.

As shown in Fig. 3, the cross member consists of
four trunnions constituting its four arms. All four
arms are assumed to be rigidly connected to the center
O. The points C, and D are on the driver side trunnions
while E, and F are on the driven side trunnions.

In the initial analysis, the trunion radius is
assumed to be negligible compared to the length of the
arm. The effect of a finite trunion radius affects only
the tangential velocity between the trunion and the
bearing cup. This will cause wear at the bearing
surfaces due to continuous contact.

A coordinate frame \( XYZ \) is defined for each
shaft, such that the \( X \) axis is always parallel to the
shaft axis and the frame is fixed with respect to shaft
rotation about its axis. When all the shafts are
aligned with no angularity between them, the \( Y \) axis
is such that it is parallel to the yoke of the first joint.
The \( Z \) axis can be derived from the cross product
of the other two axes. Each of these frames will rotate
with the shafts along the yoke rotations but do not rotate with respect to the shaft axis rotation. The rotation of points E and F can be given in terms of three possible rotations:

1) Rotations about the shaft axis. X axis rotation of β about the local fixed frame. \( T_2 \).

2) Rotation about the driven yoke axis. Y axis rotation of φ about the local fixed frame. \( T_3 \).

3) Rotation about the driven yoke axis. Z axis rotation of ξ about the local fixed frame. \( T_1 \).

Using these transformations, the position vector to the points E and F on the driven shaft/driver side yoke are determined. The position vectors for C and D can be obtained by a rotational transformation \( T_0 \) about the X axis of the driving shaft.

\[
\begin{align*}
\bar{r}_c &= T_0 \bar{r}_{c,0}, \\
\bar{r}_d &= T_0 \bar{r}_{d,0}, \\
\bar{r}_e &= T_1 T_2 T_3 \bar{r}_{e,0}, \\
\bar{r}_f &= T_1 T_2 T_3 \bar{r}_{f,0},
\end{align*}
\]

where

\[
T_i = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\theta) & -\sin(\theta) \\
0 & \sin(\theta) & \cos(\theta)
\end{bmatrix},
\]

\[
T_i = \begin{bmatrix}
\cos(\xi) & -\sin(\xi) & 0 \\
\sin(\xi) & \cos(\xi) & 0 \\
0 & 1 & 0
\end{bmatrix},
\]

\[
T_i = \begin{bmatrix}
\cos(\phi) & 0 & \sin(\phi) \\
0 & 1 & 0 \\
-\sin(\phi) & 0 & \cos(\phi)
\end{bmatrix},
\]

\[
T_i = \begin{bmatrix}
0 & \cos(\beta) & -\sin(\beta) \\
0 & 1 & 0 \\
\sin(\beta) & 0 & \cos(\beta)
\end{bmatrix},
\]

and

\[
\bar{r}_{c,0} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}, \quad \bar{r}_{d,0} = \begin{bmatrix} -R \\ 0 \\ 0 \end{bmatrix}, \quad \bar{r}_{e,0} = \begin{bmatrix} 0 \\ R \\ 0 \end{bmatrix}, \quad \bar{r}_{f,0} = \begin{bmatrix} 0 \\ -R \\ 0 \end{bmatrix}.
\]

The subscript \( \sigma \) in the radius vector refers to the value of \( \sigma \).

Using the condition of orthogonality of adjacent trunnions, Eq. (5) is obtained,

\[
\bar{r}_c \cdot \bar{r}_e = 0, \quad (5)
\]

\[
\tan(\beta - 90) = \frac{\cos(\phi)}{\cos(\xi)} \tan(\theta) + \sin(\phi) \tan(\xi). \quad (6)
\]

2.2.1 Estimation of secondary rotational motion of cross

A mathematical approach using the definition of the angular velocity vector can be used to obtain the primary and secondary rotations. The angular velocity vector of the cross can be obtained in a local frame attached to the cross center. The axes 2 and 3 pass through the two arms of the cross and axis 1 is along the normal to the plane containing the journal cross trunnions. Based on this consideration and resolving all coordinates into a fixed frame as defined previously for the input shaft, the following is obtained:

\[
\dot{\bar{r}}_c = T_0 \dot{\bar{r}}_{e,0}, \quad (7)
\]

\[
\dot{\bar{r}}_d = T_1 T_2 T_3 \dot{\bar{r}}_{e,0}, \quad (8)
\]

\[
\dot{\bar{r}}_e = \dot{\bar{r}}_1 \times \dot{\bar{r}}_1, \quad (9)
\]

where \( \dot{\bar{r}}_{e,0} \) is given by \((0,0,0)\).

A general angular displacement in terms of the partial angular displacements is given by:

\[
\dot{\bar{r}} = \theta_1 \dot{\bar{r}}_1 + \theta_2 \dot{\bar{r}}_2 + \theta_3 \dot{\bar{r}}_3.
\]

Using this approach, the angular velocity vector is given by the vector relation:

\[
\omega = \frac{\ddot{\bar{r}}_1}{dt} \times \dot{\bar{r}}_1 + \frac{\ddot{\bar{r}}_2}{dt} \times \dot{\bar{r}}_2 + \frac{\ddot{\bar{r}}_3}{dt} \times \dot{\bar{r}}_3.
\]

On the basis of this definition, and the above defined unit vectors in the 1, 2 and 3 directions, the partial velocities in the three directions are obtained:

\[
\omega_1 = \frac{\ddot{\bar{r}}_1}{dt}, \quad \omega_2 = \frac{\ddot{\bar{r}}_2}{dt}, \quad \omega_3 = \frac{\ddot{\bar{r}}_3}{dt}.
\]

where \( \omega_1 \) is the input shaft speed and \( \omega_2 \) is the output shaft speed. With this, the problem of determining the partial velocities of the trunnion head about the yoke axis is completely resolved. The resultant of the above three equations gives the angular velocity vector.

2.3 Dynamics of the Cardan Joint

Once the kinematics has been derived, one can proceed to estimate the forces on the drive line for a SDS with rigid body contact. The compliance of the system only affects the stress distribution in the elements in a quasi-static system and also for a dynamic system with larger inertia. In the case where the system is not a SDS, the forces on the multiple support bearings will adjust themselves such that the resultant is always the same as the forces on the supports of a system for a SDS. This is because the conditions of equilibrium must always be satisfied whatever the problem at hand. It is important to note here that an over constrained system is not possible even for one element. This is because the conditions of equilibrium cannot form a dependent set of equations that is necessary for such a system. Such systems can exist as static systems only if the degrees of freedom alone can satisfy all the equations, which defies the general nature of the formulation.
2.3.1 Modeling the shaft element

Consider a single shaft supported by two bearings, an input driving yoke, an output driven yoke and a spline that provides for ideal slipping. The presence of friction can be included at a later stage by using the Coulomb theory for friction. Such a shaft is considered as a generic shaft which is representative of all the possible types of shafts available. All shafts that are used in the driveline can be identified as a simplified case of such a shaft. In practice, five types of shafts are commonly used as in Table 1.

For the equilibrium equations, one such shaft with all these properties is considered. As shown in Fig. 4, the shaft has six possible contact regions. For the case of propeller shafts, two of these six contact regions are in contact with the shaft’s driving side cross (E, F) and two are in contact with the driven side cross (C, D). The other two contact regions correspond to the contact between the bearings and the shafts, when bearings are present. Since the bearings cannot take any frictional torque, the forces at the bearings are assumed to pass through the center such that no couples are present. Any axial load bearing capacity of the center bearings on the driveshafts or the coupling shafts will also be ignored. If a spline is present in a shaft, the shaft will not have the ability to resist any axial load. The contact regions for the shaft element are reduced to point contacts (Fig. 4).

Each shaft is associated with two coordinate frames, one at each end of the shaft. The frame at the driven end can be offset from the driving end frame by the phase index angle \( \beta \). For a given Cardan joint, if the driven side shaft is rotating about its local axis at an angular speed of \( \omega_{i+1} \) and the driver side shaft is rotating at \( \omega_i \), we can compute the ratio of angular speeds by differentiating the angular displacement relation (Eq. (6)).

\[
\frac{\omega_{i+1}}{\omega_i} = \frac{\cos(\phi)}{\sin(\phi)} \sin^2(\beta) \cos(\phi) \cos(\beta)
\]

(15)

For the special case \( \xi = 0 \), this reduces to the classical Poncelet equation.

\[
\frac{\omega_{i+1}}{\omega_i} = \frac{\cos(\phi)}{\sin(\phi)} \sin^2(\beta)
\]

(16)

Similarly, if the angular acceleration of the \( i \)th shaft is \( \alpha_i \) and that of the \( (i+1) \)th shaft is \( \alpha_{i+1} \), then the relation between them is given by differentiating Eq. (15) as:

\[
\alpha_{i+1} = -2 \frac{\cos(\xi)}{\cos(\phi)} \left[ \tan(\xi) \sin(\phi) \cos(\theta) + \left( \frac{\cos(\xi)}{\cos(\phi)} \sin(\theta) + \frac{\cos(\xi)}{\cos(\phi)} \cos(\theta) \right) \alpha_i \right]
\]

\[
\times \left( \tan(\xi) \sin(\phi) \sin(\theta) - \frac{\cos(\xi)}{\cos(\phi)} \cos(\theta) \right) \omega_i^{2} + \frac{\cos(\phi)}{\cos(\xi)} \frac{\alpha_i}{\cos^2(\theta) + (\tan(\xi) \sin(\phi) \cos(\theta) + \frac{\cos(\xi)}{\cos(\phi)} \sin(\theta))^2}
\]

(17)

This equation is prone to computational problems, since \( \theta \) varies from 0 to 360. Hence it has been resolved into the simplest trigonometric functions.

Note that the locations of the points C, D, E, and F are assumed to be equidistant from the center of the journal cross (= R). This condition may be easily relaxed for journal crosses of non-equal cross member lengths, i.e., CD = EF. All joint angles are much less than 90\(^\circ\) in practical applications. Hence, potential mathematical singularities in the above equations should not affect practical calculations. However, in evaluating the angular velocity and accelerations, limits must be taken to avoid singularities.

Using the transformation \( T_k \), forces along local directions on the shaft from the driving yoke and driven yokes can be resolved in the driving yoke frame \( X'Y'Z' \), and thus a force equilibrium and moment equilibrium condition can be derived using:

\[
\sum \vec{F} = \vec{F}_{\text{ext}} = 0,
\]

(18)

\[
\sum \vec{M} = \vec{M}_{\text{ext}} = I \vec{a} + \vec{a} \times I \vec{a}.
\]

(19)

The equations of force equilibrium for a shaft with yoke phase angle \( \delta \) are:

\[
-F_{\text{ex}} + F_{\text{ex}} + F_{\text{ex}} + F_{\text{ex}} = 0,
\]

(20)

\[
F_{\text{ex}} + F_{\text{ex}} + F_{\text{ex}} \sin(\delta) - F_{\text{ex}} \cos(\delta) = 0.
\]

(21)

Table 1 Different types of shafts used in drivelines

<table>
<thead>
<tr>
<th>NO.</th>
<th>SHAFT</th>
<th>DETAILS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Transmission Output Shaft</td>
<td>2 bearings, no input yoke, output yoke present.</td>
</tr>
<tr>
<td>2</td>
<td>Coupling Shaft</td>
<td>1 bearing, an input yoke and an output yoke</td>
</tr>
<tr>
<td>3</td>
<td>Drive Shaft</td>
<td>no bearing, an input yoke and an output yoke</td>
</tr>
<tr>
<td>4</td>
<td>Jack Shaft</td>
<td>2 bearings, an input yoke and an output yoke</td>
</tr>
<tr>
<td>5</td>
<td>Axle Output Shaft</td>
<td>2 bearings, an input yoke and no output yoke</td>
</tr>
</tbody>
</table>

Fig. 4 Freebody diagram of a generalized shaft
On the basis of this, the general elemental equilibrium equation can be written as:

\[ K_{ex}X = F_{ext} \]  \hfill (29)

While developing the equations for the transmission output shaft and the axle input shaft, each of these shafts will be assumed to house two bearings. If the user can provide realistic information for the locations of these bearings, appropriate bearing loads can be computed. Free body diagrams of a transmission output shaft and an axle input shaft are shown in Figs. 5 and 6.

### 2.3.2 Modeling the journal cross element

Consider the free body diagram of a journal cross in Fig. 7. There are four contact points, as mentioned previously, and ten forces resolved along the orthogonal directions as shown in Fig. 7. Typically, the trunnions of a cross have a constant length from the center O, but there are some special cases where deviations occur.

For the equilibrium equations, all the forces are resolved along the axes fixed along the initial position of the shaft attached to the driving yoke. Using the transformations given before, the equilibrium conditions Eqs. (18) and (19) are used to obtain the following equations:

\[ \cos(\xi)\cos(\phi)(F_{exc} + F_{t2}) + \sin(\xi)\cos(\beta) \]
\[ -\cos(\xi)\sin(\phi)(F_{exc} + F_{t2}) - \sin(\xi)\sin(\beta) \]
\[ + \cos(\xi)\sin(\phi)(F_{exc} - F_{t1}) + (F_{ext} + F_{a}) \]
\[ = 0, \]  \hfill (30)

\[ -\sin(\xi)\cos(\phi)(F_{exc} + F_{t2}) - \cos(\xi)\cos(\beta) \]
\[ + \sin(\xi)\sin(\phi)(F_{exc} + F_{t2}) - \sin(\xi)\sin(\beta) \]
\[ + \cos(\xi)\sin(\phi)(F_{exc} - F_{t1}) \]
\[ + \cos(\xi)\sin(\phi)(F_{exc} + F_{t1}) \]
\[ = 0, \]  \hfill (31)

\[ R(\cos(\xi)\cos(\phi)(F_{exc} - F_{t1}) + \sin(\xi)\sin(\beta))(F_{exc} - F_{t2}) \]
\[ - R(\cos(\phi)(F_{a}) + \cos(\xi))F_{exc} + F_{t2} + (F_{ext} + F_{a}) \]
\[ = 0, \]  \hfill (32)

\[ R(\cos(\phi)(F_{t2}) - \cos(\xi))F_{exc} - F_{t2} \]
\[ + R(\sin(\xi)\cos(\beta) - \cos(\xi)\sin(\beta))(F_{exc} - F_{t2}) \]
\[ - \cos(\xi)(F_{t2}) + R(\sin(\xi)\cos(\beta) - \cos(\xi)\sin(\beta))(F_{exc} - F_{t2}) \]
\[ = 0, \]  \hfill (33)

\[ R(\cos(\phi)(F_{t2}) + \cos(\xi))F_{exc} - F_{t2} \]
\[ + R(\sin(\xi)\cos(\beta) - \cos(\xi)\sin(\beta))(F_{exc} - F_{t2}) \]
\[ - \cos(\xi)(F_{t2}) + R(\sin(\xi)\cos(\beta) - \cos(\xi)\sin(\beta))(F_{exc} - F_{t2}) \]
\[ = 0. \]  \hfill (34)

Similar to the shaft, the local equilibrium equations for the cross can also be written as:

\[ F_{exc} + F_{exc} - F_{t1} \cos(\delta) - F_{e1} \sin(\delta) + F_{e2} \cos(\delta) + F_{e2} - F_{t2} = 0, \]
\[ -RF_{exc} - RF_{exc} + RF_{t1} + RF_{t2} = 0, \]
\[ -L_{b1}F_{exc} + L_{b2}F_{exc} - LF_{t1} \sin(\delta) - R \sin(\delta)(F_{exc} - F_{t2}) + L(F_{exc} - F_{t2})\cos(\delta) = 0, \]
\[ -L_{b1}F_{exc} + L_{b2}F_{exc} - LF_{t1} \cos(\delta) + R \cos(\delta)(F_{exc} - F_{t2}) + L(F_{exc} - F_{t2})\sin(\delta) - R(F_{exc} - F_{t2}) = 0. \]  \hfill (22)

In matrix form, the local equilibrium equations for the shaft is determined as:

\[ X = [F_{exc} F_{exc} F_{exc} F_{exc} F_{exc} F_{exc} F_{exc} F_{exc} F_{exc} F_{exc} F_{exc}]^T \]
\[ K_{ex} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -S_{\delta} & 0 & -S_{\delta} & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -S_{\delta} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ F_{ext} = [F_{exc} F_{exc} F_{exc} F_{exc} M_{exc} M_{exc} M_{exc} M_{exc}]^T. \]  \hfill (26)

Fig. 5 Free body diagram of transmission output shaft

Fig. 6 Freebody diagram of axle input shaft

Fig. 7 Freebody diagram of the journal cross
\[ X = \begin{bmatrix} F_{x1} & F_{y1} & F_{z1} & F_{x2} & F_{y2} & F_{z2} & F_{x3} & F_{y3} & F_{z3} \end{bmatrix}^T \]  
\[ K_e = \begin{bmatrix} 
1 & 0 & 1 & 0 & 0 & -C_xC_y & -S_xS_y & -C_xS_y & C_x \\
0 & -S_y & 0 & S_y & C_y & -S_xS_y & -C_xS_y & C_xC_y & -S_x \\
0 & C_y & 0 & -C_y & S_y & S_x & C_x & -S_xC_y & -S_x \\
0 & R & 0 & R & 0 & R(C_xC_yS_y + S_xS_y) & -R(C_xS_yC_y + S_xS_y) & -R(C_xS_yS_y + C_xC_y) & R \\
RS_y & 0 & -RS_y & 0 & 0 & R(S_xC_yS_y - S_xS_y) & -R(S_xS_yC_y - C_xS_y) & -R(S_xC_yC_y - C_xS_y) & RS_y \\
-RC_y & 0 & RC_y & 0 & 0 & RC_yC_y & C_yC_y & -C_yS_y & -C_y \\
& -C_xC_y & (S_xS_y + C_xS_y) & (S_xC_y - C_xS_y) & -C_xS_y & (S_xC_y + C_xS_y) & -C_xC_y & C_yC_y & -C_y \\
& S_xC_y & (C_yC_y + S_xC_y) & (C_yC_y - C_xS_y) & -C_yS_y & (C_xC_y + C_xS_y) & -C_yC_y & C_xC_y & -C_x \\
& -R(S_xC_yS_y + S_xS_y) & -RC_yC_y & 0 & 0 & R(S_xC_yS_y - S_xS_y) & -R(S_xS_yC_y - C_xS_y) & -R(S_xC_yC_y - C_xS_y) & R \\
& -R(S_xC_yS_y - C_xS_y) & -RC_yS_y & 0 & 0 & R(S_xC_yS_y + C_xS_y) & -R(S_xS_yC_y + C_xS_y) & -R(S_xC_yC_y + C_xS_y) & RC_y \\
& -RC_yC_y & RS_y & 0 & 0 & RC_yC_y & C_yC_y & -C_yS_y & -C_y \\
\end{bmatrix} \]  
\[ F_{ext} = \begin{bmatrix} F_{a,ext} & F_{b,ext} & F_{c,ext} & M_{x,ext} & M_{y,ext} & M_{z,ext} \end{bmatrix}^T \]  
\[ \text{where, } C \text{ and } S \text{ are to be expanded as the cosine and sine of the angles following then. On the basis of this, the general elemental equilibrium equation can be written as:} \]
\[ K_e X = F_{ext}. \]  
\[ \text{(39)} \]

2.4 Degrees of freedom and constraints capability

In order to obtain a SDS, the number of bearings, splines and the number of universal couplings must be such that they should give rise to as many variables as there will be equations arising from the equilibrium equations for each of the system elements. For a driveline with \( n \) joints, there will be \( (n+1) \) shafts of which \( (n-1) \) will be prop-shafts the other two will be the transmission output shaft and the axle input shaft. A total of \( (2n+1) \) rigid bodies \( (n \) journal crosses and \( n+1 \) shafts) will be used in assembling the system matrix. As a result, a total of \( 6(2n+1) \) equations will be obtained for solving the system. In addition, center bearings \( (CB) \) and splines will be appropriately used to render the system practically functional and the model mathematically solvable. The center bearing gives rise to two additional variables in the system. The spline gives rise to an additional constraint equation.

The number of degrees of freedom:
- Journal crosses = \( 10n \)
- Bearings = \( 2b \)
- Transmission output shaft = 5
- Axle input shaft = 6
- TOTAL = \( 10n + 2b + 11 \)

The number of equilibrium equations and constraint equations:
- Shaft equilibrium conditions = \( 6(n-1) \)
- Journal cross equilibrium conditions = \( 6n \)
- Constraints to splines = \( sp \)
- Input and output shafts = \( 12 \)
- TOTAL = \( 12n + sp + 6 \)

Hence, to maintain a statically determinate condition, the following must to be met:
\[ 10n + 2b + 11 = 12n + sp + 6 . \]  
\[ \text{(40)} \]

The above system of equations for the various driveline elements is assembled into a global 2D matrix and finally reduced to the following form, \( AF = B \),  
\[ \text{(41)} \]
where, \( B \) is a vector of externally imposed forcing conditions, \( A \) is the coefficient matrix, and \( F \) is a vector of system variables to be solved.

2.5 Inertia effects on the analysis

For normal operating speeds, the angular acceleration of the driveshafts is very high since it is proportional to the square of the angular speed. Hence even for small inertia drivelines, the \( Ia \) term is very high and is comparable to the imposed torque. Under such conditions the inertia effects cannot be neglected and must be included in the analysis.

Although the cross inertia is comparatively smaller than the inertia of the drive shafts, its effects should be included for a complete analysis. For the present, the effects of shaft inertia alone have been included. For illustration purposes, the asymmetry created for the shaft due to the presence of yokes is not considered. Due to this assumption, the product moment of inertia terms \( I_{xy} \) and \( I_{xz} \) are assumed to be 0, which is true only for truly cylindrical shafts. Since during motion the major angular motion for the shaft is rotation about its axis, the only required parameter is \( I_{xx} \) for the shaft.

2.6 Optimization in driveline design

In order to select the best possible driveline combinations, the amplitude of variation of the motion transmitted to the axle shaft, called the 'torsionals', must be minimized. This will result in increased ride comfort. In addition, the amplitude of variation of the loads determines the life of the driveline components since speed cannot be used as a control parameter. The exact operating speed is not given to the designer and hence design is carried out in consideration of the
worst case. In order to minimize the amplitude of variation of the loads, a weighted sum of the driveline angular accelerations, called the 'inertials', is computed and parameters selected to minimize this function.

An important variable in such a design process is the phase index angle of a propeller shaft. Phasing, as it is called, helps in reducing the amplitude of variations. But there are limits to phasing due to its cyclic nature. Therefore properly selected phasing for drive-shafts can effectively reduce the inertials as well as torsionals.

2.7 Numerical examples

For the case of a 1-shaft driveline, a 2-joint system is considered. As a result, there are 2 journal crosses and 3 shafts in the model. For the case of 5 rigid bodies, we will have $6 \times 5 = 30$ rigid body equilibrium equations. In addition, such systems will usually involve a spline element as part of the drive-shaft. Hence, for the 1-shaft driveline, we will have 31 equations. The following are the 31 system variables that will be solved:

- Transmission output shaft (5 variables) $[F_{y_1}, F_{y_2}, F_{x_1}, F_{x_2}, F_{x_3}]$
- Journal cross 1 (10 variables) $[F_{x_1}, F_{x_2}, F_{x_3}, F_{x_4}, F_{x_5}, F_{x_6}, F_{x_7}, F_{x_8}, F_{x_9}, F_{x_10}]$
- Journal cross 2 (10 variables) $[F_{x_1}, F_{x_2}, F_{x_3}, F_{x_4}, F_{x_5}, F_{x_6}, F_{x_7}, F_{x_8}, F_{x_9}, F_{x_10}]$
- Axle input shaft (6 variables) $[F_{y_1}, F_{y_2}, F_{y_3}, F_{y_4}, F_{y_5}, T_{output}]$

Thus, we have a set of 31 simultaneous linear algebraic equations with 31 unknowns and a unique solution can be obtained. The above formulation is a very general approach as compared to simplified calculations found in literature.

The evaluation of the periodic variation of the various forces as well as the secondary couple was carried out. Some typical results for a one-shaft drivelines are shown. Note that the force variables solved here are variables defined along local coordinate directions. The driveline is assumed to be driven by an input torque of 100 Nm at 2000 rpm.

- Transmission output torque = 100 Nm
- Transmission output speed = 2000 rpm
- Radius of cross used = 76.2 mm
- Number of system variables = 31

Other geometric input parameters are listed in Table 2.

Example 1: Example excluding inertia effects and with no phase indexing

For the case of a 1-shaft driveline, as before, a 2 joint system is used. Thus it has 2 journal crosses and 3 shafts in the model. Analyses to determine the fluctuation of various forces are carried out. Typical results are shown in Figs. 8 through 12. Note that the forces that are determined are defined along local coordinate directions.

2.7.1 Discussion of results

The behavior of the driveline forces is almost harmonic since the angular misalignment of the shafts is less than 10°. It can be observed that the torsional fluctuation of the axle shaft is absent in spite of the presence of the drive-shaft angularity. The reason behind this is that the fluctuation produced by the first joint angularity is

<table>
<thead>
<tr>
<th>PLAN</th>
<th>ELEVATION</th>
<th>LENGTH</th>
<th>BEARING POSITION</th>
<th>NATURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
<td>0.1, 0.2</td>
<td>A/S</td>
</tr>
<tr>
<td>Drive Shaft #1</td>
<td>5.00</td>
<td>1.00</td>
<td>0.50</td>
<td>None</td>
</tr>
<tr>
<td>Axle</td>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
<td>0.1, 0.2</td>
</tr>
</tbody>
</table>

Table 2 Input parameters for the single shaft analysis in section 2.7 (A refers to a shaft with load carrying capacity. S refers to a shaft with a spline.)

![Fig. 8 Axial forces in the first joint for Example 1](image-url)
Fig. 9  Tangential forces in the first joint for Example 1

Fig. 10  Transmission output bearing forces for Example 1

Fig. 11  Radial forces in the first joint for Example 1
Fig. 12 Torque in each of the shafts in the driveline for Example 1

exactly negated by the fluctuation across the second joint, which occurs at the same amplitude but at a phase difference of 180°. The tangential forces on the same side of the yoke are exactly equal and there is no difference between $F_{ct}$ and $F_{ct}$ or $F_{ct}$ and $F_{ct}$. In addition, it can be noted that the tangential forces on the driving end of the transmission output shaft show no change and are constant.

The axial forces in Fig. 8 are such that the forces at C and D cancel each other, giving no net axial load on the transmission output shaft while those at the points E and F result in a very small net axial load on the driveshaft. The difference in the axial loads at C and D which give the secondary couple on the transmission output shaft show a fluctuation of nearly 10% of the transmitted torque ($120 \times 0.072 = 9.1$ Nm). The secondary couple is an important design criterion since this couple results in the bending of the yoke and is therefore critical.

3. Conclusions

A method for systematic analysis of a driveline is developed. For the purpose of the analysis, the driveline is first isolated and a known torque and speed are given to the system. Using the kinematic conditions, the angular speed and angular accelerations of the axle output shaft and all intermediate driveshafts are computed. The forces are then computed at each contact point in the system. To perform a time domain analysis, the quasi-static equilibrium conditions are imposed on the system at a given instant. The instantaneous behavior of the system is determined as a function of the transmission output shaft rotation. For equilibrium of the shafts, bearing supports are provided to generate a statically determinate system.

Thus, the magnitude of the forces in the driveline elements has been plotted with respect to time. The general behavior of the various forces in the system has been found to behave in an oscillatory manner and these are harmonic in nature. The effects of the higher frequency terms are negligible since the joint misalignment that has been used for the purposes of the analysis are smaller and generally less than 10°. As an example, consider the angular speed in the output shaft of a 1-joint analysis. For a constant input speed and a planar configuration where Poncelet equation can be used:

$$\omega_2 = \frac{\cos(\frac{\phi}{2})}{1 - \sin^2(\phi)\sin^2(\theta)} \omega_1.$$ (42)

The Fourier expansion for this for a constant $\omega_1$ is given by (obtained by standard Fourier expansion in complex domain using the theory of residues):

$$\omega_2 = \sum_{n=0}^{\infty} \tan^2 \left( \frac{\phi}{2} \right) \cos(2n\theta) \omega_1.$$ (43)

It will be clear from Eq. (43) that the amplitude of the higher order terms reduces very rapidly and for practical applications, the first harmonic term will be the most dominant one. The effect of the higher order terms are practically negligible and this approximation is used to determine the information required for the design of drivelines.

Some of the important parameters such as torsionals and inertials can be interpreted in terms of the
angular accelerations of the individual shafts. Assuming that all the driveshafts have the same inertia, the important design criteria relating to the effects of inertia are simplified in terms of the accelerations without the need for knowing the inertia itself. Though this may not be a good approximation for a general case of driveshaft configuration, it is a widely used standard. The torsionals, which are the amplitude of the angular acceleration of the axle output shaft, do not determine actual effects of the intermediate large angularity since a similar large angularity in the subsequent shaft can negate the value, it does not constitute a complete design criteria. The inertials, which are the sum total of the angular accelerations of all the driveshafts serve as a better design criteria. This is because the effects of large intermediate angularities cause acceleration with large amplitude and thus cannot be compensated for by subsequent angularities, since they are added.

References