A numerical study of the hole-tone feedback cycle:
The influence of a closed cavity and a tailpipe

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This paper is concerned with the hole-tone feedback cycle, where an air jet issuing from a circular nozzle impinges upon a plate with a similar hole located a little downstream, generating self-sustained flow oscillations. Focus is on the influence of (i) a closed cavity and (ii) a tail pipe. The system can be viewed as a simplified model of an automotive muffler with a tailpipe.

Keywords: vortex rings; vortex sound; Fourier transform; analytical Green’s function

1. Introduction

We are concerned with the so-called hole-tone problem [4] of which the common teakettle whistle may serve as an example of utilization of the sound generation in the system. The (steam) jet, issuing from a nozzle, passes through a similar hole in a plate, placed a little downstream from the nozzle. The shear layer of the jet is unstable and rolls up into a large, coherent vortex (‘smoke-ring’). This large vortex cannot pass through the hole in the plate and hits the edge of the hole, where it creates a pressure disturbance. The disturbance is thrown back (with the speed of sound) to the nozzle, where it disturbs the shear layer. This initiates the roll-up of a new coherent vortex. In this way an acoustic feedback loop is formed.

In the present work the basic configuration is modified to include a closed cavity and a tailpipe, as shown in Fig. 1. This brings new effects into play, in addition to the fundamental hole-tone frequencies, namely the resonance frequencies of the cavity and the tailpipe.

![Fig. 1. Sketch of the hole-tone setup with closed cavity (the large-diameter middle section) and tailpipe (the downstream section). The arrow indicates the direction of the flow.](image)

The aim of the present work-in-progress is to understand how these three competing frequencies interact, and how the resultant sound generation can be controlled. The present extended abstract is however limited to a description of the analysis and the numerical approach.

2. Aeroacoustic model

The unsteady, sound-generating jet flow is simulated by employing the discrete vortex method, using vortex rings as the ‘fundamental elements’ [4]. The Reynolds number (based on the mean jet speed and the hole diameter) is relatively high \( (Re \approx 3 \times 10^4) \), yet the Mach number is relatively low \( (M \approx 0.03) \). For evaluating the sound generated by the self-sustained flow oscillations, Howe’s equation for vortex sound at low Mach numbers [2] is thus a natural starting point.

Let \( \omega = \nabla \times \mathbf{u} \) denote the vorticity. The sound pressure \( p(x, t) \) at the position \( x \) and time \( t \) is related to the vortex force \( \mathbf{L}(x, t) = \omega(x, t) \times \mathbf{u}(x, t) \) via the non-homogeneous wave equation

\[
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) p = \rho \nabla \cdot \mathbf{L},
\]

(1)

The boundary conditions are

\[
\frac{\partial p}{\partial n} = 0 \text{ on the end plate,}
\]

\[
p \to 0 \text{ for } |x| \to \infty,
\]

where \( n \) denotes the normal vector.

The frequency domain version of (1) is

\[
(\nabla^2 + k^2) P = -\rho \nabla \cdot \mathbf{L}
\]

(3)
where $L(x, \nu)$ is the Fourier transform of $L(x, t)$, and $k = \nu/\epsilon_0$ is the wave number (with $\nu$ being the frequency). To solve (3) use is made of the free-space Green’s function $G(x, y; \nu)$ which is a solution of the equation

$$ (\nabla^2 + k^2) G = -\delta(x - y), \quad (4) $$

where $\delta(x - y)$ is the delta function. $y$ denotes the location of an acoustic source and $x$ again the location of an observation point.

The pressure away from the solid surfaces can be obtained from Green’s 3rd identity [3], of which the surface integrals are discretized via the boundary element method (BEM). The surfaces are however assumed to be plates of zero thickness, and a special version of the BEM is then needed. Terai [6] has shown that the pressure at a point $x$ away from the surfaces can be expressed as

$$ P(x, \nu) = -\rho \sum_j \int \int \int \frac{\partial G(x, y)}{\partial y_j} L_j(y, \nu) d^2 y + \int \frac{\partial \tilde{P}(y_\beta, \nu)}{\partial n_\beta} G(x, y_\beta) d^2 y_\beta, \quad (5) $$

where $\tilde{P}$ is the pressure difference across a plate. This pressure difference can be computed by taking the derivative of (5) in the direction of the normal, which gives a Fredholm integral equation of the first kind for determination of the unknown $\tilde{P}$.

3. The Green’s function in cylindrical coordinates

The present problem is best formulated in terms of cylindrical polar coordinates, specified by $x = (r, \theta, z)$, $y = (r_*, \theta, z_*)$. In terms of these coordinates (4) takes the form

$$ \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial z^2} + k^2 G = \frac{-\delta(r - r_*)}{r} \delta(z - z_*). \quad (6) $$

A standard form of the Green’s function for (6) is [5]

$$ G = \frac{i}{2} \int_0^\infty J_0(\mu r) J_0(\mu r_o) \mu e^{i\sigma |z - z_*|} d\mu, \quad (7) $$

where $\sigma = \sqrt{k^2 - \mu^2}$ and $J_0(z)$ is the Bessel function of first kind, zeroth order. The time-domain form of the Green’s function is then obtained by inverse Fourier transform. But neither the ‘inner’ integral of (7) nor the ‘outer’ Fourier integral is easy to evaluate.

A new method is outlined in the following, taking (6) as the starting point. By making use of Fourier transform with respect to the longitudinal coordinate $z$ we obtain:

$$ G(x, r; x_*, r_*) = \frac{i}{2} \sum_{m=0}^\infty \frac{(-1)^m}{(m!)^2} \frac{r_*}{2} \frac{2^m}{2^m} \int_{-\infty}^\infty \kappa^{2m} H^{(1)}_0(\kappa r) e^{i\xi |z - z_*|} d\xi, \quad (8) $$

where $H^{(1)}_0(z)$ is the Hankel function of first kind, zeroth order. For the case $r < r_*$, $r$ and $r_*$ should simply be interchanged. The square root function $\kappa = \sqrt{k^2 - \xi^2}$ is specified as

$$ \kappa = \begin{cases} \sqrt{k^2 - \xi^2} & \text{for } -k < \xi < k, \\ i\sqrt{-k^2 - \xi^2} & \text{for } -k < -\xi \text{ or } \xi > k. \end{cases} $$

This form ensures convergence of (8).

Write the series as $G = \sum_m G_m$, with

$$ G_0 = \frac{i}{4} \int_{-\infty}^\infty H^{(1)}_0(\kappa r) e^{i\xi Z} d\xi, \quad (9) $$

$$ G_1 = -\frac{i}{4} \left( \frac{r_*}{2} \right)^2 \int_{-\infty}^\infty (k^2 - \xi^2) H^{(1)}_0(\kappa r) e^{i\xi Z} d\xi, $$

$$ G_2 = \frac{i}{16} \left( \frac{r_*}{2} \right)^4 \int_{-\infty}^\infty (k^2 - \xi^2)^2 H^{(1)}_0(\kappa r) e^{i\xi Z} d\xi, $$

and so on, with $Z = |z - z_*|$. These integrals can be evaluated analytically. The result for $G_0$ is

$$ G_0 = \frac{1}{2} e^{ikD}, \quad (10) $$

with $D = \sqrt{r^2 + Z^2}$. This result can be found in standard tables, e.g. [1]. The terms in the following coefficients $G_1, G_2, \ldots$, where powers of $\xi$ occur, can be obtained from this result by differentiation with respect to $Z$ on both sides of the equality sign. The wave number $k$ occurs only in the form of integral powers, giving normal (non-fractional) derivatives in the time domain.

The process of evaluating coefficients of higher and higher order can be continued until the desired accuracy has been obtained. Obviously the coefficients become more and more complicated with increasing order. In practice one would like to avoid including terms with high-order time derivatives. Numerical examples show that the first two coefficients (i.e. $G \approx G_0 + G_1$) often are sufficient.

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References


