Verification and optimization of Formula SAE suspension employing Inerter mechanism

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Abstract
In generally, a suspension system needs to be soft to insulate against road disturbances and hard to insulate against load disturbances. It cannot achieve with a traditional passive suspension that only considered to the stiffness and damper. On other hands, the formula cars need high tire grip on racing challenge by reducing rolling displacement at corner or double change lands with the simplest suspension dynamics system. In this study, the paper clarifies some issues related to suspension system with inerter to reduce displacement and rolling angle under impact from road disturbance on Formula SAE Car. In this paper, we integrate some kinds of suspension system with inerter on quarter-car and half-car models. We propose some new designs, which have some advantages for suspension system by improving dynamics. We optimize design of model based on the minimization of cost functions for roll dynamics, by reducing the displacement transfer and the energy consumed by the inerter. The base model is a passive suspension model then we carried out quarter-car and half-car model with different parameters; they show the benefit of the inerter in proposal suspension system. The advantage of research is integration a new mechanism, the inerter; this system can improve the vehicle oscillation.

Keywords: Inerter, suspension system, optimal design, Formula SAE, rolling

1. Introduction

Passive, semi-active and active suspension systems utilized to improve ride comfort of vehicles and their effectiveness demonstrated. However, it is not easy to improve comfort and dynamics stability with passive suspension systems (Bosch, SAE International 2011). To achieve it, several control methods have proposed, but most of them relate the active suspension (Marcello Chiaberge, 2011). In this case study, the passive suspension presented as the simple system that can improve rolling stability depend on the sensitivity of the system parameters that take and consider to be introduced.

To improve rolling stability, this study proposes a design method on passive suspension system taking with new component element named “inerter” into consideration the both sensitive of the sprung and un-sprung mass vehicle behavior when have road disturbance. The method can improve both the rolling and the displacement of vehicle body that is optimization modal parameters of suspension and tire. Furthermore, the optimization is in the time domain to attain the optimal values of parameters during impact period. The dynamics of road disturbance assumed for initial conditions. In order to verify the effectiveness of the proposed method, a half-car model that has variable stiffness, damping and inerter suspension system is constructed and the numerical simulations carried out.

For modeling of an inerter, it was defined to be a mechanical two-terminal, one-port device with the property that the equal and opposite force applied at the nodes is proportional to the relative acceleration between the nodes through a rack, pinion, and gears (Michael Z.Q.Chen, et al., 2009). For approximately model, the dynamics of the device, let \( r_1 \) be the radius of the rack pinion, \( r_2 \) the radius of the gear wheel, \( r_3 \) the radius of the flywheel pinion, \( \gamma \) the radius of gyration of the flywheel, \( m \) the mass of the flywheel (Malcolm C.Smith, 2002).
That the following relation holds:

\[ F = b(\dot{\psi}_2 - \dot{\psi}_1) \]  

(1)

The constant of proportionality \( b \) called the inertance and has units of kilograms:
\[ b = \frac{\alpha_1}{\alpha_3} \alpha_2^2 \]

Where \( \alpha_3 = \gamma/r_3 \) and \( \alpha_2 = r_2/r_1 \).

It stored energy equal to
\[ \frac{1}{2}b(v_2 - v_1)^2 \]  

(2)

Let us focus attention first on the familiar two-terminal modeling elements: spring, damper and inertor. Each is an ideal modeling element, with a mathematical definition. It is useful to discuss on mechanical networks, which give some hint toward the inertor idea, in order to highlight the new passive suspension system relate to rolling problems, shown in Figure 1 (Malcolm C. Smith, 2003).

![Figure 1 Theory of inertor structural correspondences for the mechanical network and rolling problems on Formula SAE car.](image)

2. Modeling of suspension system
2.1. Definition of suspension model with inertor

We summarize the approach of the suspension design problem was formulated as an optimal modal parameter to improve vertical displacement and rolling angle. The solution of the optimization problem made use structure of new quarter-car and half-car model that improve from traditional passive suspension system in adding inertor elements. In some previous researching, the good and simple structure was able to come up with new network topologies involving inertor that called suspension parallel structure (Takanori Uemura, 2009).

![Figure 2 The simple suspension parallel structure.](image)
2.2. Specification of quarter-car model

To estimating displacement of the sprung mass, we use a hump road profile and no load disturbances applied on the sprung mass. Based on previous study (Takanori Uemura, 2009), we have modal parameters for passive suspension system as shown in Table.1:

| Table.1 The specification of Formula SAE quarter-car model. |
|-------------|-------------|-------------|
| Symbols     | Parameters  | Values      |
| M           | Mass of body| 63 kg       |
| m           | Mass of tire| 12 kg       |
| k           | Stiffness coefficient| 24000 N/m |
| c           | Damping coefficient   | 1200 Ns/m  |
| b           | Mass of inertance    | 20 kg      |
| k_t         | Stiffness coefficient of tire| 70000 N/m |
| H_0         | Road disturbance hump| 0.05 m     |

We use the quarter-car model and design new structure call quarter-car suspension parallel model (Fig.3). This model will change from normal passive suspension to new suspension with stiffness, damping and inert in parallel. We study about the vertical displacement of sprung mass in some kinds of simulations.

![Fig.3 The quarter-car model.](image)

a - model without inert;  b - model with inert in parallel structure

For the quarter-car model, the suspension strut provides an equal and opposite force on the sprung and unsprung masses by means of the positive real admittance function which relates the suspension force to the strut velocity through spring, damper and inert. We define them as the following equations.

The module of sprung mass body represented by these equations:

The model without inert:

\[ M \ddot{Z}_2(t) = F_k(t) + F_c(t) \]  (3)

The model with inert:

\[ M \ddot{Z}_2(t) = F_k(t) + F_c(t) + F_b(t) \]  (4)

Where:

\[ F_k(t) = k(Z_o(t) - Z_2(t)) \]  (5)

\[ F_c(t) = c(Z_2(t) - \dot{Z}_2(t)) \]  (6)

\[ F_b(t) = b(Z_i(t) - \dot{Z}_2(t)) \]  (7)

The un-sprung mass module equation:

Without inert:

\[ m \ddot{Z}_i(t) = F_{rt}(t) - (F_k(t) + F_c(t)) \]  (8)

With inert:

\[ m \ddot{Z}_i(t) = F_{rt}(t) - (F_k(t) + F_c(t) + F_b(t)) \]  (9)

Where:

\[ F_{rt}(t) = k_t(Z_0(t) - Z_2(t)) \]  (10)
In this study, we integrated inerter mechanism that applied on passive suspension system. First, we use conventional model with initial parameter from formula SAE car. Second, base on mathematical model, we apply inerter on suspension system in parallel structure. For the last model, we design physical model with more detail of inerter components as gear, rack, pinion, flywheel and friction. Each of them has difference characteristics on working as frictions or efficient rate so we can estimate from simulation results.

Fig. 4 The quarter-car model in simulation.
   a – The model without inerter;
   b – The model with mathematical inerter;
   c – The model with physical inerter

2.3. Specification of half-car model

We summarize the approach of the suspension design problem was formulated as an optimal modal parameter to improve rolling angle through displacement variables. The solution of the optimization problem uses structure of new half-car model that improved from traditional passive suspension system in adding inerter elements. In this paper, we shall apply the training parameterization method to the half-car model.

Fig. 5 The half-car model.
In this part, we made simulations creating mathematical model with inerter in the linear half-car model. The vehicle model has an input road disturbance to right sides, and the main outputs are rolling angle ($\phi$) and sprung mass displacement ($Z_s$). Several subsystems of model performance considered such as sprung mass, un-sprung mass, suspension and tire (Fig.6). To optimizing suspension problems, we used these simulation results as the initial values that results represent for the picks of rolling angle Max$\phi$ and sprung mass displacement Max$Z_s$.

We shall apply the training parameterization method to the half-car model. The half-car model represented in Fig.5 is the simple model to consider for suspension design. It consists of the sprung mass $M$, the un-sprung mass $m$ and the tire model represented by stiffness $k$. The suspension strut provides an equal and opposite force on the sprung and un-sprung masses by means of the positive-real admittance function which relates the suspension force to the strut velocity through spring, damper and inerter.

For the half-car model, we define it as the following equations. The module of sprung mass body represented by these equations:

$$M\ddot{Z}_s(t) = (F_{KL}(t) + F_{CL}(t) + F_{BL}(t)) + (F_{KR}(t) + F_{CR}(t) + F_{BR}(t))$$

$$I\ddot{\phi}(t) = a(F_{KL}(t) + F_{CL}(t) + F_{BL}(t)) - a(F_{KR}(t) + F_{CR}(t) + F_{BR}(t))$$

$$I = Ma^2/3$$

The module of suspension systems represented as:

$$F_{KL}(t) = k(Z_{L}(t) - (Z_s(t) - a\phi(t)))$$

$$F_{CL}(t) = c(Z_{L}(t) - (Z_s(t) - a\phi(t)))$$

$$F_{BL}(t) = b(Z_{L}(t) - (Z_s(t) - a\phi(t)))$$

$$F_{KR}(t) = k(Z_{R}(t) - (Z_s(t) + a\phi(t)))$$

$$F_{CR}(t) = c(Z_{R}(t) - (Z_s(t) + a\phi(t)))$$

$$F_{BR}(t) = b(Z_{R}(t) - (Z_s(t) + a\phi(t)))$$

The un-sprung mass module as:

$$m\ddot{Z}_{L}(t) = F_{KL}(t) + F_{CL}(t) + F_{BL}(t)$$

$$m\ddot{Z}_{R}(t) = F_{KR}(t) + F_{CR}(t) + F_{BR}(t)$$

The tire module as:

$$F_{KL}(t) = k_t(Z_0L(t) - Z_{L}(t))$$

$$F_{KR}(t) = k_t(Z_0R(t) - Z_{R}(t))$$

For the evaluation of the rolling angle, we use a hump road profile. The hump has height $H_{0R}$ (on the right side) and flat on the left side. The hump initially appears after 1 second then the right wheel impacts and the left wheel has no disturbance.

Base on previous study, we have modal parameters for passive suspension system as:

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>Mass of body</td>
<td>126 kg</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass of tire</td>
<td>12 kg</td>
</tr>
<tr>
<td>$I$</td>
<td>Roll moment of inertia</td>
<td>15.1 kgm2</td>
</tr>
<tr>
<td>$k$</td>
<td>Stiffness coefficient</td>
<td>24000 N/m</td>
</tr>
<tr>
<td>$c$</td>
<td>Damping coefficient</td>
<td>1200 Ns/m</td>
</tr>
<tr>
<td>$b$</td>
<td>Mass of inerter</td>
<td>20 kg</td>
</tr>
<tr>
<td>$k_t$</td>
<td>Stiffness coefficient of tire</td>
<td>70000 N/m</td>
</tr>
<tr>
<td>$a$</td>
<td>Half length of track</td>
<td>0.6 m</td>
</tr>
<tr>
<td>$H_{0R}$</td>
<td>Road disturbance hump</td>
<td>0.1 m</td>
</tr>
</tbody>
</table>

In this part, we made simulations creating two models called: old_model on basic suspension system and new_model with inerter. The methodology used of a linear half-car model that constructed in the simulation toolboxes. The vehicle model has an input road disturbance right sides, the main output is rolling angle ($\phi$) and sprung mass displacement ($Z_s$). Several subsystems of model performance considered such as sprung mass, un-sprung mass, suspension and tire showed in Fig.6. For each aspect of performance, we will propose time-domain performance measures that evaluated after a simulation run.

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To optimizing suspension problems, we used these simulation results as the initial values. Results represent for the peaks of rolling angle $\text{Max} \theta$ and sprung-mass vertical displacement $\text{Max} Z$.

![Diagram](image)

Fig.6 The simulation of half-car model to calculate rolling and displacement.

3. Definition of optimization suspension system
3.1. Optimization quarter-car model parameters

For the optimization of displacement, in the section we will focus on a single aspect of performance related to the dynamics problem. We used the structure parameter that represented the following design variables: $k$, $c$, $b$, and $k_0$, with lower and upper boundaries showed below.

<table>
<thead>
<tr>
<th>Lower boundary</th>
<th>Parameter</th>
<th>Upper boundary</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>20000</td>
<td>$k$</td>
<td>28000</td>
<td>N/m</td>
</tr>
<tr>
<td>1000</td>
<td>$c$</td>
<td>1400</td>
<td>Ns/m</td>
</tr>
<tr>
<td>10</td>
<td>$b$</td>
<td>30</td>
<td>kg</td>
</tr>
<tr>
<td>60000</td>
<td>$k_0$</td>
<td>80000</td>
<td>N/m</td>
</tr>
</tbody>
</table>

The objective function represented for the maximum displacement value of sprung-mass as:

$$\text{Max} Z = f(k, c, b, k_0) \rightarrow \text{Min}$$

Under the synthetic assumption, we can figure out this peak-point of displacement equation:

$$\text{Max} Z = 0.052 + 1.114(e - 7)k - 6.853(e - 12)k^2 - 1.086(e - 05)c + 4.597(e - 09)c^2 + 2.413(e - 07)b - 1.049(e - 12)b^2 + 7.957(e - 04)k_0 - 6.993(e - 06)k_0^2$$

The constrained functions represented by the tire displacement:

$$\text{Min}[\text{Max} Z_1] \leq Z_1(k, c, b, k_0) \leq \text{Max}[\text{Max} Z_1]$$

To calculating $\text{Max} Z_1$, we used Orthogonal Arrays (OA) of Design of Experimental with $L_{27}(3^{13})$ then, we have:

$$\text{Min}[\text{Max} Z_1] = 0.0618 \text{ (m)}$$

$$\text{Max}[\text{Max} Z_1] = 0.0756 \text{ (m)}$$

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We introduce the approximate optimization method, Sequential Quadratic Programming (SQP) with Response Surface Method (RSM). We made RSM using the OA to optimize modal parameters of the passive suspension system with inerter. The new modal parameters represented for optimization suspension system (opt_model) and they called k_opt, c_opt, b_opt and k_i_opt parameters.

In order to verify the effectiveness of the proposed optimal method, the numerical simulations carried out. The vertical displacement optimization results measured over various fixed structure suspensions. The optimization performed for k_opt, c_opt, b_opt and k_i_opt ranging from boundary conditions. The modal parameter results obtained by fixed-structure presented in Table.4.

Table.4 The comparative modal parameters for quarter-car model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>k</th>
<th>c</th>
<th>b</th>
<th>k_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>old_model</td>
<td>24000</td>
<td>1200</td>
<td></td>
<td>70000</td>
</tr>
<tr>
<td>new_model</td>
<td>24000</td>
<td>1200</td>
<td>20</td>
<td>70000</td>
</tr>
<tr>
<td>opt_model</td>
<td>20727</td>
<td>1363</td>
<td>28</td>
<td>78182</td>
</tr>
<tr>
<td>Unit</td>
<td>N/m</td>
<td>N/m</td>
<td>kg</td>
<td>N/m</td>
</tr>
</tbody>
</table>

3.2. Optimization half-car model parameters

For the optimization of rolling, in the section we will focus on a single aspect of performance related to the rolling problem. We used the structure parameter that represented the following design variables: k, c, b, and k_i with lower and upper boundaries showed below.

Table.5 The boundary of design variables.

<table>
<thead>
<tr>
<th>Lower boundary</th>
<th>Parameter</th>
<th>Upper boundary</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>20000</td>
<td>k</td>
<td>28000</td>
<td>N/m</td>
</tr>
<tr>
<td>10000</td>
<td>c</td>
<td>1400</td>
<td>Ns/m</td>
</tr>
<tr>
<td>10</td>
<td>b</td>
<td>30</td>
<td>kg</td>
</tr>
<tr>
<td>60000</td>
<td>k_i</td>
<td>80000</td>
<td>N/m</td>
</tr>
</tbody>
</table>

The objective function represented for the rolling angle modes as:

\[
\varphi = \varphi(k, c, b, k_i) \to \text{Min}
\]

Under the synthetic assumption of basic passive half-car model, we can figure out this rolling equation:

\[
\varphi = 0.159 - 5.236(e-07)k + 2.119(e-11)k^2 - 1.35(e-5)c + 1.138(e-08)c^2 - 1.136(e-06)b + 6.555(e-12)b^2 - 7.75(e-4)b + 2.838(e-05)b^2
\]

(29)

Constrain functions were represented by the body displacement modes under boundaries above. In this case, the results show:

\[
\text{Min}(\text{Max} Z_2) \leq Z_2(k, c, b, k_i) \leq \text{Max}(\text{Max} Z_2)
\]

(31)

To calculating Max Z_2, we used Orthogonal Arrays (OA) of Design of Experimental with L_27 (3^13) then:

\[
\text{Min}(\text{Max} Z_2) = 0.0641 (m)
\]

\[
\text{Max}(\text{Max} Z_2) = 0.0755 (m)
\]

With the same optimal methods, we introduce the approximate optimization method, Sequential Quadratic Programming (SQP) with Response Surface Method (RSM). We made RSM using the OA to optimize modal parameters for half-car model. The new modal parameters represented for optimization suspension system (opt_model) and they called k_opt, c_opt, b_opt and k_i_opt.

Table.6 The comparative modal parameters for half-car model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>k</th>
<th>c</th>
<th>b</th>
<th>k_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>old_model</td>
<td>24000</td>
<td>1200</td>
<td>X</td>
<td>70000</td>
</tr>
<tr>
<td>new_model</td>
<td>24000</td>
<td>1200</td>
<td>20</td>
<td>70000</td>
</tr>
<tr>
<td>opt_model</td>
<td>20727</td>
<td>1036</td>
<td>16.63</td>
<td>78182</td>
</tr>
<tr>
<td>Unit</td>
<td>N/m</td>
<td>Ns/m</td>
<td>kg</td>
<td>N/m</td>
</tr>
</tbody>
</table>

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4. Results and discussions of suspension models

These results confirmed that the suspension system with inerter effects to the displacement of sprung-mass, the time histories of the displacement shown in Fig.7, respectively. From the results, it verified that the displacement reduced by comparison between old and new models.

The optimization result presented as circle-symbol curve suggesting that the structure of the suspension optimize from the stiffness, damper and inerter. An encouraging feature of the optimization algorithm that it allows the change in the structure of suspension parameters varies in order to obtain the minimum vertical displacement values.

![Displacement of sprung-mass on quarter-car model](image1)

Fig.7 The displacement of sprung-mass on quarter-car model.

Comparing among old-model and opt-model we found that correspondences between these peaks of displacement variables of sprung-mass is summarized below. All results verified with new suspension system accomplished that the body displacement dynamics reduced with employment inerter component.

<table>
<thead>
<tr>
<th>Displacement peak</th>
<th>Result (m)</th>
<th>Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z_m old</td>
<td>0.0752</td>
<td>0%</td>
</tr>
<tr>
<td>Z_m opt</td>
<td>0.0629</td>
<td>16.36%</td>
</tr>
</tbody>
</table>

For the half-car model, the results confirmed that the inerter added to affect the roll angle, the time histories of the body roll angles are shown in Fig.8, respectively. From the results, it verified that the rolling angle reduced by comparison among each models.

![Results of optimization rolling angle for half-car models](image2)

Fig.8 Results of optimization rolling angle for half-car models.

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The optimization result presented as circle-symbol curve suggesting that the structure used the optimal parameters of stiffness, damper and inerter. Comparing among old-model, new-model and opt-model we found that the rolling angle reduced, as summarized below.

<table>
<thead>
<tr>
<th>Rolling peak</th>
<th>Result (rad)</th>
<th>Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{old}$</td>
<td>0.114</td>
<td>0%</td>
</tr>
<tr>
<td>$\phi_{new}$</td>
<td>0.098</td>
<td>14.03%</td>
</tr>
<tr>
<td>$\phi_{opt}$</td>
<td>0.090</td>
<td>21.05%</td>
</tr>
</tbody>
</table>

The table shows that with inerter components employed to passive suspension system as a parallel structure, the rolling angle reduced. We just use for a simple and fix inerter parameter $b$, the rolling value reduce 14 percent, while it will be reduced more than 21 percent if we optimal suspension parameter. In conclusion, the inerter component has an advanced effect to the suspension system in general and in passive suspension system in particular.

5. Conclusion

This paper has described the background and application of a new element called inerter through the suspension synthesis in Formula SAE car. The passive suspension considered that is possible application of the inerter. The results showed that suspension with inerter was not only have better displacement but also have smaller rolling angle of sprung mass body on quarter-car and half-car model.

It showed that conventional spring and damper resulted in normal vibration behavior, but the use inerter can reduce the oscillation. In this studying, optimal variables stiffness, damping and inerter in suspension system achieved better results for rolling with road disturbance. These simulations confirmed that ride comfort in the same frequency domain with basic suspension system could improve rolling angle. Furthermore, base on these advanced optimization results; we will integrate other types of inerter which can be controlled, and apply on suspension system for large dynamic stability.

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