VIBRATIONS OF TURBINE BLADES.

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By

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ABSTRACT

The natural vibration of clamped-free bars is investigated, assuming the cross section to vary as a certain power of the distance from the free end, and further simplifying the calculation by taking the radius of gyration to be constant. The frequency of the lowest mode of vibration is expressed in a simple formula ready for use. Next the vibration of flexible bars attached to the periphery of a rotating disk is dealt with under the same assumption regarding the form of bars as in the preceding case. The frequency is again expressed in a formula. These two kinds of frequencies may be combined by Lamb-Southwell's method to give the approximate value of the frequency in rotation under the restoring forces, both elastic and centrifugal. The calculation is applied to a turbine blade, and the calculated frequencies of the blade at rest are compared with the values observed in the experiments, which were performed by recording the vibration on a smoked paper in one series, and on a sensitive film in another series. The article consists of four sections, of which the first gives the outlines of the calculation, while the details of the calculation, the methods of the experiments, etc., are shown in the subsequent three sections.

I. OUTLINES OF CALCULATION

It is often informed by engineers engaged in turbine practice, that moving blades are broken by the unavoidable vibrations at a certain velocity notwithstanding the precaution taken in the construction. Thus the determination of the frequency of blade vibrations has become one of the important subjects in the turbine design, especially when the velocity is high.

The usual form of turbine blades is not truly prismatic, and the graphical method seems to be preferred in the calculation, if the variation of the cross section can not be expressed by a simple expression. For this subject, the reference may be made to the well known work by
Stodola. But it seems that the variation of the cross section may be sometimes approximately expressed by a formula, and then a ready formula can be deduced for calculating the frequency. In this way we shall be able to know the general feature of the problem with particular regard to the variation of the frequency with the shape of blades.

The present note is the result of the calculation performed to meet the need in the practice as above stated, and it is based on the approximate method first developed by Lamb and Southwell in the investigation of the rotating disk. So we have to find the frequency of the elastic blade at rest, and that of the flexible blade in rotation, separately; then the sum of the squares of the individual frequencies is approximately equal to the square of the required frequency under the combined action of the restoring forces (both elastic and centrifugal), the approximate value thus found being strictly the lower limit of the true value.

When the area and the moment of inertia are proportional to a certain power \( \mu \) of the distance from the free end, the radius of gyration being therefore constant, the frequency of an elastic blade at rest may be calculated by

\[
k = \frac{ai}{\ell^2} \sqrt{\frac{E}{\rho}},
\]

where \( a \) denote by

- \( \ell \) the length of the blade,
- \( i \) the radius of gyration with reference to a principal axis of the cross section,
- \( E \) the modulus of elasticity of the blade material,
- \( \rho \) the density of the blade material,

and we may take approximately

\[
a = 3.47(1 + 1.05\mu).
\]

The above assumption that the radius of gyration remains constant along the length of the blade may be only approximately correct in an existing blade. For example, Fig. 1 shows the variations of the cross section \( f \) and of the moments of inertia, \( I_r \) and \( I_s \) along the length of a blade. In this case, (1) and (2) may be approximately used for the vibration in the axial direction of the turbine disk, and they may be applied with less accuracy for the vibration in the tangential direction. These directions are denoted as Nos. 1 and 2, respectively. A more general case in which the radius of gyration varies also with a certain power of the

distance, has been dealt with by Dr. Dorothy M. Wrinch,\(^3\) whose formula being deduced from an asymptotic expansion, does not exactly conform to the present calculation. The deduction of the equations (1) and (2) will be given in the next section.

Next let us consider a flexible blade with the variable cross section attached to the periphery of a disk, radius \(a\), rotating at the angular velocity \(\omega\); then the calculation in the third section shows the frequency of the natural vibration to be approximately as follows.

\[
k_1^2 = \beta \omega^2, \quad \text{................................. (3)}
\]

where

\[
\beta = 1 + (1.45 + 0.4\mu) a/l \quad \text{................................. (4)}
\]

From (2) and (4), we observe that \(k\) and \(k_1\) increase with \(\mu\), i.e., the frequency is higher when the blade is more sharply tapered toward the tip, as can be generally expected. Moreover, in the flexible blade, the frequency increases with the ratio \(a/l\). Thus the shorter the blade compared with the radius of the disk, the greater the frequency.

Having found the frequencies of vibrations under the elastic and centrifugal forces, respectively, we have to find the required frequency in rotation by

\[
\nu^2 = k^2 + k_1^2. \quad \text{................................. (5)}
\]

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It will be desirable to compare the result of the foregoing calculations with that obtained by other modes of procedure. For this purpose, let us refer to the result of the calculation based on Rayleigh's energy principle as given in the work by Stodola, who shows that the frequency \( \nu \) of a blade with the uniform section is given by

\[
\nu^2 = 12 \cdot 2 \frac{E b^2}{\rho k^2} + 1.57 \frac{R}{l} \omega^2, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6)
\]

where \( R \) is the mean radius of the wheel measured to the middle of the blade, i.e., \( a + l/2 \). According to the said author, the numerical coefficients on the right side were determined to make the first term become a minimum with little attention to the second term. Now in the case of a uniform section, we have from (1) to (5)

\[
\nu^2 = b^2 + k^2 = 12 \cdot 36 \frac{E b^2}{\rho k^2} + 1.45 \frac{a + 0.72 l}{l} \omega^2. \quad \ldots \ldots \ldots (7)
\]

The comparison of (6) and (7) shows that the difference is not very considerable.

II. VIBRATIONS OF ELASTIC BARS
(Elastic Blades at Rest.)

Taking the origin at the equilibrium position of the free end in Fig. 2, the original direction of the axis of the blade is denoted as the axis of \( x \), and the axis of \( y \) is taken in the plane of vibrations containing one of the principal axes of the cross section. The variations of the cross sectional area, and of the moment of inertia, along the axis of \( x \), are assumed as can be expressed in the forms \( bx^\mu \) and \( cx^\nu \), respectively, where \( \lambda, \mu, b, c \) are constants independent of \( x \). Thus, if we denote the modulus of elasticity by \( E \), and the density of the material by \( \rho \), we have

\[
E c \frac{\partial^2}{\partial x^2} \left( x^\lambda \frac{\partial^2 y}{\partial x^2} \right) + \rho bx^\mu \frac{\partial^2 y}{\partial t^2} = 0. \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]

Here we put \( y = X \cos kt \), \( X \) being a function of \( x \) only. Then the differential equation for \( X \) is

\[
\frac{d^2}{dx^2} \left( x^\lambda \frac{d^2 X}{dx^2} \right) = \frac{\rho \beta c^2}{E c} a^\mu X,
\]

or after performing the differentiation on the left side,

(4) Loc. cit.
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\[ \frac{d^3X}{dx^3} + \frac{2\lambda}{x} \frac{d^2X}{dx^2} + \frac{\lambda(\lambda-1)}{x} \frac{d^2X}{dx^2} = Bx^{\mu-\lambda}X, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 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\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 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\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \l
first term in the series of $X$ should be either $a_0$ or $a_1x$, corresponding to
$n=0$ or $1$, respectively. In the former case,
\[
a_n = \frac{B^{n/\alpha_0}}{n(n-1)(n-2+\lambda)(n-3+\lambda)(n-4)(n-5)(n-6+\lambda)(n-7+\lambda)\ldots4.3.2+\lambda(1+\lambda)}.
\]
$n$ being always a multiple of 4, and in the latter case
\[
a_n = \frac{B^{(n-1)/\alpha_1}}{n(n-1)(n-2+\lambda)(n-3+\lambda)(n-4)(n-5)(n-6+\lambda)(n-7+\lambda)\ldots5.4.3+\lambda(2+\lambda)},
\]
in which the constants $\alpha_0$ and $\alpha_1$ are still arbitrary.

Thus the two kinds of the solutions are
\[
X_1 = a_0 \sum_{m=0}^{\infty} \frac{B^{m\alpha_0}}{4^{m+1/4}m^{(3+\lambda)/4}(1+\lambda/4)^m}.
\]
\[
X_2 = a_1 \sum_{m=0}^{\infty} \frac{B^{m\alpha_1}}{4^{m+1/4}m^{(5+\lambda)/4}(2+\lambda/4)^m}.
\]

Accordingly, $X = X_1 + X_2$, in which the constants $a_0$ and $a_1$ are still arbitrary.

Next the conditions to be satisfied at the other fixed end are $y=0$ and $\partial y/\partial z=0$ for $x=l$. Writing $P$ and $Q$ for $X/a_0$ and $X/a_1$ at $x=l$, we have
\[
a_0 P + a_1 Q = 0, \quad a_0 P + a_1 Q = 0.
\]
Thus
\[
PQ = \hat{P}Q, \quad \hat{P}Q = \hat{P}Q.
\]
where $\xi = B_i^i$.

The least value of $\xi$ satisfying the equation (12) was determined by assigning some particular values to $\lambda$. In this calculation, the terms containing $\xi^2$ and higher powers were neglected in each series, but the result will be accurate enough for the present purpose as the series are rapidly convergent. Denoting the root thus determined by $\xi_1$, we have the equation for the frequency as follows.

$B_i^i = \xi_1$, ..............(14)

or by (3)

$$\frac{\rho lbk^4}{E_0} = \xi_1.$$  

Therefore with $c/b = i^i$,  

$$k = \frac{a}{b} \sqrt{\frac{E}{\rho}}, \text{........................................(15)}$$

where $a = \sqrt{\xi_1}$.

The values of $\xi_1$ and $a$ for several values of $\lambda$ or $\mu$ are given in Table I, in which the value of the coefficient $a$ for the uniform bar was supplied by the well known equation.

<table>
<thead>
<tr>
<th>$\lambda = \mu$</th>
<th>$\xi_1$</th>
<th>$a = \sqrt{\xi_1}$</th>
<th>$a$ (approximate)</th>
<th>Error % correct-approximate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>3.516</td>
<td>3.47</td>
<td>1.3</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>19.0</td>
<td>4.26</td>
<td>4.27</td>
<td>-0.2</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>27.5</td>
<td>5.24</td>
<td>5.29</td>
<td>-1.0</td>
</tr>
<tr>
<td>$\frac{3}{4}$</td>
<td>38.2</td>
<td>6.18</td>
<td>6.10</td>
<td>-0.3</td>
</tr>
<tr>
<td>1</td>
<td>51.2</td>
<td>7.16</td>
<td>7.11</td>
<td>0.6</td>
</tr>
</tbody>
</table>

If we plot the value of $a$ given in the third column as a function of $\mu$ in a diagram, the curve obtained will be as shown in Fig. 3. Although it is slightly curved, it may be approximately replaced by a straight line of the form  

$$a = 3.47 \left(1 + 1.05\mu\right). \text{........................................(16)}$$

Accordingly, the frequency increases with $\lambda$ or $\mu$ in the linear relation, showing that the tapering of the blade elevates the natural frequency, provided
that the cross sectional area and the moment of inertia vary in a similar manner. \(^{39}\)

Let us take a turbine blade with the following data (cp. Fig. 1).

\(l=13.8\, \text{cm} \) (measured from the tip to the enlarged section, \(0.7\, \text{cm} \) above the fixture), \(\lambda = \mu = 0.2,\)

\(E = 1,150,000 \, \text{kg/cm}^2 \) (by experiment),

\(\rho = 8.886 \times 10^{-3} \, \text{kg/cm}^2.\)

Then from (16) \(\alpha = 3.47(1 + 1.05 \times 0.2) = 4.20,\)

and from (15) \(k = \frac{4.201}{1.83} \sqrt{\frac{1.15 \times 10^6 \times 980}{8.886 \times 10^{-3}}} = 78,535,\)

or \(\frac{k}{2\pi} = 12,500.\)

In the present case \(\mu\) is not constant, but if we take \(\mu_1\) at 0.39 cm, and \(\mu_2\) at 0.24 cm, corresponding to a section nearer to the root than to the tip, then \(\frac{k}{2\pi} = 487\) and 300, respectively.

### III. Vibrations of Flexible Bars

(Flexible Blades in Rotation.)

Next let us consider the frequency of moving blades under the action of the centrifugal force by neglecting the flexural rigidity. Assuming the cross sectional area to vary in the same manner as already considered, the centrifugal force at any section at the distance \(x\) from the free end is

\(C = \rho b a r \left[(a + x) \frac{x^{a+1}}{a+1} - \frac{x^{a+2}}{a+2}\right].\) \hspace{1cm} (1)

and the equation of motion is

\[\frac{\partial}{\partial x} \left[C \frac{\partial y}{\partial x}\right] = \rho b x \mu \frac{\partial^2 y}{\partial t^2}.\] \hspace{1cm} (2)

Substituting for \(C\) from (1),

\[\left[(a + x) \frac{\mu + 2}{\mu + 1} - x\right] \frac{\partial^2 y}{\partial x^2} + (\mu + 2)(a + l - x) \frac{\partial y}{\partial x} = \frac{\mu + 2}{\omega^2} \frac{\partial^2 y}{\partial t^2}.\] \hspace{1cm} (3)

Assume that \(y = F \cos kx,\) \hspace{1cm} (4)

\(F\) being a function of \(x\) only. Then

\[\left[(a + l) \frac{\mu + 2}{\mu + 1} - x\right] \frac{d^2 F}{dx^2} + (\mu + 2)(a + l - x) \frac{dF}{dx} + (\mu + 2) \frac{k_1^2}{\omega^2} F = 0.\] \hspace{1cm} (5)

Particularly, when the cross section is constant, i.e., \(\mu = 0,\)

\[2(a + l) x - x^2 \frac{d^2 F}{dx^2} + 2(a + l - x) \frac{dF}{dx} + \frac{2k_1^2}{\omega^2} F = 0.\] \hspace{1cm} (6)

Put \(z = 1 - x/(a + l)\) or \(x = (a + l)(1 - z);\) then

\(6\) The expressions (15) and (16) suggest that the required frequency may be obtained by considering a uniform bar with the radius of gyration \(r = r(1 + 1.0 \mu).\)
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\[(1 - z^2) \frac{d^2 F}{dz^2} - 2z \frac{dF}{dz} + \frac{2k_z^2}{\omega^2} F = 0. \]

A solution for \( F \) is the zonal harmonic \( P_n \), provided that \( s(s+1) = \frac{2k_z^2}{\omega^2} \)

or \( k_z^2 = \frac{s}{2}(s+1)\omega^2 \).

The condition to be satisfied at the fixed end of the blade, i.e., \( z = \frac{a}{a+l} \), corresponding to \( x = l \), is that \( F = 0 \). Accordingly,

\[ P_n \left( \frac{a}{a+l} \right) = 0. \]

The zero point of \( P_n \) for an integral value of \( s \) being found from the table of zonal harmonics, we can determine the value of \( a/l \) satisfying the above condition. The numerical values are shown in the third column of Table II. The frequency corresponding to the assigned value of \( s \) may be readily found by (8), i.e., by multiplying a factor \( \beta = \frac{1}{2}s(s+1) \) into \( a^2 \).

Returning to the general case, let us transform the equation (5) into the hypergeometric equation by putting

\[ z = 1 - \frac{\mu+1}{\mu+2} \frac{x}{a+l} \quad \text{or} \quad x = (1-z)(a+l)\frac{\mu+2}{\mu+1}. \]

Thus

\[ z(1-z) \frac{d^2 F}{dz^2} + \{1 - (\mu+2)z\} \frac{dF}{dz} + (\mu+2) \frac{k_z^2}{\omega^2} F = 0. \]

The solution of this equations is of the well known form

\[ F = F(a, \beta, \gamma, x) = 1 + \frac{a^2}{1, \gamma} \frac{z}{1, 2, \gamma} + \frac{a(a+1)\beta(\beta+1) x^2}{1, 2, \gamma(\gamma+1)} + \frac{a(a+1)(\alpha+2)\beta(\beta+1)(\beta+2) x^3}{1, 2, 3, \gamma(\gamma+1)(\gamma+2)} + \cdots \]

where \( \alpha = a+1, \quad \beta = -(\mu+2) \frac{k_z^2}{\omega^2}, \quad \gamma = 1. \)

The values of \( \alpha \) and \( \beta \) may be taken as follows.

\[ a = \frac{1}{2} \left[ \frac{\mu+1}{\mu+2} + \sqrt{(\mu+1)\gamma + 4(\mu+2) \frac{k_z^2}{\omega^2}} \right] \]

\[ \beta = \frac{1}{2} \left[ \frac{\mu+1}{\mu+2} - \sqrt{(\mu+1)\gamma + 4(\mu+2) \frac{k_z^2}{\omega^2}} \right] \]

The series (12) is convergent for all values of \( z < 1 \), but as \( \gamma < a + \beta \), it diverges, when \( z = 1 \). Therefore, the series ought to terminate in finite terms, i.e., \( \beta = -s, \) \( s \) being a positive integer.

Thus putting \( \mu+1 - \sqrt{(\mu+1)^2 + 4(\mu+2) \frac{k_z^2}{\omega^2}} = 2s \), we have the ratio...
\[ k_1^2 = \frac{s(s + \mu + 1)}{\omega^2 - \mu + 2}, \]

which will be denoted by \( \beta \) in the following, (not to be confounded with \( \beta \) temporarily used in (12) and (13)).

The series (12) may be now written.

\[ F = \sum_{m=0}^{\infty} \frac{(s + \mu + 1)(-s)^m}{(m!)^2} \] \( \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (15) \]

The condition to be satisfied at the fixed end, i.e.,

\[ z = 1 - \frac{(\mu + 1)(\mu + 2)(a + l)}{s}, \]

is that \( F = 0 \). By assigning several values to \( \mu \) and \( s \), we can determine \( z \) as making \( F \) vanish, and then the ratio \( a/l \) is calculated. This ratio and the corresponding value of \( \beta \) are collected in Table II.

### Table II. Flexible Blades under Centrifugal Force.

\[ k_1^2 = \beta \omega^2. \]

<table>
<thead>
<tr>
<th>( s )</th>
<th>( \mu = 0 )</th>
<th>( \mu = \frac{1}{4} )</th>
<th>( \mu = \frac{1}{2} )</th>
<th>( \mu = \frac{3}{4} )</th>
<th>( \mu = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>( a/l )</td>
<td>( \beta )</td>
<td>( a/l )</td>
<td>( \beta )</td>
<td>( a/l )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1.36</td>
<td>2.889</td>
<td>1.19</td>
<td>2.8</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>3.44</td>
<td>5.067</td>
<td>3.00</td>
<td>5.4</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>6.19</td>
<td>9.323</td>
<td>5.83</td>
<td>8.8</td>
</tr>
</tbody>
</table>

Plotting the sets of values for \( a/l \) and \( \beta \), for definite values of \( \mu \), in a diagram, the points are found to lie nearly along straight lines through the point \( \beta = 1 \) for \( a/l = 0 \) as shown in Fig. 4. Thus we may express the relation as

\[ \beta = 1 + e(a/l), \]

where \( e \) is a numerical coefficient with the values as shown in Table III.

As \( e \) varies in a linear relation with \( \mu \), we may again express thus,

\[ e = 1.45 + 0.4\mu, \]

Accordingly, from (16) and (17)

\[ \beta = 1 + (1.45 + 0.4\mu)a/l. \]

This equation may be used with sufficient degree of accuracy for the calculation of the frequency, which is higher when the blade is more sharply tapered toward the free end. Besides, the shorter the blade compared with the radius of the disk, the greater the frequency.

Take for example a disk with the radius \( a = 62 \text{ cm} \), and a flexible
TABLE III.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.45</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>1.35</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>1.65</td>
</tr>
<tr>
<td>( \frac{3}{4} )</td>
<td>1.75</td>
</tr>
<tr>
<td>1</td>
<td>1.85</td>
</tr>
</tbody>
</table>

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blade with the same form as in the preceding example, viz., \( l = 13.8 \) cm, \( \mu = 0.2 \). Then

\[ \alpha/l = 62/13.8 = 4.49, \]

and from (18)

\[ \beta = 1 + (1.45 + 0.4 \times 0.2) \times 4.49 = 7.87. \]

Accordingly,

\[ k_1^2 = 7.87 \omega^2. \]

If we take the velocity of the rotating disk at 3,000 revolutions per min, then with \( \omega = 100 \pi \), \( k_0/2\pi = 140 \) oscillations per sec.

The calculation of the resulting frequencies will be reserved for the next section until experimental results will be introduced.

IV. EXPERIMENTAL DETERMINATION OF FREQUENCIES

Series of experiments were performed to find the frequency of the natural vibrations of a bronze blade (at rest) with the guiding dimensions as shown in the first section. The blade fixed at the enlarged end, was set in vibration by the sudden release of a lateral load acting at the tip, and the frequency was found in comparison with the known frequency of a tuning fork. Two directions of vibrations were selected in the experiments, one corresponding to the axial direction of the turbine disk, and the other being the tangential direction, as already denoted as Nos. 1 and 2, respectively. In this way it was designed to amend the result of the calculation performed in the second section, and to find the discrepancy due to simple assumptions made in the calculation, e.g., the assumptions that the cross section of the blade varies as \( x^n \), and the radius of gyration is.
constant, that the length of the blade is equal to the distance between the
tip and the enlarged section, etc. The experiments were performed by
Mr. Fukuda of my institute.

In one series, the vibrations of the blade, and of the tuning fork with
a known frequency, were recorded on a smoked paper. Here it was
necessary to attach a very light needle made of gelatin to the tip of the
blade, but the effect on the result is considered to have been negligible. As
the vibration subsided rapidly, the train of waves marked on the paper was
naturally very limited, and a microscope was used to observe the minute
waves. This method was applied to the observation of the vibration in
the direction No. 1, and the average number of oscillations per second was
477.3.

In another series of the experiments, an optical method was used, and the
record was taken on a photographic film surrounding a rotating cylinder.
An arc-lamp was used as the source of light, which passed through con-
densers, a slit in a diaphragm, and a photographic lens. Also a cylindrical
lens was placed in the front side of the film box to obtain a fine image of
the slit, and the vibration of the blade placed at one side of the diaphragm
caused the continuous change of the length of the slit image. Besides, the
tuning fork was placed in the path of light. The length of the apparatus
was greater than five meters, and the magnification was about 35 times in
the present experiment. The vibrations in both directions were observed
by this method, and the average results were 477.2 and 300.4, respectively.

All the parts of the apparatus were the ordinary equipments already
existed in my laboratory, except a long camera and a slit. The idea of
using a slit was due to Société Genevoise pour la Construction des Instru-
ments de Physique et de Mécanique, while the general arrangement,
including the fixing device of the blade in position, the use of the tuning
fork for time record, etc., was designed in my laboratory. Complete sets
of a similar apparatus may be supplied by the said firm, which is represented
by Messrs. E. Zellweger & Co in Tokyo, to whom my thanks are due for
having kindly met my desire to publish the present report. The photographs
in Figs. 5 and 6 are the examples of the optical records (reproduced in
half size).

The experimental results agree rather well with the estimated values,
though the latter values are a little higher than the former, owing among
other things to the reason that the length of the blade taken in the
calculation was shorter than the elastic part of the actual blade.

Now combining by (5) of the first section, the observed frequencies,
VIBRATIONS OF TURBINE BLADES

Fig. 5

Fig. 6

viz., 477 and 300 in round numbers, respectively, with the calculated frequency of a flexible blade in rotation, we get the required frequency of an elastic blade in rotation

$$\sqrt{\left(\frac{k}{2\pi}\right)^2 + \left(\frac{k_1}{2\pi}\right)^2} = 100\sqrt{22.75 + 1.97} = 497,$$

and

$$\sqrt{\left(\frac{k}{2\pi}\right)^2 + \left(\frac{k_1}{2\pi}\right)^2} = 100\sqrt{9.00 + 1.97} = 331.$$

The above instance should not be taken as the proof that the present calculation gives always a reliable result, because the estimated value may be more or less indefinite, according to the choice of some data, e.g., the length of blade, the radius of gyration, etc. Thus, in the present scope of the investigation, the calculation may serve as a guide of design, and the experiment can decide the matter.

Fukuoka, March, 1924.