MOVEMENT OF THE IRROTATIONAL VOLTEX IN A LIMITED FIELD

(The first report : On the flow in a limited field.)

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By

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ABSTRACT

In the course of the experiment on the flow of a fluid, there is a phenomena, often met, that a vortex occupied itself in neighbourhood of a stagnation point of flow. If the fluid be considered as nonviscous, the flow be potential one and the vortex be an irrotational one, the state in which the vortex occupies its position in the flow can be treated as a combined state of the each existency which will fulfil the field independently. Thus, in this paper, *a.* the movements of vortices are discussed, *b.* the potential functions of vortices and flows are deduced by the method of conformal representation with several numerical examples and *c.* the combined states of the vortex and flow are also discussed and are accompanied by several examples concerning to the fluid motion in the cylinder of a two cycle internal combustion engine of port scavenging type.

INTRODUCTION

(1) THE kinematic states of vortex in a limited field had been studied by so many investigators and its characteristics had been given by Helmholtz,⁽¹⁾ Kármán,⁽²⁾ Routh,⁽³⁾ Greenhill,⁽⁴⁾ Michell⁽⁵⁾ etc.⁽⁶⁾ in their formulae, but they seem somewhat inconvenient to apply directly for such an Engineering problems as the author wants to treat. The object of the present investigation is to treat a movement of an irrotational vortex which capture itself in a flow corner. Therefore, this paper consists of three sections, that is, the first the problem of vortex in the limited field, the second the problem of a potential flow that fulfils the same limited

- (1) Ueber integrale der hydrodynamischen gleichungen Welche der Wirbelbewegungen entsprechen. Crelle. tIV, p 25-55. 1858.
- (2) Flusigkeit und luft widerstand. Phy. zeitsch. t xiii, 1912.
- (3) Some application of conjugate function. Proc. Lond. Math. Soc. t xii, p. 86, 1881.
- (4) On plane vortex motion. Quart. jour. Math. t xv 1887.
- (5) The theory of free stream line. Roy. Soc. Phil. trans. A, tcl xxxi p. 389, 1890.
- (6) Lamb; Hydrodynamics
Hicks; On the condition of steady motion of two cylinder in fluid. Quart. Jour. Math. t viii p. 174 1881

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field and that contains a vortex in its corner, and the third that the case of combination of the vortex and flow in the limited field or the problem of a flow that contains a vortex.

The field, irrotational vortex and potential flows considered in this investigation, are that treated two dimensionally expressed on Gaussian plane z . Under the suitable treatments as mentioned in the following sections, the author found the point of equilibrium of vortex to be stand still, velocity potential and stream function, velocity distribution along stream lines; and stream lines with several examples for both vortex and flows. Also the author notes that, in the given limited field, the regions of vortex and flow vary with the relative strength of these existency; that the effective width is introduced from the general equation of separation line or surface between these two regions; that the points of stagnation, at this point the pressure acts on the wall and takes a maximum value in its neighbourhood are determined from velocity distribution along the walls; that the relation between K/K_T (ratio of strength of vortex to flow), l_0 (width of a vortex region on the centre line of the field) and x_1 (the position of a vortex centre on the centre line of the field) for a vortex to be stand still in a flow, and the actual example is analogically shown.

Fig. 1

SECTION I. IRROTATIONAL VORTEX IN A LIMITED FIELD

(2) Transformation of plane which include a limited field and vortex. A field, expressed on Gaussian plane z as shown in Fig. 1, limited by the straight walls $y=0$, $y=c$ and $x=0$ in a convex polygon $ABCD$, which the author calls the "limited field". The points in this field will satisfy the next relation.

$$z = x + iy \dots \dots \dots (1)$$

$$x < 0$$

$$0 < y < c$$

c = distance between $y=0$ and $y=c$.

On the plane z , P be a position of vortex filament in the field, the images of P for line symmetry and point

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symmetry are obtained as P' , P_1 and P_1' , and the images of the field II that include P are named as I, III and IIII including the points P' , P_1 and P_1' , respectively.

limit	field	vortex
$x > 0, \quad c > y > 0$	I	P'
$x < 0, \quad c > y > 0$	II	P
$x < 0, \quad -c < y < 0$	III	P_1
$x > 0, \quad -c < y < 0$	IIII	P_1'

Then, we can extend equ. (1) over this group of images changing the limit of y in $c > y > -c$ and transform this figure on plane z' from the plane z with next relations.

$$z' = \frac{\pi}{c}(x + iy) \dots \dots \dots (2)$$

$$t = \cosh z' = \xi + i\eta \dots \dots \dots (3)$$

where

$$\xi = \cosh \frac{\pi}{c} x \cos \frac{\pi}{c} \eta \dots \dots \dots (3_a)$$

$$\eta = \sinh \frac{\pi}{c} x \sin \frac{\pi}{c} \eta$$

Thus, the field I, III and II, IIII are expressed on the upper and lower half planes of t , respectively. Again, we transform this upper half figure into a unit circle which boundary is composed of the straight line $ABCD$ in the plane t' .

$$t' = -i \frac{t+i}{t-i}, \dots \dots \dots (4)$$

hence, z plane is transformed into the plane t' , changing the straight wall AB , BCD and DA into a closed circular boundary $ABCD$, and the plane t' represents the field of the odd and even ordering number with the outside and inside of the unit circle $ABCD$. Therefore, it is easily found that the points P_1 , P' , P_1 and P_1' hold the relations of images in each plane t and t' , moreover the vortex P and P' coincide with P_1' and P_1 respectively. In the investigation of the circular boundary $ABCD$ on the plane t' , P and P' having the same strength and opposite sign, a value of stream function for each vortex at a point S on $ABCD$ will result in a new value that is constant as the resultant stream line of these two vortices P and P' . Thus it is known that the boundary $ABCD$ lies along a resultant stream line of the vortices P and P' .

(3). On the plane t , let the coordinates of vortex centre P be (ξ_1, η_1) , then it is able to take a form of

$$\left. \begin{aligned} \xi_1 &= \cosh \frac{\pi}{c} x_1 \cos \frac{\pi}{c} (2qc + y_1) \\ \eta_1 &= \sinh \frac{\pi}{c} x_1 \sin \frac{\pi}{c} (2qc + y_1) \end{aligned} \right\} \dots\dots\dots (A)$$

where $q = \text{integer}$.

Their absolute values do not vary with q , only differing in amplitude and a period $2p\pi$ becoming in act. In this mean, q indicates the q th image of P with a pitch $2qc$ in the plane z , and P and P' on the plane t or t' mean an accumulation of so many points of images which occupy on the plane z with a pitch $2qc$. Innumerable images of P on the plane z arrange themselves on either one of the two rows that symmetrically apart x and $-x$ from y axis, and compose vortex pairs in the order of the adjoining vortices which change their sense alternately (Fig. 2).

(4) Equilibrium position of vortex. On the plane z , let P occupy a point $(x, 0.5c)$ on the axis CA of the field (Fig. 2), then

$$r = \sqrt{(2x)^2 + (qc)^2},$$

where $q = \text{integer}$ indicating an order number of image.

The total effect on P due to the images is an algebraic sum of each effect.

$$\left. \begin{aligned} U &= \frac{1}{2\pi} \sum \frac{K}{r} \sin \theta_q \\ V &= \frac{1}{2\pi} \sum \frac{K}{r} \cos \theta_q \end{aligned} \right\} \dots\dots\dots (5)$$

therefore, the effect due to the image vortices belong to the row I in which P situates

$$U_1 = \frac{K}{2\pi} \sum \frac{(-1)^q}{qc} \sin \theta_q \dots\dots\dots (5a)$$

$$V_1 = \frac{K}{2\pi} \sum \frac{1}{qc} \cos \theta_q \dots\dots\dots (5b)$$

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Fig. 2

where θ_q = angle between AC and a line connecting P and its q th order image, and changes its sign according to its locality for the same q .

for the row I, $\theta_q = \pm \frac{\pi}{2}$, it gives

$$U_I = 0 \text{ and } V_I = 0.$$

The effect due to the image vortices belongs to the row II

$$U_{II} = \frac{K}{2\pi} \sum_{-\infty}^{\infty} \frac{(-1)^q}{\sqrt{(2x)^2 + (qc)^2}} \sin \theta_q = 0 \dots\dots\dots (5_c)$$

$$V_{II} = \frac{K}{2\pi} \left[\sum_{-\infty}^{\infty} \frac{-2(-1)^q}{\sqrt{(2x)^2 + (qc)^2}} \cos \theta_q + \frac{1}{2x} \right] \dots\dots\dots (5_d)$$

put $\cos \theta_q = 2x / \sqrt{(2x)^2 + (qc)^2}$, $2x/c = a$

and divide q into two parts of odd $q = 2x + 1$ and even $q = 2t$, (t = positive integer including zero),

then, equ. (4a) can be rewritten in

$$V_{II} = \frac{K}{2\pi} \frac{2a}{c} \left[- \sum_{t=0}^{\infty} \frac{1}{a^2 + (2t)^2} + \sum_{t=0}^{\infty} \frac{1}{a^2 + (2t+1)^2} + \frac{1}{2a^2} \right],$$

therefore, the component velocities are given in

Fig. 3

$$U = U_I + U_{II} = 0$$

$$V = V_I + V_{II}$$

$$= \frac{K}{2\pi} \frac{2a}{c} \left[- \sum_{t=0}^{\infty} \frac{1}{a^2 + (2t)^2} + \sum_{t=0}^{\infty} \frac{1}{a^2 + (2t+1)^2} + \frac{1}{2a^2} \right]$$

$$= -\frac{K}{2} \frac{1}{c} \frac{\pi}{\sinh a\pi} \dots\dots\dots (5_e)$$

One of the conclusions that the vortex P in this system to be in equilibrium, must the $U=0$ and $V=0$.,

then, from equ. (4_e)

$$x = \pm \infty$$

The equilibrium state will be obtained with such vortices which centres occnpy at infinite point on CA . In this system, the vortex pairs, alternating its sense, are distributed all over the plane z symmetrically with their boundaries. The velocity potential and stream functions are the same that of the flow mode VIII⁽⁷⁾ deduced in the next section as given in

$$\left. \begin{aligned} \phi &= -\frac{K}{2\pi} \cosh \frac{\pi}{c} x \cos \frac{\pi}{c} y \\ \psi &= -\frac{K}{2\pi} \sinh \frac{\pi}{c} x \sin \frac{\pi}{c} y \end{aligned} \right\} \dots\dots\dots (B)$$

(5) Velocity potential and stream function. In ordinary way, let the plane w be a plane expressed the relation between velocity potential ϕ and stream function ψ ,

$$w = \phi + i\psi, \dots\dots\dots (6)$$

then, the velocity components are

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \dots\dots\dots (7)$$

Fig. 4

(6) Let the coordinate of a vortex in the field be (x_1, y_1) we can trans-

(7) see p. 762 equ. (28),

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form the plane t' into the plane t'' , in which vortex centre P stands on the centre of a circular boundary $ABCD$ (Fig. 4).

$$t'' = K \frac{t' - (k + il)}{t' - (m + in)} \dots \dots \dots (8)$$

When k and l are the coordinates of P on the plane t'
 m „ n „ „ P' „
 and are given by equs. (3_a) and (4) in

$$\left. \begin{aligned} k &= \frac{2\xi_1}{\xi_1^2 + (\eta_1 - 1)^2}, & l &= \frac{\xi_1^2 + \eta_1^2 - 1}{\xi_1^2 + (\eta_1 - 1)^2} \\ m &= \frac{k}{k^2 + l^2}, & n &= \frac{l}{k^2 + l^2} \\ \xi_1 &= \cosh \frac{\pi}{c} x_1 \cos \frac{\pi}{c} y_1, & \eta_1 &= \sinh \frac{\pi}{c} x_1 \sin \frac{\pi}{c} y, \end{aligned} \right\} \dots \dots \dots (9)$$

then, equ. (8) can be rewritten in

$$t'' = R \frac{A - iB}{C - iD} \dots \dots \dots (10)$$

$$= R \sqrt{\frac{A^2 + B^2}{C^2 + D^2}} \{ \cos \vartheta + i \sin \vartheta \} \dots \dots \dots (10_a)$$

$$\text{where } \left. \begin{aligned} A &= -k\xi + (l+1)\eta - l + 1, & B &= (l+1)\xi + k\eta - k, \\ O &= -m\xi + (n+1)\eta - n + 1, & D &= (n+1)\xi + m\eta - m. \end{aligned} \right\} \dots (10_b)$$

R is a coefficient that controls the radius of circular boundary $ABCD$ on the plane t'' (Fig.4), and it is convenient to take a value which gives unit radius, for this fact equ. (10_a) gives

$$R = \sqrt{\frac{C^2 + D^2}{A^2 + B^2}} = \sqrt{\frac{(-n+1)^2 + m^2}{(-l+1)^2 + k^2}}, \dots \dots \dots (10_c)$$

(7) The figure on the plane t'' is composed of a circular boundary $ABCD$ with centre P at which the vortex centre occupies and P' , the image of P , situates at infinity. Moreover, the boundary $ABCD$ lies along the resultant stream line of P and P' , and we can deduce this plane figure to the case that a circular vortex exists in a infinite field. So the author can obtain from equ. (10),

$$\begin{aligned} w &= \phi + i\psi \\ &= -i \frac{k}{2\pi} \log t'' \end{aligned}$$

$$= -\frac{K}{2\pi} \left\{ -\tan^{-1} \frac{AD-BC}{AC+BD} + i \frac{1}{2} \log R^2 \frac{A^2+B^2}{C^2+D^2} \right\},$$

or

$$\left. \begin{aligned} \phi &= \frac{K}{2\pi} \tan^{-1} \frac{AD-BC}{AC+BD}, \\ \psi &= -\frac{k}{2\pi} \frac{1}{2} \log R^2 \frac{A^2+B^2}{C^2+D^2}, \end{aligned} \right\} \dots\dots\dots (11)$$

(8) Special case. When a vortex centre situates on CA line, the coordinate of centre will be $(x_1, 0.5c)$ and it gives always $\xi_1=0$ and $k=0$, so that equ. (10_c) gives $R = \frac{-n+1}{-l+1}$ and from equ.(11)

$$\left. \begin{aligned} \phi &= \frac{K}{2\pi} \tan^{-1} \frac{-2\xi\eta_1}{-\eta_1^2+(\xi^2+\eta^2)}, \\ \psi &= -\frac{K}{2\pi} \frac{1}{2} \log \frac{\xi^2+(\eta-\eta_1)^2}{\xi^2+(\eta+\eta_1)^2}, \end{aligned} \right\} \dots\dots\dots (11_a)$$

Example 1. Velocity potential and stream function for a vortex which centre at $(x_1=-0.5c, y_1=0.5c)$.

From equ. (10_c) $R=2.536$

$$\begin{aligned} \text{,, equ. (11}_a\text{)} \quad \phi &= \frac{K}{2\pi} \tan^{-1} \frac{4.634\xi}{\xi^2+\eta^2-5.369} \\ \psi &= -\frac{K}{2\pi} \frac{1}{2} \log \frac{\xi^2+\eta^2+4.634\eta+5.369}{\xi^2+\eta^2-4.634\eta+5.369} \end{aligned}$$

(9) Algebraic equation of stream lines. It is clear that the radius in form of $R\sqrt{\frac{A^2+B^2}{C^2+D^2}}$ is constant for a fixed stream line on the plane t'' , and the radius μ'' is expressed in function of ψ as

$$\mu'' = \mu_0'' e^{(\psi-\psi_0)\frac{2\pi}{k}} = R\sqrt{\frac{A^2+B^2}{C^2+D^2}} < 1 \dots\dots\dots (12)$$

in which μ_0'' is a radius corresponding to a stream line ψ_0 in the plane t'' . The above equation is, at the same time, the algebraic equation of stream lines in the plane z expressing A, B, C and D in the form of a function of x and y as known in equs. (10_b) and (3_a).

Example 2. The relation between μ'' and $e^{\frac{2\pi}{k}\psi}$ given in equ. (12) shown in Fig. 5 assuming $K=-2\pi$ and the author worked out the stream lines of the vortex $P(-0.5c, 0.6c)$ as shown in Fig. 6 with next relation,

$$e^{\frac{4\pi}{k}\psi} = \frac{\xi^2+\eta^2+4.634\eta+5.369}{\xi^2+\eta^2-4.634\eta+5.369}$$

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Fig. 5

Fig. 6

Example 3. Also the author worked out the stream lines with the vortex P $(-0.23c, 0.25c)$ as shown in Fig. 7.

(10) Referred to the plane z as mentioned in (2), the distribution of the vortices is obtained from equ. (A), taking q from $-\infty$ to ∞ . For example, Fig. 8 shows the distribution of a system of vortices which centres are at $(x_1 = \pm 0.5c, y_1 = (q + 0.5c))$ and their stream lines satisfy μ'' that are given in example 2.

(11) Velocity distribution along a stream line. The component velocities along a stream line of this irrotational vortex are obtained by ordinary way as mentioned in equ. (7).

Thus, the author deduced the general and special equations for component velocities as follows (Fig. 8).

In general ;

Fig. 7

Fig. 8

$$u = \frac{K}{2c} \frac{1}{(A^2 + B^2)(C^2 + D^2)} \left[\sinh \frac{\pi}{c} x \cos \frac{\pi}{c} y \left\{ (A^2 + B^2)(Dm + Cn + 1) - (C^2 + D^2)(Bk + Al + 1) \right\} + \cosh \frac{\pi}{c} x \sin \frac{\pi}{c} y \left\{ (A^2 + B^2)(Cm - Dn + 1) - (C^2 + D^2)(Ak - Bl + 1) \right\} \right] \dots \dots \dots (13)$$

$$v = \frac{K}{2c} \frac{1}{(A^2 + B^2)(C^2 + D^2)} \left[-\cosh \frac{\pi}{c} x \sin \frac{\pi}{c} y \left\{ (A^2 + B^2)(Dm + Cn + 1) - (C^2 + D^2)(Bk + Al + 1) \right\} + \sinh \frac{\pi}{c} x \cos \frac{\pi}{c} y \left\{ (A^2 + B^2)(Cm - Dn + 1) - (C^2 + D^2)(Ak - Bl + 1) \right\} \right] \dots \dots \dots (14)$$

In special;

along the wall $A-B$,

$$u = \frac{K}{2c} \sinh \frac{\pi}{c} x \frac{2(k^2 + l^2 - 1)}{\left\{ k^2 + (l+1)^2 \right\} \cosh^2 \frac{\pi}{c} x - 4k \cosh \frac{\pi}{c} x + k^2 + (-l+1)^2} \dots (13_a)$$

$$v = 0,$$

along the wall $B-C-D$,

$$u = 0,$$

$$v = -\frac{K}{2c} \sin \frac{\pi}{c} y \frac{2(k^2 + l^2 - 1)}{\left\{ k^2 + (l+1)^2 \right\} \cos^2 \frac{\pi}{c} y - 4k \cos \frac{\pi}{c} y + k^2 + (-l+1)^2} \dots (14_a)$$

along the wall $D-A$,

$$u = -\frac{K}{2c} \sinh \frac{\pi}{c} x \frac{2(k^2 + l^2 - 1)}{\left\{ k^2 + (l+1)^2 \right\} \cosh^2 \frac{\pi}{c} x - 4k \cosh \frac{\pi}{c} x + k^2 + (-l+1)^2} \dots \dots \dots (13_b)$$

$$v = 0,$$

Fig. 9 shows the velocity distribution along boundaries $A-B$, $B-C-D$ and $D-A$ with the vortex ($x_1 = -0.5c$, $y_1 = 0.5c$) and $K = -2\pi$.

Fig. 9

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SECTION II. POTENTIAL FLOW IN THE LIMITED FIELD

(12) Potential flows treated in this section are two dimensional concerning only the cases of point source and sink situated on the boundaries of the limited field which is the same to that for the vortex. The author treats five flow modes of two cases giving general characteristics and shows several examples.

Let T_1 =source, T_2 =sink,

t =arbitrary point on which a stream line $m\pi$ passes through, and its coordinates are $-a$ and $0.5c$.

m =arbitrary constant.

Case 1. Potential flow from T_1 to T_2 in finite distance on the straight boundaries of AB and DA , respectively.

(13) Velocity potential and stream function. On the plane z , the limited field is the same to that is given in section I,

then, $z=x+iy$,

$$t = \cosh \frac{\pi}{c} z = \xi + i\eta,$$

$$t' = \frac{t - \cosh \frac{\pi}{c} l_1}{t + \cosh \frac{\pi}{c} l_2}, \dots \dots \dots (15)$$

$$t'' = -t' = \frac{\cosh \frac{\pi}{c} l_1 - t}{\cosh \frac{\pi}{c} l_2 + t} \dots \dots \dots (16)$$

	z	t	t''
A	$-\infty, -\infty+i$	∞	-1
B	0	1	$\left(\cosh \frac{\pi}{c} l_1 - 1\right) / \left(\cosh \frac{\pi}{c} l_2 + 1\right)$
C	$i 0.5 c$	0	$\cosh \frac{\pi}{c} l_1 / \cosh \frac{\pi}{c} l_2$
D	$i c$	-1	$\left(1 + \cosh \frac{\pi}{c} l_1\right) / \left(\cosh \frac{\pi}{c} l_2 - 1\right)$
T_1	$-l_1$	$\cosh \frac{\pi}{c} l_1$	0
T_2	$-l_2 + ic$	$-\cosh \frac{\pi}{c} l_2$	∞
t	$-a + i 0.5 c$	$-i \sinh \frac{\pi}{c} a$	$\frac{-\sinh^2 \frac{\pi}{c} a + \cosh \frac{\pi}{c} l_1 \cosh \frac{\pi}{c} l_2 + i \sinh \frac{\pi}{c} a \left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2\right)}{\sinh^2 \frac{\pi}{c} a + \cosh^2 \frac{\pi}{c} l_2}$

Fig. 10

in which, l_1 and l_2 are the distances apart from y axis (Fig. 10).

And assuming the total flow be $n\pi$ for the plane w , we have in common way,

$$w = \phi + i\psi,$$

$$w' = \frac{1}{n}w,$$

and

$$w'' = e^{\frac{1}{n}}w \dots \dots \dots (17)$$

Comparing $A T_1$ and T_2 on both the planes w'' and t'' , we get

$$w'' = t'' \dots \dots \dots (18)$$

and

$$w = n \log t'' \dots \dots \dots (19)$$

$$= n \left[\frac{1}{2} \log \frac{\left(\cosh \frac{\pi}{c} l_1 - \xi \right)^2 + \eta^2}{\left(\cosh \frac{\pi}{c} l_2 + \xi \right)^2 + \eta^2} + i \tan^{-1} \frac{\eta \left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right)}{\xi^2 - \xi \left(\cosh \frac{\pi}{c} l_2 - \cosh \frac{\pi}{c} l_1 \right) + \eta^2 - \cosh \frac{\pi}{c} l_1 \cosh \frac{\pi}{c} l_2} \right]$$

K_T be a strength of this flow, K_T/π becomes the intensity of the flow,

then, $n = K_T/\pi$ or $K_T = n\pi \dots \dots \dots (20)$

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therefore, the general velocity potential and stream function are

$$\left. \begin{aligned} \phi &= \frac{K_T}{\pi} \frac{1}{2} \log \frac{\left(\cosh \frac{\pi}{c} l_1 - \xi \right)^2 + \eta^2}{\left(\cosh \frac{\pi}{c} l_2 + \xi \right)^2 + \eta^2}, \\ \psi &= \frac{K_T}{\pi} \tan^{-1} \frac{\eta \left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right)}{\xi^2 - \xi \left(\cosh \frac{\pi}{c} l_2 - \cosh \frac{\pi}{c} l_1 \right) + \eta^2 - \cosh \frac{\pi}{c} l_1 \cosh \frac{\pi}{c} l_2} \end{aligned} \right\} \dots \dots \dots (21)$$

and this flow is named "flow mode I."

(14) In this equation, it is clear that the different values for l_1 and l_2 give the different modes of flow; that is

flow mode	II	III
l_1	l_1	0
l_2	l_1	l_2
ϕ_T	$\frac{K_T}{\pi} \frac{1}{2} \log \frac{\left(\cosh \frac{\pi}{c} l_1 - \xi \right)^2 + \eta^2}{\left(\cosh \frac{\pi}{c} l_2 + \xi \right)^2 + \eta^2}$	$\frac{K_T}{\pi} \frac{1}{2} \frac{(1-\xi)^2 + \eta^2}{\left(\xi + \cosh \frac{\pi}{c} l_2 \right) + \eta^2}$
ψ_T	$\frac{K_T}{\pi} \tan^{-1} \frac{2\eta \cosh \frac{\pi}{c} l_1}{\xi^2 + \eta^2 - \cosh^2 \frac{\pi}{c} l_1}$	$\frac{K_T}{\pi} \tan^{-1} \frac{\eta \left(1 + \cosh \frac{\pi}{c} l_2 \right)}{\xi^2 - \xi \left(\cosh \frac{\pi}{c} l_2 - 1 \right) + \eta^2 - \cosh \frac{\pi}{c} l_2}$

III
0
0
$\frac{K_T}{\pi} \frac{1}{2} \log \frac{(1-\xi)^2 + \eta^2}{(1+\xi)^2 + \eta^2}$
$\frac{K_T}{\pi} \tan^{-1} \frac{2\eta}{\xi^2 + \eta^2 - 1}$

.....(21_{a,b,c})

Fig. 11

Fig. 12

For numerical example, the author takes $l_1=0=l_2$, $l_1=-1.5c$ and plots the stream lines from equs. (21 a, c) as shown in Fig. 11 and 12.

(15) The next relation must be satisfied on the plane w to take $m\pi$ for a stream line which passes through a point t .

$$\begin{aligned} w_t &= i m\pi + \phi_t, \\ w''_t &= e^{i \frac{m}{n} \pi + \phi_t / n} \\ &= \lambda \left(\cos \frac{m}{n} \pi + i \sin \frac{m}{n} \pi \right), \end{aligned}$$

where $\lambda = e^{\phi_t / n}$
therefore, from equs. (2) and (18)

$$\left. \begin{aligned} \xi_t &= \frac{\left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right) \left(1 + \lambda \cos \frac{m}{n} \pi \right)}{1 + 2\lambda \cos \frac{m}{n} \pi + \lambda^2} - \cosh \frac{\pi}{c} l_2, \\ \eta_t &= - \frac{\left(\cosh l_1 \frac{\pi}{c} + \cosh \frac{\pi}{c} l_2 \right) \lambda \sin \frac{m}{n} \pi}{1 + 2\lambda \cos \frac{m}{n} \pi + \lambda^2}, \end{aligned} \right\} \dots (22)$$

are obtained and with initial assumption, $x_t = -a$, $y_t = 0.5c$ or $\xi_t = 0$, $\eta_t = -\sinh \frac{\pi}{c} a$, then, the equ. (22) gives

$$\lambda = \frac{1}{2} \left\{ \cos \frac{m}{n} \pi \left(\frac{\cosh \frac{\pi}{c} l_1}{\cosh \frac{\pi}{c} l_2} - 1 \right) \pm \sqrt{\cos^2 \frac{m}{n} \pi \left(\frac{\cosh \frac{\pi}{c} l_1}{\cosh \frac{\pi}{c} l_2} - 1 \right)^2 + 4 \frac{\cosh^2 \frac{\pi}{c} l_1}{\cosh^2 \frac{\pi}{c} l_2}} \right\} \dots (23)$$

$$a = \frac{c}{\pi} \sinh^{-1} \frac{\left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right) \lambda \sin \frac{m}{n} \pi}{1 + 2 \cos \frac{m}{n} \pi \lambda + \lambda^2} \dots (24)$$

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The point t must satisfy the equs. (23) and (24), or the position of the point t on CA may be determined by defining a ratio of flows $\left(\frac{m}{n}\pi\right)$, that is, partial flow to the total flow.

(16). Taking special values for l_1 and l_2 in equs. (23) and (24), the author defined the values of λ and a as follows :

Flow mode	II	III
l_1	l_1	0
l_2	l_1	l_2
λ	-1	$\frac{1}{2} \left\{ \cos \frac{m}{u} \pi \left(\frac{1}{\cosh \frac{\pi}{c} l_2} - 1 \right) - \sqrt{\cos^2 \frac{m}{u} \pi \left(\frac{1}{\cosh \frac{\pi}{c} l_2} - 1 \right)^2 + \frac{4}{\cos^2 \frac{\pi}{c} l_2}} \right\}$
a	$\frac{c}{\pi} \sinh^{-1} \frac{-\cosh \frac{\pi}{c} l_1 \sin \frac{m}{u} \pi}{1 - \cos \frac{m}{n} \pi}$	$\frac{c}{\pi} \sinh^{-1} \frac{\left(1 + \cosh \frac{\pi}{c} l_2\right) \lambda \sin \frac{m}{n} \pi}{1 + 2\lambda \cos \frac{m}{n} \pi \lambda^2}$

III'	IIII
l_1	0
0	0
$\frac{1}{2} \left\{ \cos \frac{m}{n} \pi \left(\cosh \frac{\pi}{c} l_1 - 1 \right) - \sqrt{\cos^2 \frac{m}{n} \pi \left(\cosh \frac{\pi}{c} l_1 - 1 \right)^2 + 4 \cosh^2 \frac{\pi}{c} l_1} \right\}$	-1
$\frac{c}{\pi} \sinh^{-1} \frac{\left(\cosh \frac{\pi}{c} l_1 + 1 \right) \lambda \sin \frac{m}{n} \pi}{1 + 2\lambda \cos \frac{m}{n} \pi + \lambda^2}$	$\frac{c}{\pi} \sinh^{-1} \frac{-\sin \frac{m}{n} \pi}{1 - \cos \frac{m}{n} \pi}$

(23) a, b, c, d (24) a, b, c, d

(17). Flow ratio $\frac{m}{n}\pi$ as mentioned in previous article is also given by equ. (21),

$$\frac{m}{n}\pi = \tan^{-1} \frac{-\sinh \frac{\pi}{c} a \left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right)}{\sinh^2 \frac{\pi}{c} a - \cosh \frac{\pi}{c} l_1 \cosh \frac{\pi}{c} l_2} \dots \dots \dots (25)$$

Thus, we can define the values of λ , a , $m\pi$, n and $\frac{m}{n}\pi$ presuming one or two values of them.

Example 4. The equation of a and λ for a pressumable value of $\frac{m}{n}\pi = \frac{\pi}{2}$ with four flow modes. It is easily known that equs. (23_{a,b,c,a}) and (24_{a,b,c,a}) are rewritten in.

Flow mode	II	III
λ	-1	$\frac{-1}{\cosh \frac{\pi}{c} l_2}$
a	$\frac{c}{\pi} \sinh^{-1} \left(-\cosh \frac{\pi}{c} l_1 \right)$	$\frac{c}{\pi} \sinh^{-1} \frac{(1 + \cosh) \cosh \frac{\pi}{c} l_2}{1 + \cosh^2 \frac{\pi}{c} l_2}$

Flow mode	III'	IIII
λ	$-\cosh \frac{\pi}{c} l_1$	-1
a	$\frac{c}{\pi} \sinh^{-1} \frac{(1 + \cosh \frac{\pi}{c} l_1) \cosh \frac{\pi}{c} l_1}{1 + \cosh^2 \frac{\pi}{c} l_1}$	$\frac{c}{\pi} \sinh^{-1} (-1)$

(18) The velocity distribution along a stream lines. The author obtained the velocities along a stream lines differentiating the velocity potential given in equ. (21).

In general ;

$$u = \frac{K_T}{c} \frac{\sinh \frac{\pi}{c} x \left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right)}{\left\{ \left(\cosh \frac{\pi}{c} l_1 - \xi \right)^2 + \eta^2 \right\} \left\{ \left(\xi + \cosh \frac{\pi}{c} l_2 \right)^2 + \eta^2 \right\}}$$

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$$\begin{aligned}
& \left[\begin{aligned} & -\cos \frac{\pi}{c} y \left\{ \xi^2 + \eta^2 + \cosh \frac{\pi}{c} l_1 \cosh \frac{\pi}{c} l_2 \right\} \\ & + \cosh \frac{\pi}{c} x \left\{ 2\xi - \cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right\} \end{aligned} \right], \\
v = & -\frac{K_T}{c} \frac{\sin \frac{\pi}{c} y \left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right)}{\left\{ \left(\cosh \frac{\pi}{c} l_1 - \xi \right)^2 + \eta^2 \right\} \left\{ \left(\xi + \cosh \frac{\pi}{c} l_2 \right)^2 + \eta^2 \right\}} \\
& \left[\begin{aligned} & -\cosh \frac{\pi}{c} x \left\{ \xi^2 + \eta^2 + \cosh \frac{\pi}{c} l_1 \cosh \frac{\pi}{c} l_2 \right\} \\ & + \cos \frac{\pi}{c} y \left\{ 2\xi - \cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right\} \end{aligned} \right] \dots \dots \dots (26)
\end{aligned}$$

In special,

along the wall $A-T_1-B_1$,

$$u = -\frac{K_T}{c} \frac{\sinh \frac{\pi}{c} x \left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right)}{\left(\cosh \frac{\pi}{c} l_1 - \cosh \frac{\pi}{c} x \right) \left(\cosh \frac{\pi}{c} x + \cosh \frac{\pi}{c} l_2 \right)},$$

$$v=0,$$

along the wall $B-C-D$,

$$u=0,$$

$$v = \frac{K_T}{c} \frac{\sin \frac{\pi}{c} y \left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right)}{\left(\cosh \frac{\pi}{c} l_1 - \cosh \frac{\pi}{c} y \right) \left(\cosh \frac{\pi}{c} l_2 + \cosh \frac{\pi}{c} y \right)},$$

along the wall $D-T_2-A$,

$$u = \frac{K_T}{c} \frac{\sinh \frac{\pi}{c} x \left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right)}{\left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} x \right) \left(\cosh \frac{\pi}{c} l_2 - \cosh \frac{\pi}{c} x \right)},$$

$$v=0,$$

and along the line CA ,

$$u = \frac{K_T \cosh \frac{\pi}{c} x}{c} \frac{\sinh \frac{\pi}{c} x \left(\cosh^2 \frac{\pi}{c} l_2 - \cosh^2 \frac{\pi}{c} l_1 \right)}{\left(\cosh^2 \frac{\pi}{c} l_1 + \sinh^2 \frac{\pi}{c} x \right) \left(\cosh^2 \frac{\pi}{c} l_2 + \sinh^2 \frac{\pi}{c} x \right)}$$

(26 a, b, c, d)

$$v = \frac{K_T}{c} \frac{\left(\cosh \frac{\pi}{c} l_1 + \cosh \frac{\pi}{c} l_2 \right) \left(\cosh \frac{\pi}{c} l_1 \cosh \frac{\pi}{c} l_2 + \sinh^2 \frac{\pi}{c} x \right) \cosh \frac{\pi}{c} x}{\left(\cosh^2 \frac{\pi}{c} l_1 + \sinh^2 \frac{\pi}{c} x \right) \left(\cosh^2 \frac{\pi}{c} l_2 + \sinh^2 \frac{\pi}{c} x \right)}$$

The numerical exmples are shown in Fig. 13 for flow mode II with $l_1 = l_2 = -1.5c$, $-1.0c$ and $-0.5c$.

Fig. 13

Case 2. Potential flow from T_1 to T_2 both at infinity on boundaries AB and DA .

(19) The coordinates of T_1 and T_2 are respectively $(-\infty, 0)$ and $(-\infty, ic)$ on the plane z , which may be transformed into the plane t as the author deduced in the case 1. It is evident that all the stream lines on the plane z pass over the line CA at right angle and also on the plane t , at the same time, they converge to A on each side of infinity. Therefore the stream lines on the plane t must be parallel to $ABCD$ (Fig. 14.). If the stream line $m\pi$ passes through a point $t(a, 0.5c)$; from equ. (2)

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Fig. 14

$$t = \sinh \frac{\pi}{c} a,$$

comparing three points A , C and t on both the planes of t and w , we obtain

$$w = \frac{m\pi}{\sinh \frac{\pi}{c} a} \cosh \frac{\pi}{c} (x + iy) \quad \dots \dots \dots (27)$$

or

$$\left. \begin{aligned} \phi_T &= \frac{m\pi}{\sinh \frac{\pi}{c} a} \cosh \frac{\pi}{c} x \cos \frac{\pi}{c} y, \\ \psi_T &= \frac{m\pi}{\sinh \frac{\pi}{c} a} \sinh \frac{\pi}{c} x \sin \frac{\pi}{c} y. \end{aligned} \right\} \quad \dots \dots \dots (28)$$

Let K_T be a strength of flow, then $\frac{K_T}{\pi}$ is the intensity of flow that is equal

to $\frac{m\pi}{\sinh \frac{\pi}{c} a}$, and therefore, the above equation may be rewritten in

$$\phi_T = \frac{K_T}{\pi} \cosh \frac{\pi}{c} x \cos \frac{\pi}{c} y, \quad \dots \dots \dots (28a)$$

$$\psi_T = \frac{K_T}{\pi} \sinh \frac{\pi}{c} x \sin \frac{\pi}{c} y,$$

from which, we know the stream line of this flow may be defined by the value of " a " which demand stream line $m\pi$ passing on the point

Fig. 15

$t(a, 0.5c)$. Relation between a and $m\pi/\sinh \frac{\pi}{c}a$ for $m=1$ is given in

Fig. 16

stream line for the profile of a draft tube of water turbine. According to his idea for the plane problem that differs from the three dimensional flow, it is rather preferable to take such a profile, namely, a part of stream lines of vortex as shown in Fig. 7 in symmetry with respect to axis y , as the author deduced in section I. But, at present, the author is not decide to propose the profile and the condition of determination of draft tube and further investigation, which is continued by the author, will give some key for these matters.

Fig. 17

Fig. 15. The numerical example for this flow mode are shown in Fig. 16.

(20) On this flow mode, that was named as "flow mode VIII", the author noticed afterwards the same result was deduced by Prasil⁽⁸⁾ who adopted this

(21) In the way of transformation of planes, we can take, on some plane, a figure being the unit circle as mentioned in section I, and it is easily found that both the cases of the vortex at A and this flow mode VIII are the same phenomena, however, they have the same figure on the plane t' on which the vortex centre at A means the point sink and source at A . So they have the same velocity

(8) Technische Hydrodynamik p 62, 176, 103. 1926.

potential and stream function which have been given in equs.(B) and (28).

(22) Velocity distribution along a stream line. By the similar way as mentioned (5) and (7), the author obtained the velocities along a stream lines (Fig. 17).

In general ;

$$\left. \begin{aligned} u &= -\frac{K_T}{c} \sinh \frac{\pi}{c} x \cos \frac{\pi}{c} y. \\ v &= \frac{K_T}{c} \cosh \frac{\pi}{c} x \sin \frac{\pi}{c} y \end{aligned} \right\} \dots\dots\dots (29)$$

In special ;

along the wall *A-B*

$$u = -\frac{K_T}{c} \sinh \frac{\pi}{c} x$$

$$v = 0$$

along the wall *B-C-D*.

$$u = 0$$

$$v = \frac{K_T}{c} \sin \frac{\pi}{c} y.$$

along the wall *D-A*

$$u = \frac{K_T}{c} \sinh \frac{\pi}{c} x$$

$$v = 0$$

along the line *C-A*

$$u = 0$$

$$v = \frac{K_T}{c} \cosh \frac{\pi}{c} x$$

..... (29_{a,b,c,d})

SECTION III. IRROTATIONAL VORTEX OCCUPIED ITSELF IN A POTENTIAL FLOW

(23) Flow and vortex. In the course of the experiment on the flow of a gas or liquid, there is a phenomenon, often met, that a vortex occupied itself in neighbourhood of a stagnation point of flow. If the fluid be considered as nonviscous, the flow be a potential one and the vortex be an irrotational one, the state in which the vortex occupies its position in the flow can be treated as combined state of the each existency which will fulfil the field independently. Therefore, the velocity potentials and stream functions of the vortices and flows in combined state are to be

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introduced by those of the vortex and flow which are obtained in section I and II. It is clear in such a flow, that the stream lines vary their length and curvature; that the width of flow, the distributions of velocities and pressures quite differ from the mono-state of the flow. And the stream function in combined state can be expressed in

$$\Psi = \psi + \psi_T \dots\dots\dots (30)$$

where $\psi =$ a stream function of a free vortex

$\psi_T =$ „ „ a flow

Example. 5. Figs. 18-27 are the several examples for the combined state and their relations are also tabulated.

Fig.	Ψ	where	$\psi_P = \frac{1}{2} \log \frac{0.294(\xi^2 + \eta^2) - 1.606\xi + 0.992\eta + 1}{0.894(\xi^2 + \eta^2) - 1.504\xi - 0.932\eta + 1}$
18	$\psi_P + \psi_{P'}$		$\psi_{P'} = \frac{-1}{2} \log \frac{0.894(\xi^2 + \eta^2) + 1.606\xi + 0.992\eta + 1}{0.894(\xi^2 + \eta^2) + 1.504\xi - 0.932\eta + 1}$
19	$-\psi_Q + \psi_P$		
20	$\psi_Q + \psi_{TII}$		$\psi_Q = \frac{1}{2} \log \frac{\xi^2 + \eta^2 + 4.634\eta + 5.369}{\xi^2 + \eta^2 - 4.634\eta + 5.369}$
21	$\psi_P + \psi_{TII}$		
22	$-\psi_P + \psi_Q + \psi_{TII}$		$\psi_{TII} = 2 \tan^{-1} \frac{111.35\eta}{\xi^2 + \eta^2 - 3099.7}$
23	$\psi_Q + \psi_{TIII}$		
24	$-\psi_Q + \psi_{TIII}$		$\psi_{TIII} = 2 \tan^{-1} \frac{2\eta}{\xi^2 + \eta^2 - 1}$
25	$\psi_Q + \psi_{TVIII}$		
26	$\psi_P + \psi_{TVIII}$		
27	$-\psi_P + \psi_Q + \psi_{TVIII}$		$\psi_{TVIII} = 2\xi$

Fig. 18

Fig. 20

Fig. 19

Fig. 21

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Fig. 22

Fig. 25

Fig. 23

Fig. 26

Fig. 24

Fig. 27

in which $\xi = \cosh \frac{\pi}{c} x \cos \frac{\pi}{c} y$, $\eta = \sinh \frac{\pi}{c} x \sin \frac{\pi}{c} y$.

(24) Surface of separation. In the case of a vortex centre positioned at an arbitrary point in the flow, the line or surface of separation which confine a region of a vortex from that of a flow is given by a equation of resultant stream function equating to zero.

$$\Psi = \psi + \psi_r = 0 \dots\dots\dots (31)$$

from which, the author knows that the line or surface of separation changes its position and configuration according to the position of sink

and source, coordinate of a vortex centre and a ratio of strength of the vortex to that of the flow.

Let coordinate of a point, at which, the line of separation intersects the centre line CA of the field be $(l_0, 0.5c)$, l_0 changes its magnitude with l_1 , l_2 , x_1 , y_1 and K/K_T , or it depends only upon K/K_T for the same flow mode and the same position of the vortex centre.

Example 6. Fig. 28 shows the variation of K/K_T due to the different values of l_0 for the flow mode II ($l_1 = -1.5c, -1.0c$ and $-0.5c$) containing the vortex which centre is at $(x_1 = -0.5c, y_1 = 0.5c)$. Putting $x = l_0$ and $y = 0.5c$ in equs. (11_a), (21_a) and (31)

Fig. 28

$$2K/K_T = \log \frac{\sinh \frac{\pi}{c} l_0 + 2.317}{\sinh \frac{\pi}{c} l - 2.317} \bigg/ \tan^{-1} \frac{2 \cosh \frac{\pi}{c} l_1 \sinh \frac{\pi}{c} l_0}{\sinh^2 \frac{\pi}{c} l_0 - \cosh^2 \frac{\pi}{c} l_1} \dots (32)$$

which gives the relation between K_T/K and l_0 .

In this figure, for $l_0 = x_1$, $2K_T/K$ becomes infinite value and the vortex coincides with the singular point that is the centre of vortex, and for $l_0 \geq x_1$ the three conditions are observed. @ Reducing the K_T/K from infinite to a certain value, there are two points which appear on each side

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of vortex centre or one is at infinite, two are at finite distance and the one is at $x=0$. In this case, both the infinite and zero points always exist on the boundaries, being not the position of separation surface, and the other two points shows that the vortex isolates and the line of separation closes itself. ⑥ When K_T/K attains just a critical value, one of the separation points coincides with $x=0$ and other remains yet on the further side of the vortex centre. It means that the self-closed separation surface touches on the boundary BCD at C . ⑦ When K_T/K goes smaller than this critical value, only one point exists and the others do not appear in free space. It means that the line of separation begins and ends from boundary to boundary, that is, the points of stagnation exist at some points on the boundaries.

(25) The width of flow. When the flow does not include any vortex in its space, the condition to be satisfied by a point t , on which some stream line $m\pi$ passes, has been given in section II (14). And, as the stream line $\psi_T=0$ situates along the boundaries, this flow has a width " a " between t and C on the centre line CA of the field, as far as it concerns to the partial flow $m\pi$. However, when the flow contains a vortex as shown Fig. 19, it becomes so complex that the region of vortex obstructs the flow and the width of flow is narrowed by the amount of non-flow space $\{l_0(\text{on the side } >x_1)-l_0(\text{on the side } <x_1)\}$ on CA .

At this moment, the point t does not hold its position and moves to a new position t' (a' , $0.5c$) along CA . The point t' has to satisfy

$$\Psi = \psi + \psi_T = m\pi \dots\dots\dots (33)$$

from which, it is known that t' takes a different value for different l_1 , l_2 , x_1 , y_1 and K/K_T . " a' " can be obtained from equ. (33), hence, the effective width of flow

$$b = a' - \{l_0(\text{on the side } >x_1) - l_0(\text{on the side } <x_1)\} \dots\dots\dots (34)$$

and the efficiency of width of flow

$$\eta = \frac{b}{a} = \frac{a' - \{l_0(\text{on the side } >x_1) - l_0(\text{on the side } <x_1)\}}{a} \dots\dots\dots (35)$$

are given by the equs. (24) and (34). But, if these considerations are taken about total flow $n\pi$, " a' " takes infinite value and η becomes indeterminate.

Example 7. In the previous example, taking $K_T=2\pi$, $l_1=-1.5c$, $-1.0c$ and $-0.5c$, eq. (33) can be rewritten as

$$K/K_T \log \frac{\sinh \frac{\pi}{c} a' + 2.317}{\sinh \frac{\pi}{c} a' - 2.317} = 2 \left\{ \frac{m\pi^2}{K_T} - \tan^{-1} \frac{2 \cosh \frac{\pi}{c} l_1 \sinh \frac{\pi}{c} a'}{\sinh^2 \frac{\pi}{c} a' - \cosh^2 \frac{\pi}{c} l_1} \right\}. \quad (36)$$

" a' " is obtained after solving each side of this equation as an independent function of a' which gives the same value in both functions with given values of K/K_T and m/n (Fig. 29). It is seen that the smaller l_1 becomes, the larger the rate of displacement of t along CA for l_1 becomes.

Example 8. The efficiency of width of flow in above example is

Fig. 29

Fig. 30

given by eqs. (24_a), (32), (34) and (35) (Fig. 30). It is noticeable that the efficiency takes the greater value for the smaller of l_1 with the same K/K_T and K_T .

(26) The displacement of point t due to the existence of a vortex in the flow is so small as far as K/K_T is small and l_1 is large in some extent that it is able to presume a' is nearly equal to a . Then, if it is allowed to consider a solid boundary lies along the stream line $\psi_T = m\pi$ in a fixed position for both states, the flow will become a potential flow of source and sink situated at adjacent corner the field which is entirely surrounded by the walls. A flow as mentioned above can be found in the fluid motion in a cylinder of two cycle internal combustion engine of port scavenging type, though it is not to be said quite similar to the two dimensional flow with point source and sink, because, in the actual case, the motion is three dimensional flow with inlet and outlet port having opening widths. But, in the first approximation, they are to be in the analogical relations. Fig. 31 is one of the excellent photographs of the

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fluid motion in a three dimensional model cylinder of internal combustion engine studied by assist. prof. K. Tomiduka⁽⁹⁾ and this photograph was snapped at the moment of the piston moved on the outer dead point and generation of the large vortex region is seen in the upper part of cylinder content. The separation surface of this region and general flow of air intersect with the centre line at a distance about $0.76 \times$ diameter of cylinder from the top and the length of the model cylinder is about 1.42 times the diameter. Therefore, the author found the position of the separation surface on CA $l_0 = 0.9c$ with $l_1 = 1.42c$, $K/K_T = 1.336^{(10)}$ and the vortex centre at $(-0.5c, 0.5c)$, where c correspond to the diameter of cylinder. And, there is discrepancy of 0.14 between them. This discrepancy is not only due to the difference of their conditions, but it is to the difference of flow mode, that is, in this case, scavenging air issued from transfer port slightly in form of jet which will have a defined projecting angle depending upon the dimension of transfer and its port, as profs. R. Mises,⁽¹¹⁾ B. Eck,⁽¹²⁾ K. Tanaka⁽¹³⁾ etc. have studied on their cases. The jet hold its equilibrium configuration under the influence of vortex, being bended from the point, on which the free jet will pass, to $0.76 \times$ dia. of cylinder. By this reason, it may be considered that the region of vortex is so much controled by the effect of jet that the region is narrowed for the large opening of the port and is enlarged for small opening from these of calculation of the contiunous potential flow. Thus, the results of the calculations treated as a two dimensional potential flow and those of the model experiments are to be said somewhat similar, but they may not be satisfactory and further investigations will be able to solve these problems.

Fig. 31

(9) Jour. Aeronautical research Inst. Tokyo Imp. Univ. fig. 5, p. 225, no. 34, 1927.

(10) See p. 777 Example 13.

(10) V. D. I. p. 447, No. 21, Bd. 61, 1917.

(11) Zeit. für Angewandte Math. and Mech. p. 474, Dec., 1924.

(13) Jour. Soc. Mech. Eng. Tokyo. p. 404, no. 111, vol. 29, 1926.

(27) Point of stagnation. At the point of stagnation, the velocities along all stream lines are zero. Therefore, the distribution of the point of stagnation along the wall is the same to that of the points on which the separation surface either begins or ends, then, it is given by the distribution of a point at which the velocity is zero.

Hence,

$$\left. \begin{aligned} \frac{\partial \Phi}{\partial y} = \frac{\partial(\phi + \phi_r)}{\partial y} = 0 & \quad \text{along boundaries } AB \text{ and } DA \\ \frac{\partial \Phi}{\partial x} = \frac{\partial(\phi + \phi_r)}{\partial x} = 0 & \quad \text{,, ,, } BCD \end{aligned} \right\} \quad (37)$$

then, it is known that the distribution of point of stagnation is also a function of l_1 , l_2 , K/K_r and coordinates of vortex centre.

(28) Distribution of velocity. Distribution of velocity is given by equ. (30)

$$\left. \begin{aligned} u &= -\frac{\partial \Psi}{\partial y} = \frac{\partial \Phi}{\partial x} = u + u_r \\ v &= \frac{\partial \Psi}{\partial x} = \frac{\partial \Phi}{\partial y} = v + v_r \end{aligned} \right\} \dots\dots\dots (38)$$

which include the point of stagnation in its system.

Example 9. Velocity distribution along the boundaries AT_1B and BCD for combined state of for vortex, $K = -2\pi$, situated at $(-0.5c, 0.5c)$ and the flow mode II, $l_1 = -1.5c$ and $K_r = 2\pi$.

Fig. 32.

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Equs. (13_a), (14_b) and (26_{a,b,c}) give Fig. 32
for the boundary AT_1B

$$u = -\frac{K}{2c} \frac{\sinh \frac{\pi}{c} x}{0.2171 \cosh^2 \frac{\pi}{c} x + 1.1503} - \frac{K_T}{c} \frac{111.352 \sinh \frac{\pi}{c} x}{3099.7 - \cosh^2 \frac{\pi}{c} x}$$

for the boundary BC

$$v = \frac{K}{2c} \frac{\sin \frac{\pi}{c} y}{0.2171 \cos^2 \frac{\pi}{c} y + 1.1503} + \frac{K_T}{c} \frac{111.352 \sin \frac{\pi}{c} y}{3099.7 - \cos^2 \frac{\pi}{c} y}$$

.....(38_a)

Along the boundary $A-T_1-B$ from A to T_1 , the velocity increases from zero to its maximum value of infinite (it is not shown in figure); passing over the point T_1 to a point S , it changes its sense and magnitude from infinite to zero, and the point S , at which the velocity is zero, is one of the points of stagnation in this combined state; it starts from S to B along the wall, which belongs to vortex region, and the velocity changes again its magnitude from zero to zero passing through a maximum value and reaches to the one of the fixed stagnation points; along the boundary $B-C-D$ velocity changes its magnitude from zero to zero with a maximum value at its midway.

The distribution of the point of stagnation is given by equ. (38_a) as follows,

$$\left. \begin{aligned} \frac{K}{K_T} &= -\frac{222.7 \left\{ 0.2171 \cosh^2 \frac{\pi}{c} x + 1.1503 \right\}}{3099.7 - \cosh^2 \frac{\pi}{c} x} \text{ along } AT_1B \\ \frac{K}{K_T} &= -\frac{222.7 \left\{ 0.2171 \cos^2 \frac{\pi}{c} y + 1.1503 \right\}}{3099.7 - \cos^2 \frac{\pi}{c} y} \text{ along } BCD \end{aligned} \right\} (37_a)$$

from which, it is clearly seen that the position of stagnation point depend upon the ratio of strength K/K_T of these existency for the same combination and K/K_T has negative value for $x < l_1$ and positive for $x > l_1$ that means the direction of general flow and that of the flow of vortex are the same along the surface of separation (Fig. 33).

(29) Distribution of pressure. Being considered that the Bernoulli theorem can be hold along a stream line, a pressure at any point on that stream line may be obtained by

$$p = \text{const.} \{ \text{const.} - q^2/2g \} \dots\dots\dots (39)$$

where q = a velocity at a point considered. The distribution of pressure is thus obtained from that of velocity. The head at the point of stagnation, in which, velocity head vanishes and statical head appears alone, has a maximum value in that neighbourhood. And it exerts powerful force against the wall which strength and constraction are to be sufficiently enduring for this pressure in actual case.

Exemple 10. The distribution of pressure in the previous example is shown also in Fig. 32 from equ. (39). It is known, that, along the boundary AT_1B , the distribution curve has three equal maximum points among which the two points are fixed stagnation points at the corners of the field and another is a variable one. A points

Fig. 33

in midway of SB is reduced to minimum value by the amount of $u^2/2g$ which corresponds to the max. velocity head of vortex stream along the boundary SB , therefore, there are two max. statical heads appearing in a part near the corner B , and such a phenomenon have been observed in prof. Y. Matsumoto's⁽¹⁴⁾ experiment, Waseda university, on the flow along a wall that abruptly changes its direction; it is one of the items what author continue in his investigation.

(30) Standing free vortex in a flow. In a combined state, the condition of a vortex to stand still at a fixed point is that the translating

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velocity of vortex at its centre must be opposite and equal magnitude to the velocity along a stream line of flow and this stream line may pass through a point, at which, the vortex centre situates, when the vortex does not exist in the flow. And then, in velocity plane, they must satisfy

$$U + iV + u + iv = 0. \quad \dots\dots\dots(40)$$

Now, considering the vortex stands still on the axis of symmetry of the field, a translational velocity on this line only appears as a y axis component which is given by equ. (5_e). Therefore, the vortex will stand still on the line in the case of a flow which has only its y axis component velocity on the line, and in other case, it can not hold its position on the line CA and flows away, with a resultant velocity at any position of a moving vortex, to some position on which the equilibrium condition may be established. The position of standing vortex varies depending upon the flow mode and the ratio of strength of both flows.

Example 11. The relation between the coordinate $(x_1, 0.5c)$ of the centre of standing vortex and the ratio of strength K/K_T in a flow of the flow mode II is obtained from eqs. (5_e), (26_a) and (40)

$$\frac{K}{K_T} = \frac{4 \cosh \frac{\pi}{c} l_1 \cosh^2 \frac{\pi}{c} x_1 \sinh \frac{\pi}{c} x_1}{\cosh^2 \frac{\pi}{c} l_1 + \sinh^2 \frac{\pi}{c} x_1}, \quad \dots\dots\dots(41)$$

from which, it is known that the strength of vortex has an opposite sign to that of a flow and that the existence of a standing vortex is confined in the limit of $x_1 < l_1$. The numerical example of equ.(41) for $l_1 = -1.5c$, $-1.42c$, $-1.0c$ and $-0.5c$ are shown in the second quadrant of Fig. 34.

Example 12. The relation between l_0 and x_1 in the above example is given by the equation of separation surface in equ. (31), which is

$$-\frac{K}{2\pi} \frac{1}{2} \log \frac{\xi^2 + (\eta - \eta_1)^2}{\xi^2 + (\eta + \eta_1)^2} + \frac{K_T}{\pi} \tan^{-1} \frac{2\eta \cosh \frac{\pi}{c} l_1}{\xi^2 + \eta^2 - \cosh \frac{\pi}{c} l_1} = 0.$$

The coordinates of separation surface on CA are l_0 and $0.5c$, and K/K_T is given by equ. (41). The above may be rewritten in

$$\frac{\cosh \frac{\pi}{c} l_1 \cosh^2 \frac{\pi}{c} x_1 \sinh \frac{\pi}{c} x_1}{\cosh^2 \frac{\pi}{c} l_1 + \sinh^2 \frac{\pi}{c} x_1} \log \left(\frac{\sinh \frac{\pi}{c} l_0 - \sinh \frac{\pi}{c} x_1}{\sinh \frac{\pi}{c} l_0 + \sinh \frac{\pi}{c} x_1} \right)^2$$

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$$= \tan^{-1} \frac{2 \sinh \frac{\pi}{c} l_0 \cosh \frac{\pi}{c} l_1}{\sinh^2 \frac{\pi}{c} l_0 - \cosh \frac{\pi}{c} l_1} \dots \dots \dots (42)$$

The third quadrant of Fig. 34 shows the above relation with $l_1 = -1.5c$.

Fig. 34

(31). The relation between K/K_T , l_0 and x_1 . In the case of a standing vortex occupies its position in a flow of the flow mode II, as the relations between K/K_T and l_0 , K/K_T and x_1 are known in eqs. (32) and (41), respectively, the author tried to find graphically two values among K/K_T , l_0 and x_1 knowing either one of them. In afterward obtaining the relation between l_0 and x_1 in equ. (42), the graphical method becomes so much simple that it does not need the trial process described as the following, but, the author begins, in order, to note the former method.

That is for the fixed flow mode or $l_1 = \text{const} = l_2$

(1) A group of the thin curves, which shows the relation between K/K_T and l_0 for various values of x_1 , is drawn in the 1st quadrant and the other group of the curves which shows the relation between K/K_T and x_1 is also drawn in the 2nd quadrant of Fig. 34.

(2) When l_0 is known.

(a) Take $l_0 = oa$ and $ob // K/K_T$ axis intersecting any arbitrary curve, which is x_1 , at b .

(b) Take the point c on x_1 axis with corresponding value x_1 in (a) and draw $cd // K/K_T$ axis intersecting a given flow mode curve at d .

(c) Make $def // \text{abscissa}$ intersecting the thin curve which has x_1 in the 1st quadrant at f .

If f coincides with the point b , the x_1 or corresponding value of the point c shows the position of a standing vortex centre and oe is the corresponding required value of K/K_T .

If it is not so, the processes of (b) and (c) are repeated until f coincide with b .

(3) When x_1 is known.

(d) Draw $ed // K/K_T$ axis, $deb // \text{abscissa}$ and $ba // K/K_T$ axis.

(e) $oe = K/K_T$ and $oa = l_0$ give the corresponding required values.

(4) When K/K_T is known

(f) Draw deb for the known value K/K_T and $dc // K/K_T$ axis, then $oc = x_1$ is the required value.

(g) b be extended to intersect a curve which has the same value of $oc = x_1$, and $ba // K/K_T$ axis, then, $oa = l_0$ is the corresponding required value.

(5) Other method.

Drawing the curve which shows the relation between l_0 and x_1 in the 3rd quadrant, the required values are easily found in any way along

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oa' , $a'b'$, $b'c$, cd and de which are parallel to either abscissa or ordinate successively.

In the first quadrant of the Fig. 34, the thick lines are loci of the points which are obtained by the process (2)-(c) for $l_1 = -1.5c$ and $-1.42c$.

Example 13. In article (25) we know that the centre of the vortex, which occupied the upper portion of the model cylinder, situates at a point about a half diameter below the top on its centre line, therefore, the chart (Fig. 34) indicates $K/K_T = 1.336$ and $l_0 = 0.9c$ tracing straight line from the 2nd quadrant to the 1st quadrant or to the 3rd quadrant as mentioned in article (31)-(3).

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