Two-Dimensional Numerical Simulation and Experiment on Large Deformation of Collapsible Tube*

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By using the tube law, which describes the relationship between the cross-sectional area of a collapsible tube and the transmural pressure, which is the difference between inside pressure and outside pressure, a coupled problem between flow in the tube and wall deformation is formulated. In this paper, a numerical method for calculating two-dimensional deformation of the tube is presented and the results are compared with experimental results. The calculation results are compared with those of Ref. (6) to show the validity of our calculation method in which tube wall thickness is taken into account in determining the contact behavior of the tube wall. In the experiment, we use a silicone rubber tube as the collapsible tube. The tube's cross-sectional shape is visualized by using the laser light sheet method. It is shown experimentally that the tube law calculated by dividing the inner volume of the tube by its length has a large error in the wall contact region. In the calculation in Ref. (6), tube wall thickness is not treated rigorously in the wall contact region. However, according to our results, the effect of wall thickness on the tube law is not negligible in the wall contact region.

Key Words: Biomechanics, Numerical Analysis, Buckling, Collapsible Tube, Visualization

1. Introduction

A soft, elastic tube with a thin wall, which is called a collapsible tube and is exemplified by a venule or a natural rubber tube, is markedly deformed under pressure, i.e., when the transmural pressure p_{tr} (inside pressure minus outside pressure) becomes negative. When there is a flow through the collapsible tube, the shape of the tube and the flow through the tube interact with each other. It is a complicated phenomenon⁽¹⁾ and not observed in a rigid tube.

In the study of this phenomenon, we often use the tube law which shows the relationship between the tube's cross-sectional area and the transmural pres-

sure. By using the tube law, the coupled problem between flow and tube wall becomes simply a one-dimensional problem^{(3),(4)}. The cross-sectional area used in the tube law is obtained not by dividing volume by length⁽²⁾, but by using the local cross-sectional shape. This is why the former causes some errors due to the restriction at the end of the tube.

Tadjbakhsh et al.⁽⁵⁾ and Flaherty et al.⁽⁶⁾ reported the calculation of cross-sectional shape when pressure is applied to a tube. Shapiro⁽³⁾ and Skalak⁽⁷⁾ examined how appropriate the numerical model is by comparing it with theoretical analysis. Dion et al.⁽⁸⁾ computed the tube law using a tube whose initial shape is elliptical.

In this study, in order to know how the thickness of the tube wall influences the model, we formulate an equation in which the model of Tadjbakhsh et al.⁽⁵⁾ and Flaherty et al.⁽⁶⁾ and the thickness are combined and considered. We compare the tube law obtained by conducting an experiment that visualizes the shape of the cross section with that obtained by dividing the volume inside the tube by the length. In addition, we consider the difference between experimental results and calculated ones.

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Fig. 1 Forces, moment and transmural pressure exerting on tube wall

2. Numerical Methods

2.1 Model

When a collapsible tube buckles, its deformation becomes different from that before buckling. Thus, we use two kinds of models. Here, the buckling transmural pressure is $-3^{(5)}$.

2.1.1 Pre-buckling model In the case of pre-buckling analysis, we calculate the deformation of the tube by using Eq. (1) which is modified the Equation⁽⁹⁾ for the small deformation of a thick tube to adapt thin tube.

$$\Delta R = \frac{(1+\nu)(R-t)p_{tr}}{Et} \{R(1-\nu)+t\}, \quad (1)$$

where p_{tr} is transmural pressure, R is mean radius when transmural pressure $p_{tr}=0$, E is Young's modulus, ν is Poisson's ratio, and t is thickness. We compute the cross-sectional area from the obtained ΔR .

2.1.2 Post-buckling model We assume the following for the model.

- 1) When transmural pressure is higher than buckling pressure, the cross section of the tube is circular.
- 2) The cross-sectional shape of the tube along the axial direction is uniform.
- 3) The perimeter of tube wall is inextensible.
- 4) The tube wall is homogeneous and elastic.
- 5) The tube is very thin and is deformed under Kirchhoff-Love's assumption.
- 6) Pressure is uniform in the circumferential direction.

Figure 1 shows internal pressure p_i , external pressure p_e , tension force F_t , shear force F_N and bending moment M on tube wall. The angle $d\theta$ is made by the unit perimeter length ds .

The equilibrium equations are as follows.

$$\frac{dF_t}{ds} + F_N \frac{d\theta}{ds} = 0, \quad (2)$$

Fig. 2 Cartesian coordinates (x, y) and curvilinear coordinates $(\tilde{s}, \theta)$

$$\frac{dF_N}{ds} - F_t \frac{d\theta}{ds} + p_{tr} = 0, \quad (3)$$

$$\frac{dM}{ds} - F_N = 0, \quad (4)$$

$$M = D \left(\frac{d\theta}{ds} - \frac{1}{R} \right), \quad (5)$$

where $D = Et^3/12(1-\nu^2)$ is the flexural rigidity modulus.

Combining Eqs. (2), (3), (4) and (5) gives

$$D \frac{d^3\theta}{ds^3} + \frac{D}{2} \left(\frac{d\theta}{ds} \right)^3 - c \frac{d\theta}{ds} + p_{tr} = 0, \quad (6)$$

where c is an integral constant. Dimensionless variables $\tilde{s} = s/R$, $\tilde{k} = R d\theta/ds$, and $\tilde{p}_{tr} = p_{tr}R^3/D$ allow us to transform Eq. (6) into Eq. (7).

$$\frac{d^2\tilde{k}}{d\tilde{s}^2} + \frac{1}{2}\tilde{k}^3 - c' \tilde{k} + \tilde{p}_{tr} = 0, \quad (7)$$

where c' is a dimensionless integral constant.

The relationship between curvature $\tilde{k}$ and deformed shape is shown in Fig. 2. Figure 2 leads to Eq. (8).

$$\begin{aligned} x(\tilde{s}) &= \tilde{R} \int_0^{\tilde{s}} \cos \left(\int_0^{\tilde{s}} \tilde{k} d\tilde{s} \right) d\tilde{s} + x(0), \\ y(\tilde{s}) &= \tilde{R} \int_0^{\tilde{s}} \sin \left(\int_0^{\tilde{s}} \tilde{k} d\tilde{s} \right) d\tilde{s} + y(0). \end{aligned} \quad (8)$$

x and y in Eq. (8) are coordinates on mean surfaces. Owing to thickness consideration, we can construct coordinates x and y of the inner or outer line from x and y in Eq. (8).

Equation (7) requires three boundary conditions for it to become two ordinary differential equations with the unknown constant c' . We only calculate for the two-lobe mode (see Fig. 2) which is observed experimentally. For periodicity exists, we may calculate for a part of perimeter of the cross section divided by four, that is, $0 < \tilde{s} < \pi\tilde{R}/2$. We need the appropriate boundary conditions, which satisfy the symmetry. We must choose different boundary

conditions according to the following three modes.

Mode 1. before contact

Mode 2. contact at one point

Mode 3. contact line

In Mode 1,

$$\left. \frac{d\tilde{k}}{d\tilde{s}} \right|_{\tilde{s}=0} = 0, \quad \left. \frac{d\tilde{k}}{d\tilde{s}} \right|_{\tilde{s}=\frac{\pi}{2}} = 0, \quad \int_0^{\pi/2} \tilde{k} d\tilde{s} = \frac{\pi}{2}. \quad (9)$$

The first and second equations in the Eq.(9) correspond to the case in which the shear force is 0 at boundary points. The third equation in Eq.(9) implies that the angle made by two tangential lines at both ends is $\pi/2$. To fulfill the inextensible condition, we set integral range to uniform.

In mode 2,

$$\left. \frac{d\tilde{k}}{d\tilde{s}} \right|_{\tilde{s}=0} = 0, \quad x\left(\frac{\pi}{2}\right) = \frac{t}{2}, \quad \int_0^{\pi/2} \tilde{k} d\tilde{s} = \frac{\pi}{2}. \quad (10)$$

Here the contact force at contact point $\tilde{s} = \pi/2$. Therefore, we can use the fixed position condition instead of the second equation in Eq.(9).

In mode 3,

$$\left. \frac{d\tilde{k}}{d\tilde{s}} \right|_{\tilde{s}=0} = 0, \quad x(\tilde{s}_c) = \frac{t}{2}, \quad \int_0^{\tilde{s}_c} \tilde{k} d\tilde{s} = \frac{\pi}{2}, \quad \tilde{k}(\tilde{s}_c) = 0, \quad (11)$$

where s_c is non-contact length. Because s_c is an unknown value, we must treat four boundary conditions.

2.2 Procedure for numerical simulation

In the case of the non-contact condition, we can eliminate c' from Eq.(7). Equation (12) is the integration of Eq.(7) from 0 to $\pi/2$.

$$\frac{d\tilde{k}}{d\tilde{s}}\left(\frac{\pi}{2}\right) - \frac{d\tilde{k}}{d\tilde{s}}(0) + \frac{1}{2} \int_0^{\pi/2} \tilde{k}^3 d\tilde{s} - c' \int_0^{\pi/2} \tilde{k} d\tilde{s} + \tilde{p}_{tr} \frac{\pi}{2} = 0. \quad (12)$$

Equation (13) is obtained by substituting Eq.(9) into Eq.(7).

$$c' = \tilde{p}_{tr} + I, \quad (13)$$

where

$$I = \frac{\int_0^{\pi/2} \tilde{k}^3 d\tilde{s}}{\pi}. \quad (14)$$

Equation (15) is obtained by substituting Eq.(13) into Eq.(7).

$$\frac{d^2\tilde{k}}{d\tilde{s}^2} + \frac{1}{2}\tilde{k}^3 - (\tilde{p}_{tr} + I)\tilde{k} + \tilde{p}_{tr} = 0. \quad (15)$$

Equation (16) is the discrete equation of Eq.(15).

$$\frac{\tilde{k}_{m+1}^i - 2\tilde{k}_m^i + \tilde{k}_{m-1}^i}{\Delta s^2} - (\tilde{p}_{tr} + I^{i-1})\tilde{k}_m^i = -\frac{1}{2}\tilde{k}_m^{i-1} - \tilde{p}_{tr}. \quad (16)$$

where Δs indicates a small section. Subscript m denotes the discrete point number which is uniformly distributed at $[0, \pi/2]$. The distribution span is expressed as perimeter length divided by N . Superscript i denotes iteration step number. Equation (16) with the first and second equation in Eq.(9) at all discrete

Table 1 Comparison of present results with those in Ref.(6)

	Present	Ref. (6)
One-point contact pressure	-5.248	-5.247
Osculation pressure	-10.340	-10.337
Non-contact length at $\tilde{p}_{tr} = -20.0$	1.261	1.261

Fig. 3 Contact length of tube wall (Wall thickness $t=0$)

points makes an $N+1$ ary band matrix. By inserting a known value of k into the non-linear term k^3 in Eq.(16), we can treat this equation as a linear one.

The computation procedure is shown below.

1) Curvature $\tilde{k}^i$ is computed from linearized simultaneous equation (16).

2) When the difference between $\tilde{k}^i$ in 1) and known value $\tilde{k}^{i-1}$ is negligible, $\tilde{k}^i$ is solution of Eq.(16).

3) When the difference is not negligible, we insert $\tilde{k}^i$ in $\tilde{k}^{i-1}$, and reform procedure 1) again. In the case of calculation for a contacting wall, it is impossible to eliminate c' in Eq.(7). This is why the boundary conditions are different.

We use the following equation for calculation.

$$\frac{\tilde{k}_{m+1}^i - 2\tilde{k}_m^i + \tilde{k}_{m-1}^i}{\Delta s^2} + \frac{1}{2}\tilde{k}_m^{i-1} - c'\tilde{k}_m^i + \tilde{p}_{tr} = 0. \quad (17)$$

Equation (17) with boundary conditions is solved using Broyden's method⁽¹⁰⁾.

3. Calculation Results

To confirm the effectiveness of the calculation method, comparison with the results ((a) one-point contact pressure, (b) osculation pressure, (c) non-contact length at $\tilde{p}_{tr} = -20.0$) in Ref.(6) is shown in Table 1. Our results are almost the same as those in Ref.(6). An accurate solution can be obtained using our proposed method.

Figure 3 shows the relationship between trans-

Fig. 4 Integral constant c' and initial curvature $\bar{k}(0)$ at various transmural pressures $\bar{p}_{tr}$

Fig. 5 Post-buckling deformation of tube (inner diameter is 1, wall thickness is 0.1)

mural pressure $\bar{p}_{tr}$ and contact length of tube wall L_c . When wall thickness is neglected, contact length can be solved using similarity solution⁽⁶⁾. Good agreement is obtained between our results and the results of the similarity solution, as shown in Fig. 3.

Figure 4 shows the plots of c' and $\bar{k}(0)$, which is k at 0, vs. $\bar{p}_{tr}$. Substituting c' and $\bar{k}(0)$ into Eq. (7) is solved by the Runge-Kutta method. A solution can be reached without using iterative calculation. We compute by using four ordinary Runge-Kutta method with $\bar{k}(0)=2.970$ and $c'=-1.017$ at $\bar{p}_{tr}=-4.0$.

$$\begin{aligned} \bar{k}'(\pi/2) &= -4.564 \times 10^{-4}, \\ \int_0^{\pi/2} \bar{k} d\bar{s} - \pi/2 &= 1.905 \times 10^{-5}. \end{aligned} \quad (18)$$

The above results are almost equal to Eq. (9).

Figure 5 shows the calculated cross-sectional shapes of the tube. The ratio of tube diameter to tube thickness is 10. The cross sectional shape is oval in

Fig. 6 Effect of tube wall thickness on tube law

Fig. 5(a). The wall span is gradually narrow in Fig. 5(b). The shape of tube is separated by one point of contact in Fig. 5(c). When $\bar{p}_{tr}$ is below -9.649 , the contacting part is not a one-point contact but a line contact. (Fig. 5(e)) When 3-D fluid flow through the cross-sectional shapes in Fig. 5 is analyzed, flow is found to widely change at the markedly deformed part. Therefore, the calculation grid resolution requires very high accuracy. Since the contact point is a singular point in the case of grid generation, it is difficult to keep the grid rectangular.

Figure 6 shows the relationship between non-dimensional transmural pressure and nondimensional area which is defined as the ratio of tube area to lumen area A_0 at transmural pressure $\bar{p}_{tr}=0$. In order to investigate the influence of wall thickness, we calculate the tube law when $d_i/t=10$ and 100. The double-dashed line represents the pre-buckling region, the solid line represents the buckling region before wall contact, the broken line represents the one-point contact regions and the dashed line represents the line contact region. The triangle, white circles and black circles show each boundary point. In the region from the start of buckling to contact, very small changes in transmural pressure induce large changes in cross-sectional area. Large changes in incline exist on boundary point, which is both the buckling point and contacting point. There is no difference in tendency between the two curves. However, in the contact region, nondimensional lumen cross-sectional area of $d_i/t=10$ is smaller.

4. Experimental Method

We use a silicone rubber tube (Fuji Systems Co.) as the collapsible tube. The nominal diameter and thickness of the tube are as follows: (a) inner diameter=7.6 mm, thickness=0.25 mm; (b) inner diameter=6.0 mm, thickness=0.3 mm; and (c) inner diameter=6.0 mm, thickness=0.6 mm. A tube of

Fig. 7 Calculated shapes and experimental images of deformed collapsible tube (inner diameter 7.6 mm, wall thickness 0.25 mm)

length 160 mm, is placed in an acrylic box. Both ends of the tube are connected to two circular rigid tubes. We let the outer diameter of the acrylic tube be equal to the inner diameter of the collapsible tube. The concentration of sucrose solution that surrounded the tube is 40.6%⁽¹¹⁾. The tube is also filled with the same solution. The outside pressure of the tube is computed from the liquid level and the inside pressure is measured by a manometer, which is mounted on the acrylic tube. Tube deformation is caused by the inner pressure change of the tube. The illuminated cross-sectional shape by a laser light sheet is recorded using a CCD camera⁽¹¹⁾, where tube axis is perpendicular to the plane of the laser light sheet.

5. Experimental Results

Figure 7 shows bit map images of tube cross section and the calculated results (solid curves). Transmural pressure is (a) -0.13 kPa, (b) -0.42 kPa, (c) -0.97 kPa, and (d) -1.37 kPa, while non-dimensional pressure is (a) -1.46 , (b) -4.88 , (c) -11.28 , and (d) -15.95 . It was impossible to record from the direction perpendicular to the tube cross section in order that rigid tube and manometer is set on its direction. For images that were recorded from a diagonal direction, the images become wide. Images

Fig. 8 Tube laws obtained using different cross-sections from acrylic tube and that obtained by dividing tube volume by tube length

in Fig. 7 are, however, corrected⁽¹²⁾. It is shown experimentally that the cross-sectional shape is also largely deformed by transmural pressure changes. Because of the unclear outline, the obtained lumen cross-sectional areas from these images have errors.

In the case of the tube with an inner diameter of 7.6 mm and a wall thickness of 0.25 mm, measurements were made on its cross section such that the distance from the part of the connection between the collapsible tube and the rigid tube is 5 mm, 15 mm, 40 mm or 80 mm. Figure 8 shows the plots of non-dimensional axis $\tilde{p}_{tr}$ vs. $\tilde{A}$. Agreement of the results for $z = 15$ mm, 40 mm and 80 mm indicates two-dimensional deformation at the center of the cross section. To test numerical model, which is based on two-dimensional problem, it is necessary to compare its results with experimental results obtained from the center of the tube cross section. Results of the tube law, which is obtained by dividing inner volume V by its tube length $L^{(2),(3)}$, are shown in Fig. 8. In the contact region, the pressure at the center of the tube cross section is higher than that obtained by V/L . Tube deformation is a three-dimensional phenomenon, therefore, one-dimensional analysis using the tube law would lead to error around the part of connection. In the case of inner flow analysis, its error influences the inlet and/or outlet boundary of the tube.

6. Comparison of Numerical Results with Experimental Results

For the collapsible tube of different sizes, we compare the numerical results with experimental ones. We use a Young's modulus of 3.0 Mpa, which is measured by an autograph (AGS-1 kNG, Shimadzu

Fig. 9 Calculated and experimental tube laws for different ratios of inner diameter to wall thickness

Co.), and a Poisson's ratio of 0.5 is based on incompressible assumption. In the contact region ($\tilde{A}$ smaller than 2), the numerically obtained cross-sectional area is larger than the experimentally obtained one. The difference is due to initial ellipticity⁽⁸⁾, error of the measurement area and asymmetrical deformation. Initial ellipticity is defined as the major-to-minor axis ratio. The ellipticity for a tube of inner diameter 7.6 mm and wall thickness 0.25 mm is 1.15, that for a tube of inner diameter 6.0 mm and wall thickness 0.3 mm is 1.48, and that for a tube with inner diameter 6.0 mm and wall thickness 0.6 mm is 1.20. Because the outline of the tube wall is not clear, it is difficult to recognize the boundary between the liquid and the tube wall. Especially, in the case of a very narrow wall span, the estimated area is smaller than the actual one. Therefore, the error occurs in smaller areas in Fig. 9. When connecting the collapsible tube to the acrylic tube, non-uniformity caused by asymmetrical deformation is not observed in the calculated results. This is why the numerical model is based on the assumption of symmetry.

7. Conclusions

We calculated changes in cross-sectional shape of a collapsible tube when pressure applied to the tube changes (i.e., tube law). Changes in the cross-sectional shape are measured by a visualization experiment using the laser light sheet method, too. The obtained results are as follows:

(1) In the case of the tube law having wall thickness, analysis of the contact region is necessary when the numerical method is used so that the similarity

solution does not exist. An accurate solution can be obtained using our proposed method.

(2) In post-buckling analysis, the governing equations for tube deformation are non-linear ordinary differential equations. Using an unknown constant and initial value in this paper, the equations are solved by the Runge-Kutta method without the need for iterative calculation.

(3) In the case of a silicone rubber tube of 160 mm length, 0.25 mm thickness and 7.6 mm inner diameter, the tube law at more 15 mm from connecting parts is almost the same.

(4) For the above-mentioned tube, the result obtained by dividing tube volume by tube length has a large error in contact region.

(5) The computed lumen area from bit map images is smaller than the actual one. This is why unclear part of bit map images overlap when contact occurs.

(6) There is a difference between experimental and calculated results. In the experimental results, ellipticity is observed and the estimated area is small.

(7) Tube thickness is not considered in previous theoretical analysis. However, we show by experiments and calculation that wall thickness influences the contact region of the tube law.

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