Motion Control of an Inverted Pendulum via Final-State Control*

Hidekazu NISHIMURA**

In this paper, the author proposes a control method that can bring a system to an arbitrary final state even if the input is already saturated, by adopting a hyperbolic tangential function that describes the input limitation. Furthermore, by using the derivative of the function, the obtained input becomes continuous at terminal points of an initial time and a final time. This formulation of the input limitation can be described by a linear parameter-dependent system and the final-state control with error learning can be applied to the system. The effectiveness of this method is verified by carrying out both simulations and experiments of the swing-up motion control of a cart and inverted pendulum system, whose cart displacement is limited. This method does not require precise determination of the final time and can generate a continuous, smooth, feedforward control input when the output of the actuator is limited.

Key Words: Motion Control, Learning Control, Optimal Control, Final-State Control, Input Constraint, Linear Parameter-Varying System, Inverted Pendulum

1. Introduction

An actuator exhibits generation torque or force limitation. The actuator is usually controlled so that saturation does not take place. Since the linear control theory cannot take into account the limitation of the actuator output, control performance becomes conservative. In order to improve control performance, larger actuators are necessary. However, if the actuator output limitation was taken into account in the controller design, improvement of control performance would be achieved without changing actuator size.

The bang-bang control method⁽¹⁾ can be used to obtain a control input that moves an initial state to an objective state in a predetermined time. The two-point boundary value problem can be solved using various nonlinear programs that take into account the limitation of the input amplitude, the state constraints

and the final time^{(2),(3)}. These control methods, however, generate a discontinuous input, which, in turn, excites vibration modes in a mechatronic system. It is necessary to obtain a smooth control input that will not excite the vibration modes and that satisfies the limitation of the input amplitude.

The final-state control method for obtaining a feedforward input in order to bring a system from a certain initial state to a specified final state in a predetermined time has been proposed⁽⁴⁾ and applied to a three-link brachiation robot, that exhibits nonlinearity depending on its motion⁽⁵⁾. Those studies, however, did not determine the appropriate final time so that the obtained input does not exceed the limitation of the input amplitude of the actuators used in the experiment.

Based on these viewpoints, the author proposes a control method that can bring a system to an arbitrary final state even if the saturation of input takes place, by adopting a hyperbolic tangential function that describes the input amplitude limitation. Furthermore, because the hyperbolic tangential function can be differentiated, the derivatives of the function make the obtained input continuous even at the terminal

* Received 15th November, 1999

** Department of Electronics and Mechanical Engineering, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan. E-mail: nism@meneth.tn.chiba-u.ac.jp

Fig. 1 Hyperbolic tangential function

points of an initial time and a final time. A feedforward input with saturation is obtained for the swing-up motion control of a cart and inverted pendulum system. The results of simulations and experiments verify the usefulness of the proposed final-state control method.

2. Final-State Control with Input Constraint

2.1 Linear parameter-dependent system with input limitation

Let us consider an n -dimensional and multiple-input linear parameter-dependent system :

$$\begin{aligned} \dot{x}_f &= A_f(x_f)x_f + B_f(x_f)u, \\ y &= C_f(x_f)x_f, \end{aligned} \quad (1)$$

where u is an $m \times 1$ vector $[u_1 \ u_2 \ \dots \ u_m]^T$ and each input has an input constraint whose upper limit is $\pm \alpha_i$ ($i=1, 2, \dots, m$). Although this constraint is generally described by a discontinuous saturation function, the author employs the following hyperbolic tangential function shown in Fig. 1

$$u_i = f(u_{oi}) = \alpha_i \tanh\left(\frac{u_{oi}}{\alpha_i}\right), \quad i=1, 2, \dots, m, \quad (2)$$

because the hyperbolic tangential function is continuous and can be differentiated. Using Eq. (2), the state equation (1) can be rewritten as follows:

$$\dot{x}_f = A_f(x_f)x_f + B_f(x_f) \begin{bmatrix} \alpha_1 \tanh\left(\frac{u_{o1}}{\alpha_1}\right) \\ \vdots \\ \alpha_m \tanh\left(\frac{u_{om}}{\alpha_m}\right) \end{bmatrix}. \quad (3)$$

Applying the final-state control^{(4),(5)} to Eq. (3) makes the obtained input discontinuous at terminal points of the initial time and the final time. Thus, the author employs the derivative input $\dot{u}_{oi} = du_{oi}/dt$ as the input to the augmented system, in order to avoid the sudden change in input u_{oi} at terminal points, which may excite the vibration system included in the

Fig. 2 Input constraint with integrator

Fig. 3 Function of sech^2

mechanical system.

From Eq. (2), the time derivative $\dot{u}_i = du_i/dt$ is given by :

$$\frac{du_i}{dt} = \frac{df(u_{oi})}{du_{oi}} \frac{du_{oi}}{dt} = \text{sech}^2\left(\frac{u_{oi}}{\alpha_i}\right) \frac{du_{oi}}{dt}. \quad (4)$$

The block diagram of Eq. (4) is shown in Fig. 2 and the state-space description can be written as follows :

$$\dot{u} = \frac{df(u_o)}{du_o} \dot{u}_o, \quad (5)$$

where

$$\begin{aligned} \frac{df(u_o)}{du_o} &= \text{diag}\left[\frac{df(u_{o1})}{du_{o1}}, \dots, \frac{df(u_{om})}{du_{om}}\right] \\ &= \text{diag}\left[\text{sech}^2\left(\frac{u_{o1}}{\alpha_1}\right), \dots, \text{sech}^2\left(\frac{u_{om}}{\alpha_m}\right)\right]. \end{aligned} \quad (6)$$

Figure 3 shows the function of the square of hyperbolic secant.

From Eqs. (1) and (5), the augmented system using the virtual feedforward input $\dot{u}_o$ is

$$\dot{x} = A(x)x + B(u_o)\dot{u}_o, \quad (7)$$

where

$$x = \begin{bmatrix} x_f \\ u \end{bmatrix}, \quad A(x) = \begin{bmatrix} A_f(x_f) & B_f(x_f) \\ 0 & 0 \end{bmatrix},$$

$$\text{and } B(u_o) = \begin{bmatrix} 0 \\ \frac{df(u_o)}{du_o} \end{bmatrix}.$$

If $\dot{u}_o$ is determined, the actual feedforward input u given to the real system is obtained from the augmented state variables x . It is worth noting that

the amplitude of the input u is limited by a hyperbolic tangential function of u_o .

2.2 Final-state control with error learning⁽⁵⁾

The n -dimensional discrete-time system, which is digitized from the state equation of a linear parameter-varying system such as Eq. (7) in sampling period Δt using the Padé approximation, is described as follows:

$$x_{k+1} = A_k x_k + B_k u_{dk}, \quad (8)$$

where

$$\begin{aligned} A_k &= \left(I - \frac{1}{2} A(x_k) \Delta t \right)^{-1} \left(I + \frac{1}{2} A(x_k) \Delta t \right), \\ B_k &= \Delta t \left(I - \frac{1}{2} A(x_k) \Delta t \right)^{-1} B(u_o(k \Delta t)), \\ x_k &= x(k \Delta t) = \begin{bmatrix} x_f(k \Delta t) \\ u(k \Delta t) \end{bmatrix}, \text{ and } u_{dk} = \dot{u}_o(k \Delta t). \end{aligned}$$

When the initial state is x_o and an arbitrary input u_o is given to the augmented system (8), the state transition is described as follows:

$$\begin{aligned} x_1 &= A_0 x_0 + B_0 u_{d0} \\ x_2 &= A_1 x_1 + B_1 u_{d1} \\ &= A_1 A_0 x_0 + A_1 B_0 u_{d0} + B_1 u_{d1} \\ &\vdots \\ x_N &= A_{N-1} A_{N-2} \cdots A_0 x_0 + U_{NL} U_d \end{aligned} \quad (9)$$

where

$$U_{NL} = [A_{N-1} A_{N-2} \cdots A_1 B_0, A_{N-1} \cdots A_2 B_1, \cdots, A_{N-1} B_{N-2}, B_{N-1}] \quad (10)$$

and

$$U_d = \text{col}(u_{d0}, u_{d1}, \cdots, u_{dN-1}). \quad (11)$$

Now, let the desired state be

$$x_N = x^o. \quad (12)$$

Then, the last equation (9) can be expressed as a linear equation:

$$y_{NL} = U_{NL} U_d, \quad (13)$$

where $y_{NL} = x^o - A_{N-1} A_{N-2} \cdots A_0 x_0$. Equation (13) means that if the input vector U_d which satisfies this equation has been given to the system, the system reaches state x^o exactly at time step N . The solution of Eq. (13) is not unique, but the following solution,

$$U_d = U_{NL}^T (U_{NL} U_{NL}^T)^{-1} (x^o - A_{N-1} A_{N-2} \cdots A_0 x_0) \quad (14)$$

is to minimize the squared norm of input among all solutions. If the transition matrices A_k ($k=0, 1, \cdots, N-1$) do not change, Eq. (14) is the solution of the virtual feedforward input. The actual feedforward input u can be given from the augmented state variables⁽⁵⁾. However, if the feedforward input (14) is given to the parameter-varying system (8), the final state may be different from the desired state, because by feedforward input (14), the transition matrices A_k and the input matrices B_k change from those in Eq. (9) obtained by an arbitrary feedforward input. By adding feedforward input (14) into system (8), the final state becomes as follows from the same procedure as Eq. (9):

where the superscript ' explicitly denotes the change in vector or matrix via the change in state transition and

$$x'_N = A'_{N-1} A'_{N-2} \cdots A'_0 x_0 + U'_{NL} U_d, \quad (15)$$

From Eq. (15), the following solution can be obtained again:

$$U'_d = [A'_{N-1} A'_{N-2} \cdots A'_1 B'_0, A'_{N-1} \cdots A'_2 B'_1, \cdots, A'_{N-1} B'_{N-2}, B'_{N-1}].$$

These changes cause the following final error:

$$U_d = U_{NL}^T (U_{NL} U_{NL}^T)^{-1} (x'_N - A'_{N-1} A'_{N-2} \cdots A'_0 x_0) \quad (16)$$

$e = x^o - x'_N$.

$$e = x^o - x'_N. \quad (17)$$

In order to modify the error, the following error compensation input is added to Eq. (16):

$$\Delta U_d = U_{NL}^T (U_{NL} U_{NL}^T)^{-1} e, \quad (18)$$

and, by substituting the feedforward input (19)

$$U_d + \Delta U_d = U_{NL}^T (U_{NL} U_{NL}^T)^{-1} \times (x^o - A'_{N-1} A'_{N-2} \cdots A'_0 x_0) \quad (19)$$

for U_d in Eq. (15), the final state reaches the desired state, if the state transition matrices and the input matrices do not change. However, if input (19) $U_d + \Delta U_d$ becomes markedly different from U_d , the change in state transition occurs and consequently the final error cannot converge to a tolerance value by the error compensation input (18).

We employ the learning coefficient γ ($0 \leq \gamma \leq 1$) in order to suppress the change in input and state transition. By using the input

$$U_d \leftarrow U_d + \gamma \Delta U_d \quad (20)$$

the final state x''_N becomes

$$x''_N = A''_{N-1} A''_{N-2} \cdots A''_0 x_0 + U''_{NL} (U_d + \gamma \Delta U_d). \quad (21)$$

If, $A''_k \cong A'_k$ and $U''_{NL} \cong U'_{NL}$, Eq. (21) is rewritten as $x''_N = (1 - \gamma) x'_N + \gamma x^o$. Therefore, the convergence property of the final error after learning of iteration times h is as follows:

$$x_N^{(h+1)} \cong (1 - \gamma)^h x'_N + \left(\sum_{i=1}^h (1 - \gamma)^{(i-1)} \gamma \right) x^o, \quad (22)$$

where it can be assumed that the change in the input and state transition is small, that is,

$$U_{NL}^{(h+1)} \cong U_{NL}^{(h)}. \quad (23)$$

The superscript (h) of the vectors and the matrices in Eqs. (22) and (23) denotes the h th renewal and the superscript ' and '' in Eqs. (15)–(21) means $h=1$ and $h=2$, respectively. Figure 4 shows the flowchart for feedforward input calculation. From Eq. (22), when the renewal times h become sufficiently large, the final state $x_N^{(h+1)}$ converges to the objective final state x^o . When the change in state transition due to the change in input is small, the learning coefficient γ near 1 can be used and the convergence speed becomes high. In contrast, if the change in state transition is large, the learning coefficient γ should approach zero.

Fig. 4 Flowchart for calculation of feedforward input

3. Application to Inverted Pendulum

This study treats the cart and inverted pendulum system, as shown in Fig. 5. The notations are as follows.

- m_c : equivalent mass of cart
- G_m : gain coefficient of motor
- u : control voltage input to motor
- d : damping coefficient of cart
- m : mass of pendulum
- j : moment of inertia of pendulum
- l : distance from rotational center to center of gravity
- L : length of pendulum

The specifications of the system are shown in Table 1.

The governing equation can be written as follows:

$$M(\theta)\ddot{X} + C\dot{X} + K(\theta, \dot{\theta})X = Gu, \quad (24)$$

where

Table 1 Specifications of the cart and inverted pendulum system

parameter	value
m_c	1.57 kg
G_m	11.3 kg · m/s ² · V
d	4.55 N · s/m
m	0.11 kg
j	0.00086 kg · m ²
l	0.15 m
L	0.3 m
g	9.81 m/s ²
Δt	0.01 s

Fig. 5 Cart and inverted pendulum system

$$M(\theta) = \begin{bmatrix} m_c + m & ml \cos \theta \\ ml \cos \theta & j + ml^2 \end{bmatrix}, \quad C = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix},$$

$$K(\theta, \dot{\theta}) = \begin{bmatrix} 0 & -ml\dot{\theta}^2 \frac{\sin \theta}{\theta} \\ 0 & gml \frac{\sin \theta}{\theta} \end{bmatrix}, \quad G = \begin{bmatrix} G_m \\ 0 \end{bmatrix},$$

$$\text{and } X = \begin{bmatrix} s \\ \theta \end{bmatrix}$$

using the extended linearization method⁽⁶⁾ and when $\theta=0$, it is assumed that

$$\frac{\sin \theta}{\theta} = 1.$$

This linearization method transforms the nonlinear pendulum system to the linear parameter-dependent system. $M(\theta)$ and $K(\theta, \dot{\theta})$ include nonlinear terms; in particular, the term $K(\theta, \dot{\theta})X$ is produced by the extended linearization method.

From Eq.(24), the state-space description is given by:

$$\dot{x}_f = A_f(x_f)x_f + B_f(x_f)u, \quad (25)$$

Fig. 6 Response by final-state control without input constraint from 0 rad to π rad

where

$$A_f(x_f) = \begin{bmatrix} 0_{2 \times 2} & I_{2 \times 2} \\ -M^{-1}(\theta)K(\theta, \dot{\theta}) & -M^{-1}(\theta)C \end{bmatrix},$$

$$B_f(x_f) = \begin{bmatrix} 0_{2 \times 1} \\ -M^{-1}(\theta)G \end{bmatrix}, \text{ and } x_f = \begin{bmatrix} X \\ \dot{X} \end{bmatrix}.$$

The swing-up motion control of the pendulum system is to bring the pendulum from $\theta=0$ rad to $\theta=\pi$ rad. Figure 6 shows the result of motion control by the final-state control without an input constraint. Although the pendulum is brought to $\theta=\pi$ rad, the cart displacement exceeds 1.2 m. Since the possible displacement of the cart is from -0.5 m to 0.5 m, it may be difficult to perform the swing-up motion at once. Also, when the final-state control with an input constraint is used, no feedforward input can be obtained, which satisfies the input and displacement constraints. Thus, the purpose of this study is to bring the pendulum from the initial state to the objective state in two steps. In the first step, the control input brings the pendulum to $\theta=2\pi/3$ rad, reducing the objective state by one-third, and in the second step, another control input brings the pendulum to the inverted state. Even if the final state is $\theta=2\pi/3$ rad, the solution obtained from the final-state control without input constraints is not appropriate for the cart and inverted pendulum system, as shown in Fig. 7. The displacement of the cart exceeds the limit. Although the objective final state and the final time are arbitrary, as an example, the final state is selected to be $2\pi/3$ rad and the final time is 2.3 sec. Even if the final state were $\pi/2$ rad, the main results would not change.

Thus, the final-state control with input constraint was applied to the swing-up motion control of the cart and inverted pendulum system. Assuming that the limitation of input amplitude is ± 0.8 V, learning was

Fig. 7 Response by final-state control without input constraint from 0 rad to $2\pi/3$ rad

Fig. 8 Error convergence property

iterated with the learning coefficient $\gamma=0.1$. The convergence property of the squared error between the final state and the objective state is shown in Fig. 8. At about the 130th learning iteration, the squared error converges to 10^{-6} . The time response is shown in Fig. 9. The amplitude of the input voltage is limited to ± 0.8 V and the limitation of the displacement of the cart is satisfied. This final-state control can be used in the experiment.

4. Experimental Results

Figures 10(a) and (b) show the front view and the side view of the experimental setup used, respectively. In the experimental setup, the cart is supported by a linear-slide bearing and driven by a motor through a timing belt. Two rotary encoders were used, one detecting the rotational angle of the pendulum and the other, the cart displacement. Figure 11 shows the experimental result of the swing-up motion of the cart and inverted pendulum system. The dotted

Fig. 9 Response by final-state control with input constraint

(a) Front view

(b) Side view

Fig. 10 Experimental setup

Fig. 11 Experimental result of swing-up motion

lines in the cart displacement s and the swing angle θ indicate the simulation result shown in Fig. 9. In the initial approach, the constrained feedforward input by 0.8 V brings the rotational angle of the pendulum to $2\pi/3$ rad in 2.3 sec and finally this state is brought to the inverted state by the appropriate feedforward input in 0.8 sec. In the final approach, final-state control is performed without input constraint. After the final state of 3.1 sec, a linear quadratic regulator is used for stabilizing the pendulum. From Fig. 11, it is seen that there are differences between the experimental results and the simulation results of the cart displacement and the swing angle. These differences are due to the error between the mathematical model and the experimental setup. Although feedforward control is generally less robust than feedback control, it is verified that the final-state control is sufficiently robust to bring the nonlinear system to the objective final state.

5. Conclusions

The design methodology of the final-state control is shown, taking into account input constraint by using the hyperbolic tangential function. The introduction of the time derivative of the continuous function generated a feedforward input without sudden changes at the terminal points of the initial time and the final time. This control method was applied to the swing-up motion control of the cart and inverted pendulum system, and the results verified the usefulness of the method in the motion control of the system with input constraints.

References

- (1) Lewis, F., Optimal Control, (1986), p. 252-282, Wiley-Interscience.
- (2) Furukawa, T., Time-Subminimal Trajectories for

- Discrete Nonlinear Systems, The 75th JSME Spring Annual Meeting, No. 98-1 (1998), p. 125-126.
- (3) Furukawa, T., Lee, K.Y. and Dissanayake, M.W. M.G., A Method for Sub-Minimal-Time Trajectory Planning of Redundant Dual Manipulator Systems, Journal of the Robotics Society of Japan, Vol. 13, No. 4 (1995), p. 532-538.
- (4) Totani, T. and Nishimura, H., Final-State Control Using Compensation Input, Trans. Soc. Instrument and Control Engineers, Vol. 30, No. 3 (1994), p. 253-260.
- (5) Nishimura, H. and Funaki, K., Motion Control for Three-Link Brachiation Robot by Using Final-State Control with Error Learning, IEEE/ASME Transactions on Mechatronics, Vol. 3, No. 2 (1998), p. 120-128.
- (6) Frieland, B., Advanced Control System Design, (1996), p. 110-112, Prentice Hall.
-