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非線形問題における形状最適化手法について

Shape Optimization Method for Nonlinear Problems

正 野口 裕久 (慶大)

Hirohisa NOGUCHI, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama

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1. Introduction

This study summarizes a novel method of shape optimization for general nonlinear problems⁽¹⁾. The proposed method follows the idea of the traction method proposed by Azegami⁽²⁾ but is extended to handle any kinds of nonlinearity. It is also formulated in the complete discretized form and, therefore, it can be incorporated easily in the existing general-purpose nonlinear finite element method or meshfree method.

The proposed method utilizes the adjoint method and the Lagrangian functional is defined so that the objective function is minimized under the constraint conditions for the equilibrium and the volume. Instead of direct seeking of the stationary points where the variation of the Lagrangian functional becomes zero, the sensitivities of the Lagrangian functional with respect to the initial geometrical coordinates are evaluated.

It is well known that if these sensitivities with respect to the initial geometrical coordinates are used without any smoothing technique for the updating the geometry, the non-smoothed geometry is occasionally obtained. In order to avoid this situation, Azegami proposed the traction method⁽²⁾ that utilizes the inverse of linear stiffness matrix as a smoother. The traction method has been applied to many kinds of practical problems and well established in the field of linear analyses. However, the application to the general nonlinear problem is still left to be unsolved.

The key-issues of the proposed methods are as follows. First, the sensitivities of the Lagrangian functional are computed numerically by using the semi-analytical method. In the semi-analytical method, only the conditional partial derivative of the internal force with respect to the initial geometry is evaluated by the finite difference method numerically. While the adjoint variables (Lagrange multipliers) are computed consistently by using the tangent stiffness matrix, which is readily available if the Newton-Raphson method is adopted for solving nonlinear equations.

Second, in the proposed method, the tangent stiffness matrix may become a smoother for the sensitivities with respect to the initial geometry. Although this is valid if the stiffness matrix is positive definite, there seems no problem in case that optimization is carried out for design of initial geometry. Here, again, if the triangular decomposed tangent stiffness matrix is available, the smoothed sensitivities can be computed efficiently.

Finally, in the presentation, a numerical example for nonlinear fluid-structure interaction problem⁽³⁾ is demonstrated to clarify the validity of the proposed method

2. Adjoint Method

Let π be objective function to be minimized under constraint conditions for equilibrium and volume. In case the objective function is evaluated by the finite element method, the Lagrangian functional L can be expressed in the discretized form as shown in Eq.(1).

$$L(\mathbf{u}, \mathbf{w}, \Lambda, \mathbf{b}) \equiv \pi - \{\mathbf{w}^T \cdot \mathbf{Q}(\mathbf{u}, \mathbf{b}) - \mathbf{w}^T \cdot \mathbf{F}\} + \Lambda \cdot (m - M) \quad (1)$$

Where $\mathbf{Q}$: internal force vector, $\mathbf{F}$: external force vector and $\mathbf{u}$: displacement vector, m , M : current and given mass, $\mathbf{w}$: adjoint variable vector or the Lagrange multipliers for equality constraint condition, Λ : the Lagrange multiplier for in-equality constraint condition. $\mathbf{b}$ is initial nodal coordinate vector or design variables. By taking the variation of Eq.(1), the followings are obtained

$$\begin{aligned} \delta L = \delta \mathbf{u}^T & \cdot \left\{ \frac{\partial \pi}{\partial \mathbf{u}} - \left(\frac{\partial \mathbf{Q}}{\partial \mathbf{u}} \right)^T \mathbf{w} \right\} \\ & + \delta \mathbf{w}^T \cdot \{\mathbf{F} - \mathbf{Q}(\mathbf{u}, \mathbf{b})\} \\ & + \delta \Lambda \cdot (m - M) \\ & + \delta \mathbf{b}^T \cdot \left\{ \frac{\partial \pi}{\partial \mathbf{b}} + \frac{\partial (\mathbf{w}^T \cdot \mathbf{F})}{\partial \mathbf{b}} - \frac{\partial (\mathbf{w}^T \cdot \mathbf{Q}(\mathbf{u}, \mathbf{b}))}{\partial \mathbf{b}} + \Lambda \cdot \frac{\partial m}{\partial \mathbf{b}} \right\}^T \end{aligned} \quad (2)$$

From the stationary condition of Eq.(2): the following equations are obtained for $\delta \mathbf{u}$, $\delta \mathbf{w}$, $\delta \Lambda$ and $\delta \mathbf{b}$.

$$\left(\frac{\partial \mathbf{Q}}{\partial \mathbf{u}} \right)^T \mathbf{w} = \frac{\partial \pi}{\partial \mathbf{u}} \quad (4)$$

$$\mathbf{F} = \mathbf{Q}(\mathbf{u}) \quad (5)$$

$$\Lambda \cdot (m - M) = 0, \quad m - M \leq 0 \quad (6)$$

$$\frac{\partial \pi}{\partial \mathbf{b}} + \frac{\partial (\mathbf{w}^T \cdot \mathbf{F})}{\partial \mathbf{b}} - \frac{\partial (\mathbf{w}^T \cdot \mathbf{Q}(\mathbf{u}, \mathbf{b}))}{\partial \mathbf{b}} + \Lambda \cdot \frac{\partial m}{\partial \mathbf{b}} = 0 \quad (7)$$

Equation (4) is the nonlinear equilibrium equation and has already been solved before the optimization procedure. If the Newton-Raphson method is adopted for solving the nonlinear problem, the following equation is solved repeatedly until convergence is achieved, where $\mathbf{R}$ shows residual force vector.

$$\frac{\partial \mathbf{Q}}{\partial \mathbf{u}} \Delta \mathbf{u} = \mathbf{R} \quad (8)$$

As the tangent stiffness $\partial \mathbf{Q} / \partial \mathbf{u}$ is already triangular-decomposed, Eq.(4) can be solved efficiently and the adjoint vector $\mathbf{w}$ is obtained. Note that the transpose matrix of tangent stiffness appears in Eq.(4), the adjoint vector can be computed more efficiently in case the tangent stiffness is symmetric. Eqs.(5) and (6) are the conditions that the current mass (volume) is equal or less than the given mass (volume).

3. Discrete Force Method

If Eqs (4)-(7) hold, the Lagrangian functional can be rewritten equivalently as in Eq.(9).

$$\delta L = -\delta \mathbf{b}^T \cdot \left\{ \frac{\partial \pi}{\partial \mathbf{b}} - \frac{\partial (\mathbf{w}^T \cdot \mathbf{F})}{\partial \mathbf{b}} + \frac{\partial (\mathbf{w}^T \cdot \mathbf{Q}(\mathbf{u}, \mathbf{b}))}{\partial \mathbf{b}} - \Lambda \cdot \frac{\partial m}{\partial \mathbf{b}} \right\} \quad (9)$$

In the case of steepest decent method, the variation of $\mathbf{b}$ is set to the vector parallel to the derivative of the Lagrangian functional with respect to the design variables, as

$$\begin{aligned} \Delta \mathbf{b} &= -\lambda \cdot \frac{\partial L}{\partial \mathbf{b}} \\ &= \lambda \cdot \left\{ \frac{\partial \pi}{\partial \mathbf{b}} - \frac{\partial (\mathbf{w}^T \cdot \mathbf{F})}{\partial \mathbf{b}} + \frac{\partial (\mathbf{w}^T \cdot \mathbf{Q}(\mathbf{u}, \mathbf{b}))}{\partial \mathbf{b}} - \Lambda \cdot \frac{\partial m}{\partial \mathbf{b}} \right\} \quad (10) \end{aligned}$$

where λ is a properly defined positive constant. Substituting Eq.(10) into Eq.(9), it is shown easily that the Lagrangian function always decreases with the variation of $\mathbf{b}$.

$$\begin{aligned} \Delta L &\equiv \left(\frac{\partial L}{\partial \mathbf{b}} \right)^T \cdot \Delta \mathbf{b} \\ &= \left(\frac{\partial L}{\partial \mathbf{b}} \right)^T \cdot \left(-\lambda \cdot \frac{\partial L}{\partial \mathbf{b}} \right) \\ &= -\lambda \cdot \left| \frac{\partial L}{\partial \mathbf{b}} \right|^2 \\ &< 0 \end{aligned} \quad (11)$$

Referring to the traction method proposed by Azegami⁽²⁾, the vector for the shape gradient function is newly defined as:

$$\mathbf{L}_G(\mathbf{b}) \equiv \frac{\partial \pi}{\partial \mathbf{b}} + \frac{\partial (\mathbf{w}^T \cdot \mathbf{F})}{\partial \mathbf{b}} - \frac{\partial (\mathbf{w}^T \cdot \mathbf{Q}(\mathbf{u}, \mathbf{b}))}{\partial \mathbf{b}} + \Lambda \cdot \frac{\partial m}{\partial \mathbf{b}} \quad (12)$$

Then, in the proposed discrete force method, the variation vector of design variable is determined by solving the following equation.

$$\frac{\partial \mathbf{Q}}{\partial \mathbf{u}} \cdot \Delta \mathbf{b}_{DFM} = -\lambda \mathbf{L}_G(\mathbf{b}) \quad (13)$$

$$\Delta \mathbf{b}_{DFM} = \lambda \left\{ \frac{\partial \mathbf{Q}}{\partial \mathbf{u}} \right\}^{-1} \cdot \{ -\mathbf{L}_G(\mathbf{b}) \} \quad (14)$$

Here, the tangent stiffness is again utilized as a smoother to improve the irregularity of original sensitivity vector $\Delta \mathbf{b} = \mathbf{L}_G(\mathbf{b})$. If the tangent stiffness is positive definite, it is again shown that the Lagrangian function always decreases with $\Delta \mathbf{b}_{DFM}$.

$$\begin{aligned} \Delta L &\equiv \left(\frac{\partial L}{\partial \mathbf{b}} \right)^T \cdot \Delta \mathbf{b}_{DFM} \\ &= \lambda \{ \mathbf{L}_G(\mathbf{b}) \}^T \cdot \left\{ \frac{\partial \mathbf{Q}}{\partial \mathbf{u}} \right\}^{-1} \cdot \{ -\mathbf{L}_G(\mathbf{b}) \} \\ &= -\lambda \mathbf{L}_G(\mathbf{b})^T \cdot \left\{ \frac{\partial \mathbf{Q}}{\partial \mathbf{u}} \right\}^{-1} \cdot \mathbf{L}_G(\mathbf{b}) \\ &= -\frac{1}{\lambda} \Delta \mathbf{b}_{DFM}^T \cdot \frac{\partial \mathbf{Q}}{\partial \mathbf{u}} \cdot \Delta \mathbf{b}_{DFM} \\ &< 0 \end{aligned} \quad (15)$$

It is also noted that in some classes of nonlinear problems the tangent stiffness loses positive definiteness and Eq.(15) can not be applied directly and should be modified appropriately. On the other hand, the original traction method utilizes the linear stiffness matrix $\mathbf{K}_L$ instead of $\partial \mathbf{Q} / \partial \mathbf{u}$ to ensure the positive definiteness of the matrix.

If the objective function π and the external force $\mathbf{F}$ are independent of $\mathbf{b}$ and m is less than M , the vector for shape

gradient function $\mathbf{L}_G(\mathbf{b})$ is simply reduced to:

$$\mathbf{L}_G(\mathbf{b}) \equiv -\frac{\partial (\mathbf{w}^T \cdot \mathbf{Q}(\mathbf{u}, \mathbf{b}))}{\partial \mathbf{b}} \quad (16)$$

The right hand side of Eq.(16) are computed in the nonlinear finite element method for structural problem as follows. By multiplying arbitrary nodal velocity vector $\dot{\mathbf{b}}$ from the left side, the rate of shape gradient function can be written as:

$$\begin{aligned} \dot{L}_G &\equiv \mathbf{L}_G(\mathbf{b}) \cdot \dot{\mathbf{b}} = -\frac{\partial (\mathbf{w}^T \cdot \mathbf{Q}(\mathbf{u}, \mathbf{b}))}{\partial \mathbf{b}} \cdot \dot{\mathbf{b}} \\ &= \left(\int_V e_{ij}^* T_{ij} dv \right)^{\cdot} \end{aligned} \quad (17)$$

Where e_{ij}^* denotes strain tensor referring the current configuration calculated from the adjoint vector $\mathbf{w}$, T_{ij} the Cauchy stress and v current volume. By using the Reynolds transform theorem and the divergence theorem and neglecting the term of the Eulerian time derivative, Azegami⁽²⁾ transformed Eq.(18) into

$$\left(\int_V e_{ij}^* T_{ij} dv \right)^{\cdot} = \int_S e_{ij}^* T_{ij} n_k \dot{b}_k ds, \quad (18)$$

where s is surface of the volume and $\mathbf{n}$ the normal vector to the surface. As $\dot{\mathbf{b}}$ is arbitrary, the right hand side of Eq.(18) can be regarded as the virtual work by the surface traction force. Then Azegami computed the variation of $\mathbf{b}$ by solving linear stiffness equation with the traction force evaluated from Eq.(18). Thus, the Azegami's method is termed as 'Traction Method'.

In the present formulation, not the surface traction force as in the original 'Traction method', but $\mathbf{L}_G(\mathbf{b})$ is evaluated by using the following numerical differentiation as,

$$-\frac{\partial (\mathbf{w}^T \cdot \mathbf{Q}(\mathbf{u}, \mathbf{b}_k))}{\partial \mathbf{b}_k} = -\mathbf{w}^T \frac{(\mathbf{Q}(\mathbf{u}, \mathbf{b}_k + \Delta \mathbf{b}_k) - \mathbf{Q}(\mathbf{u}, \mathbf{b}_k))}{\Delta \mathbf{b}_k} \quad (19)$$

Then the computed $\mathbf{L}_G(\mathbf{b})$ is utilized in the right hand side of Eq.(13) as the body force vector, therefore the method is termed as 'Force method'. The formulation is applied to general nonlinear problems of which equilibrium equation can be expressed in discrete form as in Eq.(5). As for shape optimization, any kinds of single objective function π can be defined as a function of $\mathbf{u}$ and $\mathbf{b}$.

3. Conclusion

In this study, a novel nonlinear generalized discrete force method for shape optimization is presented. The method follows the idea of the traction method proposed by Azegami but is formulated in the complete discretized form and, therefore, it can be incorporated easily in the existing general nonlinear software. The method is numerically effective as the tangent stiffness matrix is already triangular decomposed and can be utilized repeatedly not only for solving the nonlinear equations but also for computing the adjoint variable vector and the smoothed sensitivities. In order to validate the proposed method; the method should be applied to practical numerical problems in the future.

References

- (1) Azegami, H., Wu, Z. C., Domain Optimization Analysis in Linear Elastic Problems, JSME International, Series A, 39-2 (1996), pp. 272-278.
- (2) Noguchi, H., Nonlinear Generalized Discrete Force Method for Shape Optimization, Proc. IJK-OSM 2, (2002), pp. 603-608.
- (3) Umeda, H., Noguchi, H., Shape Optimization for Bluff Body Placed In Fluid Flow, Proc. IJK-OSM 2, (2002), pp. 791-796.