ABSTRACT

In this paper, constitutive equation to describe electric yielding behavior of a piezoelectric material is given out first. Then, the characteristics of \( J_e \)-integral, that is, path-independent expression of CED when electric yielding is considered are discussed and, in the light of \( J_e \)-integral, the physical meaning and applicable ranges of \( J \)-integral is clarified. Finally, how electric yielding influences fracture load estimation is fundamentally studied.

1. INTRODUCTION

For electro-mechanically coupled piezoelectric material, fracture process is believed to be influenced by electrical nonlinearity. In previous papers\(^\text{1,2}\), electric yielding model for piezoelectric materials has been studied, however, the physical standpoint of electric yielding behavior was not interpreted so clearly. In this paper, with a special emphasis on piezoelectrics, some physical interpretation of electrical nonlinearity is made. Then an idealized electric yielding model is proposed according to the hysteresis relation between electric displacement and electric field. Using piezoelectric \( J_e \)-integral, we explain the physical meaning of \( J \)-integral which was derived by Pak\(^\text{3}\) and correct the error that the mechanical part of \( J \)-integral for piezoelectric solid is still path independent in the literature\(^\text{4}\). Consequently, the applicable range of \( J \)-integral for ordinary material and piezoelectric material is clarified. Finally, electric yielding effect on fracture load estimation is fundamentally investigated through FEM. The fracture load estimation based on mechanical CED is compared with experimental result\(^\text{5}\) for PZT-4 materials and they are found to be in qualitative agreement.

2. A PERFECT ELECTRIC YIELDING MODEL

The multi-scale singularity concept was introduced first by Gao et al.\(^\text{6}\). It’s believed that the electric polarization of a crystal is accomplished by the ionic movement in perovskite based ceramic piezoelectrics. With the increase of electric field, the movement of ionic sensitive is expected to be sluggish or “lazy”, which will lead to electric saturation. Under small scale yielding condition, both mechanical and electrical crack tip-fields exhibit singularities at the global length scale. The electric yielding occurs before mechanical yielding if the solid is electrically more ductile. At the length scale that is smaller than electric nonlinear zone but larger than mechanical yielding zone, the mechanical fields are still crack-singular while the electrical fields vary smoothly due to electric yielding. In other word, even if elastic cleavage processes remain confined to a tiny region at the crack tip, electrical yielding can spread out over larger scales of the material. Similar to a plastic zone around crack tip for ordinary plastic-plastic materials, yielding behavior will strongly influence piezoelectric fracture process.

By adopting above multi-scale singularity concept, a perfect electric yielding model of piezoelectric materials is to be described. As shown in Fig. 1, an idealized hysteresis relation between electric displacement and electric field is illustrated based on the experimental observation. Under a lower electric field, the electric displacement obeys a linear law and the slope is equal to the dielectric permittivity \( \varepsilon_0 \). When the applied electric field reaches the coercive field \( E_c \) (or \( -E_c \)), electric displacement reaches a perfect yielding value \( D_0 \) (or \( -D_0 \)), called electric yielding strength. In Fig. 1(a), the remnant polarization \( D_r \) is involved. The electric yielding strength, \( D_0 \), is calculated from infinite plate problem with no mechanical loading. \( E_r \) for PZT-4 is 1MV/m and \( D_0 \) is calculated to be 0.01CM. For simplicity, the remnant polarization is considered to be sufficiently small, then, Fig. 1(a) reduces to Fig. 1(b). Only monotonic electric loading condition is considered in present model; therefore, the initial state point is at the point without electric loading.

To completely formulate an electric yielding model mathematically, three requirements have to be met. The first is an explicit relationship between stress, electric displacement, strain and electric field to describe material behavior before the onset of electric yielding, i.e., linear electro-elastic constitutive equation. The second one is an electric yielding criterion indicating the stress and/or electric displacement level at which yielding flow commences. The third one is the post-yield constitutive equation when the field is made up of both elastic and yielding components. These three requirements is to be given as follows.

With the positive \( x_2 \)-direction being the poling direction, the plane strain constitutive relation of 2-dimensional problem can be expressed as:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix} = \begin{bmatrix}
c & \epsilon^T_x \\
\epsilon^T_y & -\kappa
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y
\end{bmatrix} = \begin{bmatrix}
E_x \\
E_y
\end{bmatrix}
\]

(1)

where \( \sigma = [\sigma_11 \sigma_22 \sigma_{12}]^T, D = [D_1 D_2]^T, \varepsilon = [\varepsilon_11 \varepsilon_22 \varepsilon_{12}]^T \) and \( E = [E_x E_y]^T \) are stress, electric displacement, strain and electric field components, respectively, and \( c, \epsilon \) and \( \kappa \) are elastic, piezoelectric and dielectric coefficients, respectively.

Generally speaking, the electric yielding criterion for piezoelectric material should be a function of stress and electric displacement by drawing parallel with ordinary elastic-plastic materials. For a preliminary study on electric field influence on piezoelectric fracture behavior, the criterion for electric yielding is assumed to be a function of the electric displacement only and can be expressed as\(^\text{2}\):

\[
F(D, k) = 0
\]

(2)

then

\[
\frac{\partial F}{\partial D} \frac{D}{ek} = \frac{\partial F}{\partial k} \frac{d}{ek}
\]

(3)

in which \( k \) is a parameter governing the electric yielding behaviors between \( c \) and \( d \) and between \( e \) and \( f \) in Fig. 1. Then, as a result, the “elasto-plastic” material matrices has the form\(^\text{2}\):

\[
c^P = c + \frac{1}{\kappa_{33}} \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\kappa & 0 \\
0 & \kappa
\end{bmatrix} = -\kappa - \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

(4)

Consequently, the post-yield constitutive equation can be written in incremental form as
The three requirements have been presented so far.

3. PATH-INDEPENDENT EXPRESSION OF CED WHEN ELECTRIC YIELDING IS CONSIDERED

For linear and nonlinear electro-elastic continuous piezoelectric material, neglecting body force and body charge, \( \varepsilon \)-integral can be derived without any restriction according to energy conservation law as follows:

\[
\begin{align*}
\varepsilon_j & = \int \left[ \left( \frac{dW}{d\sigma} - \sigma \frac{d\varphi}{d\varepsilon} \right) d\Gamma + \int \left[ \frac{d\varphi}{d\varepsilon} \frac{d\varphi}{d\varepsilon} \right] dA 
\end{align*}
\]

\[
\begin{align*}
& - \int_{A_0} \left( e_{ij} \sigma_{ij} - e_{ij} d\sigma_{ij} + f_{j,i} dD_j - D_{j,i} dE_j \right) dA 
\end{align*}
\]

(6)

Appropriately, \( \varepsilon \) can be partitioned into mechanical and electrical parts as:

\[
\begin{align*}
\varepsilon^m_j & = \int \left[ \frac{dW}{d\sigma} - \sigma \frac{d\varphi}{d\varepsilon} \right] d\Gamma + \int \left[ \frac{d\varphi}{d\varepsilon} \frac{d\varphi}{d\varepsilon} \right] dA 
\end{align*}
\]

(7)

\[
\begin{align*}
\varepsilon^f_j & = \int \left( e_{ij} \sigma_{ij} - e_{ij} d\sigma_{ij} + f_{j,i} dD_j - D_{j,i} dE_j \right) dA 
\end{align*}
\]

(8)

\[
\begin{align*}
\varepsilon^s_j & = \int \left( e_{ij} \sigma_{ij} - e_{ij} d\sigma_{ij} + f_{j,i} dD_j - D_{j,i} dE_j \right) dA 
\end{align*}
\]

(9)

We take it into consideration that the definition of CED⁵ is given as an extremum value in case that the crack radius \( r \) approaches 0, the estimation of \( \varepsilon \) by applying \( \varepsilon \)-integral for a notched specimen of which the crack radius \( r \) is sufficiently small is considered realistic. Then the CED and its derivatives for piezoelectric material can be written by \( \varepsilon \)-integral

\[
\varepsilon = \lim_{r \to 0} \varepsilon^m_j \\
\varepsilon^m = \lim_{r \to 0} \varepsilon^m_j \\
\varepsilon^f = \lim_{r \to 0} \varepsilon^f_j \\
\varepsilon^s = \lim_{r \to 0} \varepsilon^s_j
\]

(10)

In 1990, Pak⁵ derived a path-independent integral \( J_p \) integral for piezoelectric materials by using Ershbe's method of deriving conservation laws.

\[
\begin{align*}
J_p^m & = \int \left[ \left( H_n \right)_{n} - T_n \left( u_{n} + q_s \varphi_{n} \right) \right] d\Gamma \\
& = \lim_{r \to 0} \int \left( H_n \right)_{n} d\Gamma \\
& = \lim_{r \to 0} \int \left( H_n \right)_{n} d\Gamma
\end{align*}
\]

(11)

He also pointed out that assuming all the electric field quantities vanished, the above equation is reduced to

\[
\begin{align*}
J_p^m & = \int \left( W_n \right)_{n} d\Gamma \\
& = \lim_{r \to 0} \int \left( W_n \right)_{n} d\Gamma
\end{align*}
\]

(12)

In fact, this is the path independent integral for ordinary materials originally derived by Rice⁶ in 1968. Here superscript \( n \) or \( c \) indicates the quantities are for notch or crack model, respectively. From Eqs. (11), (12), (13) and (14), we can obtain the relations between \( J_c^m \) and \( J_c^m \) and between \( J_f^m \) and \( J_f^m \) as follows:

\[
\begin{align*}
J_c^m & = \int \left( W_n \right)_{n} d\Gamma \\
& = \lim_{r \to 0} \int \left( W_n \right)_{n} d\Gamma
\end{align*}
\]

(13)

For linear electro-elastic piezoelectric material, \( J_c^m = J_f^m \) absolutely holds because the area integral part is zero, i.e. \( J_c^m = J_f^m \) is path-independent and \( J_c^m \) becomes the parameter with the same meaning as \( J_f^m \). While \( J_f^m \) is not equal to \( J_c^m \) because the area integral part in Eq. (15) is not equal to zero. Therefore, within the linear electro-elastic framework, \( J_f^m \) is nothing but \( J_c^m \) while \( J_f^m \) is different from \( J_c^m \) because the area integral can not be canceled in a linear piezoelectric body.

When an electric yielding problem is dealt with, the second item of the right hand side of Eq. (14) is considered sufficiently small because the area integral parts are high-order terms of infinitesimal though it can not be testified in general to be negligibly small quantity mathematically. Therefore, \( J_c^m = J_f^m \) holds approximately when unloading does not occur and the constitutive law is given by incremental strain theory based on associated flow rule.⁷ Therefore, the \( J_c^m \) to an electric yielding piezoelectric material can be evaluated approximately by Eq. (14) under the condition that unloading does not occur and that the constitutive law is given by incremental strain theory.

4. INFLUENCE OF ELECTRIC YIELDING ON FRACTURE LOAD ESTIMATION

When we think of the applicability of \( J_c^m \) parameter for a mode I piezoelectric crack, we can naturally expect that the piezoelectric crack begins to extend when \( J_c^m \) reaches a critical value. A critical \( J_c^m \) is regarded as a material property and independent of the crack length for pure brittle or small scale electric yielding conditions. The critical \( J_c^m \) can be determined using purely mechanical loading through FEM. Then the fracture load under various electric loading can be estimated through FEA. If the fracture-load estimation based on mechanical CED can satisfy the experimental results, the mechanical CED is possible to be regarded as a fracture parameter.

As shown in Fig. 2, the fracture load prediction considering the electrical nonlinearity is improved compared with the result from linear theory and qualitatively same as the experimental result for PZT-4 material. Therefore, the validity of mechanical CED regarded as a crack parameter for piezoelectric materials seems to be confirmed. However, the influence of loading procedure is pointed out and it should be studied in the future.

REFERENCES

(5) Pak and Sun, J. Am. Ceram. Soc. 78(6) 1475-80, 1995