1. Introduction
The primary objective of this paper is to develop a stable and accurate numerical model for wave propagation in heterogeneous media. It is well known that classical zero-order homogenization is valid for the case that wavelength $\lambda$ of traveling wave is much larger than characteristic length $l$ of the heterogeneity. However, if the wavelength $\lambda$ is comparable to the characteristic length $l$, the wave motion is affected by the heterogeneity due to successive reflection and refraction on interfaces, which can be observed as dispersion and polarization effects in macro scale. For such a case, high-order homogenization has been used to capture these effects\(^1\).

Fish and Chen\(^2,3\) have shown that high-order homogenization with two spatial scales produces bounded solution of stress as the time approaches infinity. To alleviate the problem of secularity they introduced multiple slow time scales in addition to two spatial scales. One solution of procedures for the resulting equations of macro scale is based on nonlocal approach, in which slow time scale can be eliminated. Fish et al.\(^4\) have validated the approach in one-dimensional case and for macroscopically isotropic medium in multiple dimensions.

In this paper, we propose a stabilized nonlocal model that is valid for macroscopically orthotropic case.

2. Summary of high-order homogenization theory
We start by briefly reviewing high-order homogenization theory with multiple spatial and time scales. Attention is restricted to macroscopically orthotropic case. The microstructure is assumed to be locally periodic. Let us consider macro coordinate system $x$ and micro coordinate system $y$ related by

\[
y=x/\varepsilon
\]

where $0<\varepsilon<1$. In addition, the following multiple slow time scales are introduced.

\[\tau_0 = t, \quad \tau_1 = \varepsilon t, \quad \tau_2 = \varepsilon^2 t\]

Asymptotic expansion with respect to displacement $u$ is considered as follows:

\[
u(x, y, \tau_0, \tau_1, \tau_2) = u^0(x, y, \tau_0, \tau_1, \tau_2) + \varepsilon u^1(x, y, \tau_0, \tau_1, \tau_2) + \varepsilon^2 u^2(x, y, \tau_0, \tau_1, \tau_2) + \cdots
\]

Substituting Eq. (3) into the governing equation of wave motion, taking perturbation with respect to $\varepsilon$ gives a set of micro and macro equilibrium equations. Due to the linearity of the micro equations and periodicity, the following decompositions of $u^0$, $u^1$, and $u^2$ are made:

\[
u^0(x, y, \tau_0, \tau_1, \tau_2) = u^0(x, \tau_0, \tau_1, \tau_2)
\]

\[
u^1(x, y, \tau_0, \tau_1, \tau_2) = u^1(x, \tau_0, \tau_1, \tau_2)
\]

\[
u^2(x, y, \tau_0, \tau_1, \tau_2) = u^2(x, \tau_0, \tau_1, \tau_2)
\]
next section, we present the C0 continuous finite element formulation with stabilization for macroscopically orthotropic case.

4. Stabilized nonlocal model

For simplicity, attention is restricted to constant mass density. In this case, Eq. (8) can be simplified as follows:

\[ \rho \ddot{U}_i - D_{ij}^0 (e_{cm}(U))_{x_j} - \epsilon^2 D_{ijkl}^0 (e_{cm}(U))_{x_j x_k} + O(\epsilon^0) = 0 \]  

(9)

We now focus on approximation of Eq. (9) in \( O(\epsilon^0) \) with \( C^0 \) continuous interpolation with stabilization. Multiplying the first of Eq. (5) by \( \epsilon^2 D^2 \) yields the relation

\[ \epsilon^2 D_{ijkl}^0 (e_{cm}(U))_{x_j x_k} = \epsilon^2 \rho \partial_i \epsilon_{ijkl}^0 (e_{cm}(U))_{x_j} + \epsilon^2 R_{ijkl}^0 (e_{cm}(U))_{x_j} + O(\epsilon^0) \]

(10)

Assuming that \( e_{\text{inel}}(U)_{x_j x_k} \) is differentiable, only symmetric part of the sixth-order tensor \( R_{ijkl} \) with respect to \( \rho \) and \( \tau \) affects the solution. We further decompose the tensor \( R \) as follows:

\[ \text{sym}(R) = R^* + R^* \]

(11)

\[ R_{ijkl}^* = \frac{1}{2} \left( (e_{\text{sym}}(P_{ijkl}^*) \sigma_{ijkl}^* + e_{\text{sym}}(P_{ijkl}^*) \sigma_{ijkl}^*) D_{ijkl} \right) \]

where \( R_{ijkl} \) and \( R_{ijkl}^* \) hold. Introducing assumed strain \( \epsilon_{ij} \) and assumed stress \( \sigma_{ij} \), the following approximation of Eq. (9) can be obtained.

\[ \left\{ \begin{array}{l}
\rho \ddot{U}_i - (\epsilon^2 \rho \partial_i \epsilon_{ijkl}^0 (e_{cm}(U))_{x_j}) = 0 \\
\epsilon_{ij} = D_{ijkl}^0 \epsilon_{kl}^0 + \epsilon^2 \partial_i \epsilon_{ijkl}^0 \sigma_{ijkl}^* \text{sym}(\partial_i \epsilon_{ijkl}^0) \sigma_{ijkl}^* \\
\end{array} \right. \]

(12)

where \( C^0 \) is compliance of the elastic tensor \( D^0 \). The corresponding Hamilton’s principle of Eq. (12) is given as

\[ \int_{\Omega} \left[ - \delta L(U, \dot{U}, \dot{\epsilon}, \sigma) - \int_{S_\Omega} \delta U_i \xi_i dS - \int_{S_\Omega} \delta \epsilon_{ij} \epsilon_{ijkl} \sigma_{ijkl} dS + \int_{S_\Omega} \delta \sigma_{ij} \epsilon_{ijkl} \sigma_{ijkl} dS \right] d\Omega = 0 \]

(13)

\[ L(U, \dot{U}, \dot{\epsilon}, \sigma) = \int_{\Omega} \left[ \frac{1}{2} \rho \dot{U}_i \dot{U}_i + \frac{1}{2} \epsilon^2 \rho \epsilon_{ijkl} \epsilon_{ijkl}^0 (e_{cm}(U))_{x_j} \right] d\Omega - \int_{S_\Omega} \frac{1}{2} \rho \partial_i \epsilon_{ijkl}^0 \epsilon_{ijkl}^0 \sigma_{ijkl}^* dS + \int_{S_\Omega} \frac{1}{2} \epsilon^2 \partial_i \epsilon_{ijkl}^0 \sigma_{ijkl}^* dS \]

where, \( \Omega \) and \( S \) denotes the domain of macrostructure and its boundary defined by outward normal vector \( n_i \), respectively, and \( S_\Omega \) corresponds to the boundary where tractions are prescribed. Note that Eq. (13) coincides with the Hu-Washizu principle in dynamics when \( \epsilon \) approaches to zero. For Eq. (12) \( C^0 \) continuous interpolation is used. In this paper, all boundary conditions in Eq. (13) except for the external force \( f_i \) are assumed to be zero in the subsequent numerical examples.

5. Numerical examples

5.1. Stability validation by plane harmonic analysis

To show the stabilization effect, Eq. (9) and (10) are discretized by Fourier spectral method with macroscopically periodic boundary condition. Microstructure is shown in Fig. 1(a) and macrostructure is a square of length 100. Phase velocity spectra of wave in the horizontal direction, which quantify dispersion effect, are plotted in Fig. 2. It can be seen that the phase velocities computed by Eq. (9) for wave number 50 (corresponding to wavelength 2) and higher are imaginary. On the other hand, Eq. (12) is stable for all excitation frequencies.

5.2. Accuracy validation

Same microstructure shown in Fig. 1(a) is used and macrostructure is shown in Fig. 1(b). Its right edge is subjected to vertical force \( q(t) \) like one sine wave. Eight-nodes serendipity interpolation is used for the nonlocal displacement \( U_i \) and bi-linear interpolations for both the assumed strain \( \dot{\epsilon}_{ij} \) and the assumed stress \( \sigma_{ij} \). Newmark’s \( \beta \) method \( (\beta=1/4, \gamma=1/2) \) is employed for time integration. Time histories of vertical nonlocal displacement at the center are plotted in Fig. 3. Reference solution was obtained by solving the heterogeneous system. Good agreement between the present model and the reference solution can be seen, while the solution obtained by the classical homogenization err badly.

References

1) Boutin, C. and Auriault, J.L., Int. J. Engng. Sci. 31(12), 1993, pp.1669-1689
2) Fish, J. and Chen, W., ASCE J. Eng. Mech., 2000 (accepted)
3) Fish, J. et al., Int. J. Num. Meth. Eng., 2001 (accepted)