110 Parametrically Excited Simply Supported Beam
(Formulation by Cosserat Theory and Nonlinear Analysis)

In-So Son and Hiroshi Yabuno (Member)
Institute of Engineering Mechanics and Systems
University of Tsukuba, Tsukuba city, 305-8573 Japan

This paper presents the nonlinear characteristics of the parametric resonance of a simply supported beam which is inextensible beam. For the beam model, the order-three expanded equation of motion has been determined in a form amenable to a perturbation treatment. The equation of motion is derived by a special Cosserat theory. The method of multiple scales is used to determine the equations that describe the first-order modulation of the amplitude of simply supported beam. The stability and the bifurcation points of the system are investigated applying the frequency response function.

Key Words: Special Cosserat theory, Nonlinear analysis, Parametric excitation, Method of multiple scales

A1. Introduction
Research on the dynamics of large-amplitude of the beam structures is important for engineering and its applications. The most comprehensive theory today available to describe overall motions of rods is the special Cosserat theory. Parametric resonance, which is characterized by the harmonic variation of coefficients of differential equations, is well known and has been studied by numerous researcher authors. Therefore, in this paper, we theoretically analyzed the nonlinear characteristics of a simply supported beam under the parametric resonance for the first mode. The equation of motion is derived by a special Cosserat theory.

A2. Analytical model
We consider a simply supported beam subjected to a periodic excitation as shown in Fig. A1. The periodic excitation is \( \xi = a \cos \Omega t \), where \( a, \Omega \) and \( t \) are the excitation amplitude, frequency and the time, respectively. The notation employed in this analysis is as follows: \( \rho A \) is mass of the beam per unit length; \( EI \) is the bending stiffness coefficient; \( m \) and \( l \) are the tip mass and the length of the beam.

![Fig. A1 Analytical model of the beam](image)

The strain vector and curvature tensor are (Fig. A2)

\[
\varepsilon = \frac{R'(u - du''')}{|dX|}, \quad K = R'R'^{-1}
\] (A1)

The equilibrium equations are

\[
\begin{align*}
\mathbf{n}'(x,t) + b(x,t) &= 0 \\
\mathbf{m}'(x,t) + \mathbf{x}'(x,t) \times \mathbf{n}(x,t) + \mathbf{c}(x,t) &= 0
\end{align*}
\] (A2)

\[
M^* + k \left[ M' \tan \theta + m \dot{U} \sec \theta \right] + k \int km' dx + \int b_1 dx = b_2 \quad (A3)
\]

Incorporating the longitudinal motion into the inertial forces, substituting them into Eq. (A3), expanding the resulting equation and retaining terms up to third order yield the equation of motion of the inextensible beam\((\text{strain} \varepsilon = 0)\).

A3. Nonlinear analysis
The method of multiple scales is used to determine the equation that scribe to first order the equation of frequency-response.

\[
a = \left[ \frac{8m}{\beta \Gamma} \left( \frac{\sigma}{2} \pm \sqrt{[\alpha, \beta, \beta \alpha] - \mu^2} \right) \right]^{1/2} \quad (A4)
\]

We can obtain the frequency-response curve using Eq. (A4). The frequency-response curve of the simply supported beam consists of two branches; the left one is unstable and the right one is stable.

A4. Conclusions
This paper presents the nonlinear characteristics of the parametric resonance of a simply supported beam. The beam, incorporating the inextensibility and unshearability constraints, describes bending motions only; hence, it is suitable for beam that is either axially unrestrained or weakly restrained. For the beam model, the order-three expanded equation of motion has been determined in a form amenable to a perturbation treatment. The equation of motion is derived by a special Cosserat theory and analyzed by the method of multiple scales.
1. Introduction

Research on the dynamics of large-amplitude of the beam structures is important for engineering and its applications. The large-amplitude motions are excited around resonances with finite displacements and rotations whereas the strains often remain small.

The most comprehensive theory today available to describe overall motions of rods is the special Cosserat theory of rods \(^{(1)}\). The beam is mathematically conceived as a one-dimensional continuum with a local rigid structure. Because of the postulated local rigidity, the sections cannot undergo distortion and warping deformations; therefore, the theory is mainly restricted to beams with closed cross sections.

There has been much research on elastic beam \(^{(2)}\). He obtained a constitutive equation relating the curvature and the bending couple. In Euler's theory, however, it was assumed that the axial force was not defined constitutively.

Couple stress tensors were introduced and a micropolar generalization was presented by the Cosserat brothers\(^{(3)}\). Although many attempts have been made along this line, it is difficult to solve Cosserat's basic equations rigorously because of its high generality and nonlinearity \(^{(4)}\). Nishinari \(^{(5)}\) studied about a simple constitutive equations for the axial force and moment using a special version of Cosserat's basic equations. Typically, the analytical approaches have employed approximate mechanical models that account for geometric and inertia nonlinearities, often using a variational formulation based on a truncated kinematic model \(^{(6,8)}\).

Parametric resonance, which is characterized by the harmonic variation of coefficients of differential equations, is well known and has been studied by numerous authors \(^{(9,10)}\). The response of linear and nonlinear system subjected to multi-frequency parametric excitation have been investigated \(^{(11,12)}\). Evensen \(^{(13)}\) has investigated the parametric resonance of a column under time dependent axial loading. Nayfeh and Pai theoretically and experimentally show that the inertia and curvature nonlinearities have significant influence on the nonlinear characteristics of the frequency-response \(^{(14)}\). Also, Yabuno et al. \(^{(15)}\) studied the nonlinear analysis of a parametrically excited cantilever beam. In particular, the effect of the tip mass on the nonlinear characteristics of the frequency-response is theoretically and experimentally presented.

In the present research, we theoretically analyzed the nonlinear characteristics of the simply supported beam under the parametric resonance for the first-order modulation. The equation of motion is derived by a special Cosserat theory and analyzed by the method of multiple scales.

2. Analytical model and equation of motion

We consider a simply supported beam subjected to a periodic excitation as shown in Fig. 1. The periodic excitation is \(\ddot{X} = a_1 \cos \omega t\), where \(a_1\), \(\Omega\) and \(t\) are the excitation amplitude, frequency and the time, respectively. The notation employed in this analysis is as follows: \(\rho A\) is mass of the beam per unit length; \(EI\) is the bending stiffness coefficient; \(m\) and \(l\) are the tip mass and the length of the beam, respectively.

![Analytical model of the beam](image)

Denoting with \(e_j (j=1,2,3)\) the orthonormal vectors of a fixed inertial reference frame such that \(e_1\) is parallel to the beam base curve, the position of a material point along the beam axis is represented by the vector \(X(x,t) = xe_1\), where \(x\) indicates the coordinate along the straight undeformed beam axis with the origin \(O\) fixed at the left end of the beam. Thus, the material section at \(P\) is specified by the pair of orthonormal vectors \(a_j (j=1,2,3)\).

We set \(a_i = a_j \times a_k\) so that \(\{a_j\}\) are a right-handed orthonormal basis for the 3-dimensional Euclidean space.

2.1 Prestressed equilibrium states

The displacement in the \(x\)-axis is assumed to be defined by the position vector \(x(x,t) = X(x) + u(x,t)\) with \(u = u_1 e_1 + u_2 e_2\) denoting the displacement vector from \(P\) to \(P'\) and by the pair of orthonormal directors \(d_1\) where \(U = \dot{x} + u\). The directors \(d_1\) and \(d_2\) are obtained from \(a_j\) via a finite rotation about the \(d_1\)-axis, described by the proper orthogonal rotation tensor \(R(x,t)\), restricted to the plane spanned by \(a_1\) and \(a_2\), as

\[
R = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]  

(1)

Therefore \(a_j\) and \(d_j\) are as follows, respectively:

\[
a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \quad \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}
\]  

(2)

2.1.1 Curvature tensor

The generalized beam strains are calculated in the reference configuration as the components of the strain vector and curvature tensor. First of all, the curvature tensor is defined as

\[ K = R^T R \]  

(3)

where \([\cdot]'\) stands for the differentiation with respect to \(x\) and \([\cdot]^T\) indicates the transpose. Substituting Eq. (1) into Eq. (3), the curvature tensor can be obtained as
The differentiation with respect to $x$, $d_1'$ and $d_2'$ are

$$d_1' = [-\theta' \sin \theta \hspace{1cm} \theta' \cos \theta] = k d_2', \quad d_2' = \begin{bmatrix} -\theta' \cos \theta \hspace{1cm} \theta' \sin \theta \end{bmatrix} = -k d_1,$$  \hspace{1cm} (5) where $k = \theta'$. 

### 2.1.2 Strain vector

Fig. 2 shows the strain of the beam element. In Fig. 2, the strain vector of the beam can be calculated as

$$\varepsilon = \frac{R}{\lvert dX \rvert} (\delta u - u''(R))$$  \hspace{1cm} (6)

where superscript $(R)$ denotes the deformation of rigid body. In addition, we can obtain the geometric relationship as follow:

$$dX^{(R)} = R dX, \quad \delta u = d\varepsilon - dX, \quad dX^{(R)} = dX^{(R)} - dX$$  \hspace{1cm} (7)

Substituting Eq. (7) into Eq. (6) yields

$$\varepsilon = \frac{R}{\lvert dX' \rvert} x' - x'$$  \hspace{1cm} (8)

Therefore the strain, on account of $x' = \varepsilon + u'$, becomes

$$\varepsilon(x,t) = \varepsilon_0 + \gamma a_2,$$  \hspace{1cm} (9)

where

$$\gamma(x,t) = (1 + u') \cos \theta + v' \sin \theta - 1 \hspace{1cm} (10)$$

$$\varepsilon_0(x,t) = -((1 + u') \sin \theta + v' \cos \theta) \hspace{1cm} (11)$$

The beam unshearability is enforced via the internal kinematic constraint $\gamma = \varepsilon_0 a_2 = 0$. This constraint, in turn, is solved to yield the rotation of the beam section as

$$\theta = \tan^{-1} \left( \frac{v'}{1 + u'} \right) \hspace{1cm} (12)$$

where $\tan^{-1}$ denotes the inverse of the tangent function and

$$\sin \theta = \frac{v'}{1 + u'}, \quad \cos \theta = \frac{1 + u'}{1 + \varepsilon_0} \hspace{1cm} (13)$$

Consequently, the only nontrivial strains are the axial strain and the bending curvature, the non-zero component of the curvature tensor $K$, and are expressed as

$$\varepsilon_0 = \sqrt{(1 + u')^2 + v'^2} - 1, \quad k = \theta' = \frac{v' + u' v' \sin \theta - u' v'}{(1 + \varepsilon_0)^2} \hspace{1cm} (14)$$

### 2.2 Equilibrium equation

To obtain the equilibrium equations using the components of the contact force, body force and couple which are expressed using the current set of directors $d_j$, as

$$K = \begin{bmatrix} 0 & -\theta' \\ \theta' & 0 \end{bmatrix}$$  \hspace{1cm} (4)

where $K$ is the curvature tensor. The equilibrium equations, after filtering out the shear force $H$, become

$$N' + \frac{k}{1 + \varepsilon_0} M' + \frac{k}{1 + \varepsilon_0} c + b_i = 0 \hspace{1cm} (21)$$

$$\left( \frac{M'}{1 + \varepsilon_0} \right) + \left( \frac{c}{1 + \varepsilon_0} \right) - kN - b_i = 0 \hspace{1cm} (22)$$

By virtue of D'Alembert's principle, the body forces and the corresponding couple are expressed as

$$b(x,t) = -\rho A \left( U a_i + \ddot{v} a_i \right) \hspace{1cm} \varepsilon_0 \dot{\theta} \left( U \cos \theta + \dot{v} \sin \theta \right) d_1 + \left( -\dot{U} \sin \theta + \dot{v} \cos \theta \right) d_2 \hspace{1cm} (23)$$

$$c(x,t) = -\rho \dot{\theta} \dot{\theta} a_3 \hspace{1cm} (24)$$

where $\left[ \cdot \right]$ indicates the differentiation with respect to time $t$. To inertially uncouple the equations of motion, we project them into the $(a_i,a_2)$ basis and obtain
\[\rho A(\ddot{x} + \ddot{u}) - \left[ N + \frac{k}{1 + \varepsilon_t} M + \frac{k}{1 + \varepsilon_t} (-\rho l \hat{\theta}) \right] \cos \theta - \left[ \frac{M}{1 + \varepsilon_t} - kN + \left( \rho \frac{\theta^2}{1 + \varepsilon_t} \right) \sin \theta \right] \cos \theta = 0 \]  
\[\rho A \bar{u} - \left[ N + \frac{k}{1 + \varepsilon_t} M + \frac{k}{1 + \varepsilon_t} (-\rho l \hat{\theta}) \right] \sin \theta + \left[ \frac{M}{1 + \varepsilon_t} - kN + \left( \rho \frac{\theta^2}{1 + \varepsilon_t} \right) \sin \theta \right] \cos \theta = 0 \]  

The boundary conditions are
\[u(0,t) = 0, \quad v(0,t) = v(l,t) = 0, \quad M(0,t) = M(l,t) = 0 \]
(27)

where \(m\) is the tip mass. From a constitutive point of view, because the axial strains are assumed small and the curvature is finite but moderately large, linear constitutive equations are assumed in the standard uncoupled form
\[N(x,t) = EA(x)\varepsilon(x,t), \quad M(x,t) = EI(x)\theta(x,t) \]
(29)

where \(E\) stands for Young's modulus, \(A\) and \(I\) denote the area and moment of inertia of the cross section of the beam, respectively.

### 2.3 Equation of motion for inextensible beam

When the beam is axially unrestrained to enforce vanishing of the axis elongation, \(\varepsilon_t = 0\), which leads to
\[(1 + u')^2 + v^2 = 1 \]  
(30)

On account of \(\varepsilon_t = 0\), the exact bending curvature, \(k = \theta'\), and its third-order expansion become
\[k = v' + u'v' - u'v' = v' + \frac{1}{2} v'^2 \cos \theta \]
(31)

Moreover, \(\sin \theta = v'\) and \(\cos \theta \approx 1 + u' = 1 - 1/2(v')^2\).

The equilibrium equation is Eq. (21) with \(\varepsilon = 0\). Therefore, the axial force along the beam can be obtained by integrating Eq. (21) with \(\varepsilon = 0\) as follows:
\[N(x,t) = N(l,t) - \int k M' \, dx - \int b_0 \, dx \]
(32)

Using the boundary mechanical condition in the axial direction at \(x = l\) given by Eq. (12), the axial force and the equilibrium equation in the transverse direction become
\[N(x,t) = -\int k M' \, dx - \int b_0 \, dx - M'(l,t) \tan \theta(l,t) - \frac{M(l,t)}{\cos \theta(l,t)} \]  
(33)

\[M' + k \left[ M \tan \theta + m \int \sec \theta \right]_{x=l} + k \int k M' \, dx + \int b_0 \, dx + b_1 = 0 \]  
(34)

Incorporating the longitudinal motion into the inertial forces, substituting them into Eq. (34), expanding the resulting equation and retaining terms up to third order yield the following equation of motion:
\[\rho A \ddot{v} + \ddot{v} \left[ (v'^2 + v''v') \right] dx - \frac{1}{2} \dot{v}^2 + 2 \ddot{v} \dot{v}' + v'^2 + 3v''v'v'' \]  
+ \ddot{v} \left[ (EI\varepsilon')' \right] dx - m \dot{v}' \left[ (v' + v''v') \right] dx \]  
(35)

where \(\dot{v} = \frac{\dot{v}}{\rho A l} \), \(\varepsilon = \frac{\dot{v}}{l}m\), \(m = \frac{m}{\rho AI}\), \(\Omega = \frac{\Omega}{l}\)  
(37)

The remaining mechanical boundary conditions, within third order, are
\[EI \left[ v' + \frac{1}{2} (v')^2 \right] = 0, \quad at \quad x = 0, l \]
(36)

### 3. Nonlinear analysis

To obtain the dimensionless equation of motion, the following dimensionless variables and parameters are introduced:
\[t' = \frac{t}{T}, \quad x' = \frac{x}{l}, \quad v' = \frac{v}{\dot{v}}, \quad \varepsilon_t' = \frac{\varepsilon_t}{l}, \quad m' = \frac{m}{\rho AI}, \quad \Omega = \frac{\Omega}{l} \]

where \(T^2 = \rho AI / E I\). The resulting dimensionless equation of motion by taking into account a viscous damping effect, dropping the star for sake of notational simplicity and neglecting the rotary term along with the distributed couples, is
\[ \ddot{v} + \ddot{v} \left[ (v'^2 + v''v') \right] dx - \frac{1}{2} \dot{v}^2 + 2 \ddot{v} \dot{v}' + v'^2 + 3v''v'v'' \]  
+ \ddot{v} \left[ (EI\varepsilon')' \right] dx - m \dot{v}' \left[ (v' + v''v') \right] dx \]  
(38)

where \(\mu\) is the damping ratio in the transverse direction. The boundary conditions are
\[v(0) = v'(0) = v(l) = v'(l) = 0 \]
(39)

For the case of principal parametric resonance of the first mode of the beam, we put
\[\Omega = 2 \varepsilon_t + \sigma, \quad \sigma = \varepsilon \sigma \]
(40)

where \(\varepsilon\) is a small parameter \(|\varepsilon| < 1\) of book-keeping device and \(\sigma\) is a detuning parameter and \([\cdot]\) denotes \(O(1)\). We also set \(a_0\) and \(\mu\), as \(a_0 = \varepsilon \dot{a}_0\) and \(\mu = \varepsilon \dot{\mu}\), respectively. By the method of multiple scales \((16)\), we analyze the equation of motion Eq. (38). The uniform expansions of the solution of Eq. (38) is sought in the
\[ v = e^{i\omega t} v_0 + e^{i\omega t} v_1 + \cdots \]  
(41)

Multiple time scales are introduced as follows:
\[ t_0 = t, \quad t_1 = \varepsilon t \]  
(42)

We substitute Eq. (42) into the system of equation of motion Eq. (38) and boundary conditions Eq. (39), use the independence of the time scales, equate coefficients of like powers of \( \varepsilon \), and obtain the following:
\[
\begin{align*}
O(e^{i\omega t}) : \\
D_\varepsilon^n v_1 + i v_1 &= 0 \\
O(e^{i\omega t}) : \\
D_\varepsilon^n v_1 + i v_1 &= -2D_{\varepsilon x}v_1 - v_1^2 D_{\varepsilon x}v_1 - 2D_{\varepsilon x}v_1 = 0 \\
\int \left[ \frac{\partial}{\partial x} \left( D_{\varepsilon x}v_1 + \frac{i}{2} v_1^2 D_{\varepsilon x}v_1 - 3v_1^2 v_1' - \frac{i}{2} v_1^2 v_1'' \right) \right] dx \\
+ 2D_{\varepsilon x}v_1 = 0 \\
\end{align*}
\]  
(43)

With this approach it turns out to be convenient to write the solution of Eq. (43) in the complex form
\[ v_0 = A e^{i\omega t} + \bar{A} e^{-i\omega t} \]  
(44)

where \( \bar{A} \) denotes the complex conjugate of the preceding term.

We can obtain the mode shape as \( \phi(x) = \sqrt{2} \sin \pi x \). Also, we can assume the particular solution of Eq. (44) in the form:
\[ v_1 = e^{i\omega t} \phi(x) + \bar{A} e^{-i\omega t} \]  
(45)

By considering the boundary condition Eq. (45) and substituting Eqs. (46) and (47) into Eq. (44), we obtain the solvability condition as follows:
\[
2i\omega A + \beta_1 A + \beta_2 A^\top = 0 \\
\]  
(48)

where
\[
\begin{align*}
\beta_1 &= 1/2 \int (\phi)'^2 dx \\
\beta_2 &= \int \phi \left[ \phi' + (x - m - 1) \phi'' \right] dx \\
\end{align*}
\]  
(49)

and
\[ \Gamma = 3\Gamma_0 + \Gamma_1 \]  
(50)

where \( \Gamma_0 \) relates to the contribution from the geometric and curvature nonlinearities whereas, \( \Gamma_1 \) is the contribution arising from the nonlinear inertia forces.

Substituting the form \( A = B(t_2) e^{i\alpha t/2} \) and the polar form
\[ \beta = \frac{1}{2} e^{i\alpha t/2} \]  
(51)

therefore, we get
\[
\left( \frac{d\alpha}{dt} + i\alpha \frac{d\beta}{dt} \right) + \frac{\sigma}{2} a + i\alpha \beta - \frac{i}{2} a^2 \beta = 0 \\
\]  
(52)

\[
\frac{d\beta}{dt} = -\frac{\sigma}{2} a + i\alpha \beta - a^2 \beta = 0 \\
\]  
(53)

Then, the first-order expansion of the solution of Eq. (38) is given by
\[ v = a \cos \left( \frac{\Omega}{2} t + \beta \right) \phi(x) + cc \]  
(54)

where \( a \) and \( \beta \) are defined by Eqs. (52) and (53). Further solving for the fixed points of the real-valued modulation equations resulting from the linearized form, the following frequency-response equation is obtained:
\[ \alpha = \frac{8\sigma}{\beta \Gamma_1} \left( \frac{\sigma}{2} \pm \sqrt{\alpha_1 \beta_1 \beta_2} - \mu^2 \right)^{1/2} \]  
(55)

3.1 Trivial solution

To determine the stability of the trivial solutions, we investigate the solutions of the linearized form of Eq. (48). Letting \( \lambda = a + i\alpha \) and \( e^{i\alpha t/2} \), we have
\[
2i\omega a + i\alpha a + \mu(a + i\alpha) + \frac{\sigma}{2} (a + i\alpha) e^{i\alpha t/2} + 2(a - i\alpha) \omega_0 \beta a e^{i\alpha t/2} = 0 \\
\]  
(56)

Dividing the above equation by \( e^{i\alpha t/2} \) and separating real and imaginary parts, we obtain
\[
\frac{d\alpha}{dt} = -\mu a + \left( \omega_0 \beta a, \frac{\sigma}{2} \right) a, \\
\frac{d\beta}{dt} = -\mu a \left( \omega_0 \beta a, \frac{\sigma}{2} \right) a, \\
\]  
(57)

Then, we can obtain the eigenvalue equation of the above system; that is
\[ \lambda^2 + 2\mu \lambda + \mu^2 = 0 \]  
(58)

where \( \lambda \) is the eigenvalue of the system of Eq. (56). Consequently, a trivial solution is stable if \( \lambda < 0 \) and otherwise it is unstable. Fig. 3 shows the stability of a trivial solution for the dimensionless parameters \( \mu = 0.06 \) and \( m = 5.6 \). The hatched region indicates the unstable solutions. Moreover, when the excitation amplitude \( a_0 = 0 \), the stable(unstable) solution is represented by solid(dashed) line in Fig. 3.
When the excitation amplitude $a_0 = 3.11 \times 10^{-4}$, we can find the trivial bifurcation point, $\sigma = 0.22$.

3.2 Non-trivial solution

To determine the stability of the non-trivial solutions, we let

$$a = a_n + \Delta a, \quad \beta = \beta_n + \Delta \beta \quad (59)$$

where $a_n$ and $\beta_n$ correspond to the non-trivial solutions, $\Delta a$ and $\Delta \beta$ are perturbations which are assumed to be small compared with and substituting Eq. (59) into Eqs. (52) and (53) yield

$$\frac{d}{dt} \begin{bmatrix} \Delta a \\ \Delta \beta \end{bmatrix} = \begin{bmatrix} 0 & 2\alpha_0 a_n \beta_n \cos(2\beta) \\ 4\alpha_0 \end{bmatrix} \begin{bmatrix} \Delta a \\ \Delta \beta \end{bmatrix} \quad (60)$$

The eigenequation of the above system is

$$\lambda^2 + 2\mu \lambda + \frac{4\alpha_0}{4\alpha_0} \left[ \sigma^2 \gamma_1 - \sigma \right] = 0 \quad (61)$$

Then, we can determine stability of the steady-state solutions.

Fig. 4 shows the influence of tip mass in the frequency-response of the inextensible beam for the case of damping ratio $\mu = 0.06$, and the dimensionless excitation amplitude $a_0$ is $3.11 \times 10^{-4}$. In Fig. 4, the solid(dashed) line shows the stable(unstable) branches.

4. Summary and Conclusions

This paper presents the nonlinear characteristics of the parametric resonance of a simply supported beam. The beam, incorporating the inextensibility and unshearability constraints, describes bending motions only; hence, it is suitable for beam that is either axially unrestrained or weakly restrained. For the beam model, the order-three expanded equation of motion has been determined in a form amenable to a perturbation treatment. The equation of motion is derived by a special Cosserat theory and analyzed by the method of multiple scales.

References