General paper

A Notion of Strain and Stress Tensor for Finite Deformation of Elastic-Plastic Continua

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Abstract: As a notion of strain compatible with the theory of plasticity having the plastic potential of von Mises type with the normality principle as flow rule, the strain tensor is defined by integrating the rate of deformation tensor considering the material spin with the rotation tensor. For a group of elastic-plastic continua exhibiting infinitesimal elastic and finite plastic deformation, elastic-plastic decomposition of the strain tensor is established, and it is proved that the elastic strain tensor has the exact physical meaning and that the stress tensor can be calculated through the generalized Hooke's law with the elastic strain tensor without any ambiguity.

Key words: *Elastic-plastic continua, Finite strain tensor, Rate of deformation tensor, Infinitesimal elastic deformation, Finite plastic deformation, Generalized Hooke's law*

1. INTRODUCTION

Advantage of finite element analysis is widely recognized not only in the field of machine and structure designing but also in the field of the research and engineering of mechanical working processes accompanied with large deformation. To save the computational cost for such problems with large deformation, an implicit finite element method is preferable because in each time increment it seeks the solution of displacement field which satisfies the equilibrium equations at the end of the time increment and hence fairly large time increment can be adopted. In order to consider the equilibrium equations at the end of the time increment, rigorous notion of finite strain is needed. Theoretically speaking, to define the notion of finite strain, the relationship between displacement field and finite strain should be established. However, implicit finite element codes, which are currently available commercially, carefully avoid explicit definition of finite strain by directly obtaining stress tensor through integrating constitutive equations for the rate of stress tensor over the time increment considering the adequate material spin. With this type of formulation, technical problems arise, for example, in handling residual error in equilibrium equations at the end of the time increment. Solutions to these technical problems often go into empirical methods, because definition and calculation of finite strain associated with displacement field has been bypassed. Typical one of such problems is unreasonable stress behavior observed for simple shear deformation when Jaumann rate of stress tensor is utilized. Although there have been a lot of discussions made about this problem[8], most of them seem to have been devoted for stress behavior without considering stress-strain relationship in the context of finite deformation. Recently Shizawa[9] made thermomechanical approach to this problem, and Watanabe[10,11] made discussion in the context of stress-

strain relationship. Present work is in the line of Watanabe's discussion. The motivation, however, lies in the strain space plasticity proposed by Naghdi et al.[1-4], and the goal is to propose more rigorous and practical numerical scheme for an implicit finite element formulation than conventional ones.

As a preparatory step for establishing an implicit finite element formulation with an explicit definition of finite strain, this paper presents a notion of strain and stress tensor which is capable of describing finite deformation of elastic-plastic continua.

2. EXISTING NOTIONS OF FINITE STRAIN

In this section, the existing notions of finite strain are investigated in view of usefulness for numerical schemes such as finite element analysis. Classically, there are two notions of finite strain currently available with clear definition associated with the displacement field. One is the Lagrangian strain and the other is the Eulerian strain. Among them, the Lagrangian strain is of particular importance, for it is adopted as the notion of finite strain in the theory of plasticity proposed by Naghdi et al.[1-4], who have restructured the classical theory of plasticity using the notion of strain space spanned by the finite strain tensor. The simplest form of constitutive equations for plastic strain rate in the Naghdi's theory of plasticity can be expressed in the following form:

$$\dot{e}_{AB}^P = \lambda \tau_{AB}^S, \quad (1)$$

where $\dot{e}_{AB}^P$ are plastic components of the Lagrangian strain rate, τ_{AB}^S are deviatoric components of the symmetric Piola-Kirchhoff stress tensor, and λ is a scalar parameter. Throughout the paper, lower case Latin indices are associated with the spatial coordinates x_i

and assume the values 1, 2, 3. Similarly, upper case Latin indices are associated with the material coordinates X_A and take the values 1, 2, 3. We also adopt the usual convention of summation over repeated indices. Without any further statements, the material coordinates X_A are assumed to indicate the position of a generic particle of a body in the reference configuration, which is also assumed to be identical to the initial configuration.

Recalling the definition of deviatoric stress tensor, we can readily derive the following equation from Eqs. (1):

$$\dot{e}_{AA}^P = 0. \quad (2)$$

In the context of the infinitesimal theory of plasticity, Eq. (2) ensures conservation of volume throughout the plastic deformation. In the context of finite strain plasticity, however, there have been no literatures in which the conservation of volume is confirmed starting from Eq. (2). The conservation of volume due to plastic deformation is one of the most important stipulations even in the context of finite strain plasticity, and hence we shall check whether Eq. (2) is compatible with the conservation of volume. Since Eq. (2) always holds according to Eqs. (1), the following equation holds after any arbitrary deformation:

$$e_{AA}^P = 0. \quad (3)$$

The question now becomes whether Eq. (3) is compatible with the conservation of volume. As one of the simplest deformation, consider the deformation of uniaxial elongation. Suppose we have a cubic with the side length L in the reference configuration, and the cubic is subjected to the motion described as follows:

$$x_1 = (1+t)X_1, \quad x_2 = \frac{X_2}{\sqrt{1+t}}, \quad x_3 = \frac{X_3}{\sqrt{1+t}}, \quad (4)$$

where t is the time variable. This motion is uniform elongation in the x_1 -direction. Though elastic strain may be induced during loading, the elastic strain completely vanishes after unloading due to the uniformity of the deformation. Hence the deformation described in Eqs. (4) can be regarded as purely plastic deformation. From Eqs. (4), Jacobian is calculated as $J = 1$ everywhere, which assures $\dot{J} = 0$ i.e. the motion is isochoric. Having these observations, the motion is considered as an kinematically admissible plastic deformation. Considering the deformation when $t = 1$, trace of the the Lagrangian strain is given by

$$e_{11} + e_{22} + e_{33} = \frac{3}{2} - \frac{1}{4} - \frac{1}{4} = 1. \quad (5)$$

From this result, it is concluded that the Eqs. (1) are not compatible with the conservation of volume through plastic deformation. In other words, the constitutive equations (1) for plastic strain rate are not adequate for describing the volume preserving plastic deformation. Currently, as far as the isotropic materials are concerned, the simplest and reasonable way

of formulating the constitutive equations for plastic deformation is considered to adopt the plastic potential of von Mises type with the normality principle as flow rule. With this type of formulation, like Eqs. (1), plastic strain rate tensor becomes proportional to the deviatoric stress tensor, and this inevitably leads trace of the plastic strain tensor to zero; Equations (1) are one of the simplest examples. Hence the fact that Eqs. (1) are not compatible with the conservation of volume means that the Lagrangian strain is not suitable for describing plastic deformation accompanied by finite strain.

Now, it is the Eulerian strain to be tested. For the deformation described by Eqs. (4) when $t = 1$, the trace of the Eulerian strain tensor is given by

$$e_{11}^E + e_{22}^E + e_{33}^E = \frac{3}{8} - \frac{1}{2} - \frac{1}{2} = -\frac{5}{8}. \quad (6)$$

This means that the Eulerian strain is not adequate for describing plastic deformation accompanied by finite strain either.

Recently Heidushke[12] introduced the logarithmic strain space description into Naghdi's theory of plasticity. The logarithmic strain tensor does not have above mentioned difficulty concerning volume preserving plastic deformation, and its definition has clear physical background. In this paper the logarithmic strain is deemed exact strain definition, and more practical strain definition is introduced for numerical purpose.

3. DEFINITION OF A FINITE STRAIN TENSOR

The rate of deformation tensor D_{ij} is defined by

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad (7)$$

where $v_{i,j}$ means the partial derivative of velocity vector v_i of a typical particle in a body with respect to the spatial coordinates x_j . The rate of deformation tensor is frequently used as a notion of strain rate tensor in the context of finite deformation. It is well known that zero trace of the rate of deformation tensor represents isochoric motion. This fact is readily understood from the following relationship between the material derivative of Jacobian J and the rate of deformation tensor:

$$\dot{J} = J v_{i,i} = J D_{ii}. \quad (8)$$

Therefore, in view of the conservation of volume during plastic deformation, it is confirmed that the rate of deformation tensor is a promising candidate for plastic strain rate tensor. Thus, in this paper, the rate of deformation tensor is adopted for describing plastic strain rate tensor. Question now becomes how physically valid notion of finite strain is obtained from the rate of deformation tensor.

Define the following notion of strain through integrating the rate of deformation tensor:

$$\varepsilon_{ij}(X_K, t) = R_{iA}(X_K, t)R_{jB}(X_K, t)\hat{\varepsilon}_{AB}(X_K, t) \quad (9)$$

and

$$\hat{\varepsilon}_{AB}(X_K, t) = \int_0^t R_{mA}(\eta) R_{nB}(\eta) D_{mn}(\eta) d\eta, \quad (10)$$

where R_{iA} is the rotation tensor obtained by the polar decomposition of the deformation gradient, and the second argument t or η is the time variable. $\eta = 0$ is associated with the reference configuration before deformation, and current configuration is the one at $\eta = t$. It should be noted that strain tensors defined by Eqs. (9) and (10) describe strain accumulated at a typical particle located by the material coordinates X_K , and the rate of deformation tensor D_{mn} is integrated as a function of position in the reference configuration and time. The rotation tensors R_{mA} , R_{nB} are also functions of position in the reference configuration and time. In this paper, the strain tensor defined by Eqs. (10) is called the *referential strain tensor* to distinguish from the *current strain tensor* or, simply, the *strain tensor* defined by Eqs. (9).

This definition of strain can be seen in the work of Stören and Rice[5], in which it is introduced as not a strain-measure but a deformation-measure because it is path-dependent. In the present work, however, it will be shown that the entities defined by Eqs. (9) and (10) can be treated as strain-measures under the certain circumstances.

3.1. Physical Interpretation of the Strain Tensor

Since the rate of deformation tensor D_{mn} is regarded as strain rate tensor, one can expect that strain tensor can be obtained by integrating the rate of deformation tensor in some way. In the context of finite strain continuum mechanics, however, integration of each component of D_{mn} with respect to time is not always physically meaningful because there may be significant local rigid body rotation due to large deformation. In Eqs. (9) and (10) the rotation tensor is adopted as the adequate material spin for integrating the history of deformation at a typical particle of the body. This definition of strain can be interpreted in the following way.

The strain component ε_{11} , for example, has to be the sum of deformation which has been received in the pertinent period of time by the infinitesimal line segment $d\mathbf{x}_1(t)$ currently lying in the x_1 direction. Suppose $d\mathbf{x}_1$ is embedded in the deforming material, then the direction of the vector $d\mathbf{x}_1(\eta)$ has not always been in the x_1 direction. Hence the rate of deformation tensor has to be integrated while tracing the direction change of the vector $d\mathbf{x}_1(\eta)$ to obtain the correct notion of strain tensor.

Generally speaking, the rotation of the vector embedded in the deforming material can be expressed by the rotation tensor $\mathbf{R}$, and then the following relations are obtained:

$$\begin{aligned} d\mathbf{x}_1(t) &= \mathbf{R}(t) d\mathbf{X}_1, \\ d\mathbf{x}_1(\eta) &= \mathbf{R}(\eta) d\mathbf{X}_1, \end{aligned} \quad (11)$$

where $d\mathbf{X}_1$ is the reference configuration of the vector $d\mathbf{x}_1$, and the pure deformation has been neglected be-

cause the direction change of the vector embedded in the material is of our concern in this discussion. Using the relationship $\mathbf{R}^{-1} = \mathbf{R}^t$ where $\mathbf{R}^t$ denotes the transposed matrix of $\mathbf{R}$, Eqs. (11) can be written as

$$\begin{aligned} d\mathbf{x}_1(t) &= \mathbf{R}(t) \mathbf{R}^t(\eta) d\mathbf{x}_1(\eta), \\ d\mathbf{x}_1(\eta) &= \mathbf{R}(\eta) \mathbf{R}^t(t) d\mathbf{x}_1(t). \end{aligned} \quad (12)$$

Since the rotation tensor $\mathbf{R}$ represents the local rigid body rotation, not only the vector $d\mathbf{x}_1$ but also all vectors embedded in the material are subject to the rigid body rotation $\mathbf{R}$ before or after the pure deformation. Hence, if we observe the material deformation with reference to the base vectors rotating in accordance with Eqs. (12), pure deformation can be extracted and observed. It follows that, to evaluate the finite strain as a consequence of the history of the rate of deformation tensor, the following entities should be integrated:

$$A_{im}(t, \eta) A_{jn}(t, \eta) D_{mn}(X_K, \eta), \quad (13)$$

where

$$A_{im}(t, \eta) = R_{iA}(t) R_{mA}(\eta). \quad (14)$$

Thus the definition of the finite strain tensor, Eqs. (9) and (10), has been motivated.

3.2. Evaluation of the Strain Tensor for Uniaxial Isochoric Motion

In Chapter 2, the existing notions of finite strain are tested by one of the simplest isochoric motions described by Eqs. (4). Hence the finite strain tensors defined by Eqs. (9) and (10) are tested below in the same manner.

In case of this uniaxial motion, the deformation gradient is expressed by diagonal matrix, and the rotation tensor becomes the unit matrix. As a result the strain tensor defined by Eqs. (9) and (10) reduces to time integration of each component of the rate of deformation tensor, and the following components of the strain tensor are obtained:

$$\begin{aligned} \varepsilon_{11}(t) &= \ln |1 + t|, \\ \varepsilon_{22}(t) &= \varepsilon_{33}(t) = -\frac{1}{2} \ln |1 + t|, \\ \varepsilon_{12}(t) &= \varepsilon_{23}(t) = \varepsilon_{31}(t) = 0. \end{aligned} \quad (15)$$

From the results in Eqs. (15), the strain tensors defined by Eqs. (9) and (10) can be interpreted as extended notion of logarithmic strain which is commonly used for uniaxial finite deformation. Equations (15) also indicate that, for the isochoric motion given by Eqs. (4), $\varepsilon_{ii} = 0$ holds for the strain tensor. Starting from the definition (9) and (10), this characteristic is generalized for arbitrary isochoric motions as follows:

$$\begin{aligned} \varepsilon_{ii}(t) &= R_{iA}(t) R_{iB}(t) \int_0^t R_{mA}(\eta) R_{nB}(\eta) D_{mn}(\eta) d\eta \\ &= \int_0^t D_{mm}(\eta) d\eta. \end{aligned} \quad (16)$$

Since $D_{mm} = 0$ always holds for arbitrary isochoric motions, $\varepsilon_{ii} = 0$ also holds for any of isochoric motions.

3.3. Evaluation of the Strain Tensor for the Motion of Simple Shear

Now consider another typical motion, simple shear, expressed by the following equation:

$$x_1 = X_1 + tX_2, \quad x_2 = X_2, \quad x_3 = X_3, \quad (17)$$

where t is the time variable.

For this motion, the deformation gradient $\mathbf{F}$ and the rate of deformation tensor $\mathbf{D}$ are given as follows:

$$\mathbf{F} = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (18)$$

The right stretch tensor $\mathbf{U}$ is obtained by extracting the square root of the Green's deformation tensor $\mathbf{C} = \mathbf{F}^t \mathbf{F}$. And then finding the rotation tensor $\mathbf{R}$ through the relation $\mathbf{R} = \mathbf{F} \mathbf{U}^{-1}$, polar decomposition of the deformation gradient $\mathbf{F}$ completes. Thus $\mathbf{R}$ and $\mathbf{U}$ are given by

$$\mathbf{R} = \begin{bmatrix} 2/\sqrt{t^2+4} & t/\sqrt{t^2+4} & 0 \\ -t/\sqrt{t^2+4} & 2/\sqrt{t^2+4} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} 2/\sqrt{t^2+4} & t/\sqrt{t^2+4} & 0 \\ t/\sqrt{t^2+4} & (t^2+2)/\sqrt{t^2+4} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (19)$$

Using Eqs. (10) and Eqs. (18), (19), the referential strain tensor $\hat{\varepsilon}$ is obtained as

$$\hat{\varepsilon} = \begin{bmatrix} -\ln(1+t^2/4) & 2\tan^{-1}(t/2) - t/2 & 0 \\ 2\tan^{-1}(t/2) - t/2 & \ln(1+t^2/4) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (20)$$

Hence the components of the strain tensor defined by Eqs. (9) are given as follows:

$$\begin{aligned} \varepsilon_{11} &= (t^2-4)/(t^2+4) \ln(1+t^2/4) \\ &\quad + 4t/(t^2+4) [2\tan^{-1}(t/2) - t/2], \\ \varepsilon_{12} &= 4t/(t^2+4) \ln(1+t^2/4) \\ &\quad - (t^2-4)/(t^2+4) [2\tan^{-1}(t/2) - t/2], \\ \varepsilon_{22} &= -\varepsilon_{11}, \quad \varepsilon_{21} = \varepsilon_{12}, \\ \varepsilon_{33} &= \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{31} = \varepsilon_{32} = 0. \end{aligned} \quad (21)$$

Equation (20) suggests that the motion of simple shear described by Eqs. (17) consists of pure shear within the X_1 - X_2 coordinate plane, compression in the X_1 direction, and extension in the X_2 direction. This is readily understood by the fact that, in the X_1 - X_2 coordinate plane, an initially regular square becomes a parallelogram after the motion of simple shear. In spite of this rather apparent observation, above-mentioned feature of simple shear cannot

be recognized if one observes the deformation only through the rate of deformation tensor, which cannot distinguish the simple shear from the pure shear. It is thus concluded that the rotation tensor plays a substantial role in calculating physically meaningful strain tensor in the context of the finite deformation.

Since the referential strain tensor defined by Eqs. (10) can be interpreted as the accumulated strain observed in the reference coordinate system, it should accord with the Green's deformation tensor $\mathbf{C}$, or more directly, with the right stretch tensor $\mathbf{U}$. Recalling that the strain tensor defined by Eqs. (10) coincides with the logarithmic strain in case of uniaxial deformation, it may be instructive to observe the referential strain tensor $\hat{\varepsilon}$ with reference to the principal axes of the right stretch tensor $\mathbf{U}$, and to compare it with the logarithmic strain obtained from the principal stretches which are the eigenvalues of $\mathbf{U}$. In order to do this, firstly find the eigenvalues and the eigenvectors of $\mathbf{U}$, and secondly diagonalize $\mathbf{U}$ obtaining the diagonalized right stretch tensor $\bar{\mathbf{U}}$. Between $\mathbf{U}$ and $\bar{\mathbf{U}}$, the following relationship holds with the relevant orthogonal matrix $\mathbf{Q}$:

$$\mathbf{U} = \mathbf{Q} \bar{\mathbf{U}} \mathbf{Q}^t. \quad (22)$$

According to the above-mentioned procedure, $\bar{\mathbf{U}}$ and $\mathbf{Q}$ are obtained as follows:

$$\bar{\mathbf{U}} = \begin{bmatrix} (\sqrt{t^2+4}-t)/2 & 0 & 0 \\ 0 & (\sqrt{t^2+4}+t)/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (23)$$

$$\mathbf{Q} = \begin{bmatrix} \sqrt{(1+t\alpha)/2} & \sqrt{(1-t\alpha)/2} & 0 \\ -\sqrt{(1-t\alpha)/2} & \sqrt{(1+t\alpha)/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (24)$$

where

$$\alpha = 1/\sqrt{t^2+4}. \quad (25)$$

Fig.1. Comparison of strain components for the motion of simple shear.

The referential strain tensor observed with reference to the principal axes of the right stretch tensor $\mathbf{U}$ is then given by $\hat{\epsilon}' = \mathbf{Q}^t \hat{\epsilon} \mathbf{Q}$, which renders

$$\begin{aligned}\hat{\epsilon}'_{11} &= -\alpha\{4\tan^{-1}(t/2) - t[1 - \ln(1 + t^2/4)]\}, \\ \hat{\epsilon}'_{12} &= \alpha\{t[2\tan^{-1}(t/2) - t/2] - 2\ln(1 + t^2/4)\}, \\ \hat{\epsilon}'_{22} &= -\hat{\epsilon}'_{11}, \quad \hat{\epsilon}'_{21} = \hat{\epsilon}'_{12}, \\ \hat{\epsilon}'_{33} &= \hat{\epsilon}'_{13} = \hat{\epsilon}'_{23} = \hat{\epsilon}'_{31} = \hat{\epsilon}'_{32} = 0.\end{aligned}\tag{26}$$

Let the coordinate system $\bar{X}_A$ be defined by the eigenvectors of the right stretch tensor $\mathbf{U}$. Then the logarithmic strain in $\bar{X}_2$ direction is given by $\ln(\bar{U}_{22})$, and it should be compared with $\hat{\epsilon}'_{22}$. The shear component $\hat{\epsilon}'_{12}$ should be zero to have exact accordance with $\bar{U}$. Figure 1 shows the result of the comparison. Although the referential strain tensor does not exactly coincide with the logarithmic strain calculated through the right stretch tensor, it gives fairly good approximation of the logarithmic strain. It gives practically the same value of strain as the logarithmic strain tensor does until the time $t = 1$ when the deformation is quite large, $\ln(\bar{U}_{22}) \approx 0.5$.

3.4. Evaluation of the Strain Tensor for Infinitesimal Deformation with Finite Rotation

Before proceeding to the evaluation of the strain tensor for arbitrary infinitesimal deformation with finite rotation, discussion would be worthwhile about the reason why the strain tensor defined by Eqs. (10) does not exactly coincide with the logarithmic strain tensor for the motion of simple shear. Figure 2 is a schematic diagram of the reference and the current configurations of a small generic part of a body. The deformation is well analyzed according to the polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{R}\mathbf{U}$.

The eigenvectors $\mathbf{A}$, $\mathbf{B}$ expressing the directions of principal stretches of the right stretch tensor $\mathbf{U}$ are considered to be embedded in the part of the body. They rotate for the amount given by the rotation tensor $\mathbf{R}$ after receiving the stretches according to the eigenvalues of the right stretch tensor $\mathbf{U}$, and then they become the vectors $\mathbf{a}$, $\mathbf{b}$ in the current configuration. On the other hand, if the coordinate axes X_1 , X_2 in the reference configuration are considered to be embedded in the body, before rotating due to the rotation tensor $\mathbf{R}$, they receive not only the stretches but also some shear deformation because, in general, their directions in the reference configuration do not coincide with the ones of the eigenvectors of the right stretch tensor $\mathbf{U}$. Hence the relative angles between the original coordinate axes and the eigenvectors observed in the current configuration differ from the ones observed in the reference configuration. It means that the angle between the embedded coordinate axes X_1 , X_2 changes due to the deformation. It is, therefore, concluded that, even if the rate of deformation tensor is observed with reference to the corotational coordinate system defined by the rotation tensor, which is the idea of the definition of the referential strain tensor given by Eqs. (10), the corotational coordinate system does not exactly coincide with the coordinate axes embedded in the body. This is the reason why

the referential strain tensor does not exactly coincide with the logarithmic strain tensor for the motion of simple shear.

The above discussion suggests that the referential strain tensor holds the exact physical meaning in case of infinitesimal deformation with finite rotation, because in this case the shear deformation, which has caused the difference between the embedded coordinate axes and the corotational coordinate system, is infinitesimal. The proof of this is given below.

Consider arbitrary infinitesimal deformation with finite rotation. Let the right stretch tensor $\mathbf{U}$ be expressed by

$$\mathbf{U} = \mathbf{I} + \epsilon, \tag{27}$$

where $\mathbf{I}$ is the unit tensor and all components of the strain tensor ϵ is infinitesimal because the deformation is assumed to be infinitesimal. Substituting Eq. (27) into the equation of the polar decomposition, the deformation gradient is given by

$$\mathbf{F} = \mathbf{R}(\mathbf{I} + \epsilon). \tag{28}$$

Using Eq. (28), the velocity gradient tensor $\mathbf{L}$ is given as follows:

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{R}}\mathbf{R}^t + \mathbf{R}\dot{\epsilon}(\mathbf{I} + \epsilon)^{-1}\mathbf{R}^t. \tag{29}$$

Since the strain tensor ϵ is composed of infinitesimal components, neglecting the higher order of ϵ , the following equation can be obtained:

$$(\mathbf{I} + \epsilon)^{-1} = \mathbf{I} - \epsilon + O(\epsilon^2) \approx \mathbf{I} - \epsilon. \tag{30}$$

Substituting Eq. (30) into Eq. (29), and neglecting the higher order of ϵ , the velocity gradient can be written as

$$\mathbf{L} = \dot{\mathbf{R}}\mathbf{R}^t + \mathbf{R}\dot{\epsilon}(\mathbf{I} - \epsilon)\mathbf{R}^t \approx \dot{\mathbf{R}}\mathbf{R}^t + \mathbf{R}\dot{\epsilon}\mathbf{R}^t. \tag{31}$$

Since $\mathbf{R}$ is orthogonal, ϵ is symmetric, and $\dot{\mathbf{R}}\mathbf{R}^t$ is skew-symmetric, the rate of deformation tensor $\mathbf{D}$ is given by

$$\mathbf{D} = (\mathbf{L} + \mathbf{L}^t)/2 = \mathbf{R}\dot{\epsilon}\mathbf{R}^t. \tag{32}$$

Now using the definition of the referential strain tensor, Eqs. (10), and Eq. (32), the referential strain tensor $\hat{\epsilon}$ is calculated as follows:

$$\hat{\epsilon} = \int_0^t \mathbf{R}^t(\mathbf{R}\dot{\epsilon}\mathbf{R}^t)\mathbf{R}d\eta = \int_0^t \dot{\epsilon}d\eta = \epsilon. \tag{33}$$

It is concluded that the referential strain tensor coincides with the strain tensor defined from the right stretch tensor in case of infinitesimal deformation with finite rotation, and hence the current strain tensor defined by Eqs. (9) is just another expression of the strain tensor with reference to the current coordinate system.

Moreover it is found from Eq. (32) that one cannot capture the pure deformation if one observes the deformation only through the rate of deformation tensor because it contains not only the correct strain but also the rigid body rotation.

Fig.2. Schematic diagram of the reference and the current configurations of a small part of a body.

4. ELASTIC-PLASTIC DECOMPOSITION OF THE STRAIN

The elastic-plastic decomposition of the strain tensor is discussed in this section, which is indispensable for elastic-plastic finite element analysis. The rate of deformation tensor $\mathbf{D}$ is assumed to be decomposed into the elastic component $\mathbf{D}^e$ and the plastic component $\mathbf{D}^p$ as follows:

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p. \quad (34)$$

Lee[6] and Lubarda and Lee[7] derived Eq. (34) assuming the existence of the plastically deformed configuration for infinitesimal elastic deformation with finite plastic deformation. Here in this paper infinitesimal elastic deformation with finite plastic deformation is also assumed as is the case for most elastic-plastic continua. However we interpret Eq. (34) in a slightly different way from Lee[6] and Lubarda and Lee[7].

Deformation is decomposed into pure deformation and rigid body rotation through the polar decomposition theorem. The pure deformation which is described by the right stretch tensor $\mathbf{U}$ is assumed to be decomposed into elastic component $\mathbf{U}^e$ and plastic component $\mathbf{U}^p$. Let the elastic strain tensor with reference to base vectors in the reference coordinate system be denoted by ϵ^e . Then the following equation holds because ϵ^e is infinitesimal:

$$\mathbf{U}^e = \mathbf{I} + \epsilon^e, \quad (35)$$

and then $\mathbf{U}^e$ turns out to be symmetric. There are following two ways of decomposing the right stretch tensor $\mathbf{U}$ into the elastic component $\mathbf{U}^e$ and the plastic component $\mathbf{U}^p$:

$$\mathbf{U} = \mathbf{U}^e \mathbf{U}^p = (\mathbf{I} + \epsilon^e) \mathbf{U}^p, \quad (36)$$

$$\mathbf{U} = \dot{\mathbf{U}}^p \mathbf{U}^e = \dot{\mathbf{U}}^p (\mathbf{I} + \epsilon^e). \quad (37)$$

Since $\mathbf{U}$ is symmetric, the relationship $\dot{\mathbf{U}}^p = (\mathbf{U}^p)^t$ can be obtained from above two equations. Thus elastic-plastic decomposition of the deformation gradient can be written as

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{R}\mathbf{U}^e \mathbf{U}^p = \mathbf{R}(\mathbf{U}^p)^t \mathbf{U}^e. \quad (38)$$

Generally speaking, since $\mathbf{U}^p$ is not symmetric, it is not consistent with the definition of the right stretch tensor. However if infinitesimal elastic strain is neglected, $\mathbf{U}^p$ becomes symmetric. Hence the asymmetric component of $\mathbf{U}^p$ is negligible compared to the plastic strain which is assumed to be finite.

Lubarda and Lee[7] derived the following equation of the elastic-plastic decomposition:

$$\mathbf{F} = \mathbf{V}^e \mathbf{F}^p. \quad (39)$$

Equation (38) differs from Eq. (39) only in describing elastic deformation with the right stretch tensor instead of the left stretch tensor. This is readily understood if one applies the polar decomposition theorem to $\mathbf{F}^p$ in Eq. (39).

Starting from Eq. (35) and Eq. (38), the velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is given by

$$\begin{aligned} \mathbf{L} = & \dot{\mathbf{R}}\mathbf{R}^t + \mathbf{R}\dot{\epsilon}^e(\mathbf{I} + \epsilon^e)^{-1}\mathbf{R}^t \\ & + \mathbf{R}(\mathbf{I} + \epsilon^e)\dot{\mathbf{U}}^p(\mathbf{U}^p)^{-1}(\mathbf{I} + \epsilon^e)^{-1}\mathbf{R}^t. \end{aligned} \quad (40)$$

Neglecting ϵ^e compared with $\mathbf{I}$, the following equation can be obtained:

$$\mathbf{L} = \dot{\mathbf{R}}\mathbf{R}^t + \mathbf{R}\dot{\epsilon}^e\mathbf{R}^t + \mathbf{R}\dot{\mathbf{U}}^p(\mathbf{U}^p)^{-1}\mathbf{R}^t. \quad (41)$$

Using Eq. (41), the rate of deformation tensor can be written as

$$\mathbf{D} = \mathbf{R}\dot{\epsilon}^e\mathbf{R}^t + \mathbf{R}\dot{\mathbf{U}}^p\mathbf{R}^t = \mathbf{D}^e + \mathbf{D}^p, \quad (42)$$

where $\dot{\epsilon}^P$ is defined by

$$\dot{\epsilon}^P = [\dot{U}^P(U^P)^{-1} + \{\dot{U}^P(U^P)^{-1}\}^t]/2. \quad (43)$$

Since it is apparent that D^e and D^P satisfy the invariance requirement under superposed rigid body motion, the elastic-plastic decomposition of the rate of deformation tensor completes.

According to the elastic-plastic decomposition of the rate of deformation tensor, the definitions of the referential elastic strain tensor $\hat{\epsilon}_{AB}^e$, the referential plastic strain tensor $\hat{\epsilon}_{AB}^P$, the elastic strain tensor ϵ_{ij}^e , and the plastic strain tensor ϵ_{ij}^P are given as follows:

$$\hat{\epsilon}_{AB}^e = \int_0^t R_{mA}(\eta) R_{nB}(\eta) D_{mn}^e(\eta) d\eta, \quad (44)$$

$$\hat{\epsilon}_{AB}^P = \int_0^t R_{mA}(\eta) R_{nB}(\eta) D_{mn}^P(\eta) d\eta, \quad (45)$$

$$\epsilon_{ij}^e = R_{iA} R_{jB} \hat{\epsilon}_{AB}^e, \quad (46)$$

$$\epsilon_{ij}^P = R_{iA} R_{jB} \hat{\epsilon}_{AB}^P. \quad (47)$$

Using Eqs. (10), (34), (44) and (45), the following relationship holds:

$$\hat{\epsilon}_{AB} = \int_0^t R_{mA} R_{nB} (D_{mn}^e + D_{mn}^P) d\eta = \hat{\epsilon}_{AB}^e + \hat{\epsilon}_{AB}^P. \quad (48)$$

As for the current strain tensor, the following relationship is readily obtained using Eqs. (9) and (48):

$$\epsilon_{ij} = R_{iA} R_{jB} (\hat{\epsilon}_{AB}^e + \hat{\epsilon}_{AB}^P) = \epsilon_{ij}^e + \epsilon_{ij}^P. \quad (49)$$

Hence it is confirmed that, with the definition of strain tensor given so far in this paper, the additivity of the elastic strain tensor and the plastic strain tensor holds for infinitesimal elastic deformation accompanied by finite plastic deformation.

5. CONSTITUTIVE EQUATIONS FOR STRESS EVALUATION

We confine our attention to a group of elastic-plastic continua which exhibit infinitesimal elastic deformation while the plastic deformation could be fairly large. For such material, as is the case for most metals, stress is determined by the elastic strain through the generalized Hooke's law with sufficient accuracy. Hence in this paper the generalized Hooke's law is utilized for stress evaluation excluding non-linear elastic behavior.

The deviatoric components τ_{ij} of the Cauchy stress tensor σ_{ij} , the mean normal stress $\bar{\sigma}$, the deviatoric components γ_{ij}^e of the elastic strain tensor ϵ_{ij}^e , and the elastic mean normal strain $\bar{\epsilon}^e$ are defined as follows:

$$\bar{\sigma} = \sigma_{ii}/3, \quad \tau_{ij} = \sigma_{ij} - \delta_{ij} \bar{\sigma}, \quad (50)$$

$$\bar{\epsilon}^e = \epsilon_{ii}^e/3, \quad \gamma_{ij}^e = \epsilon_{ij}^e - \delta_{ij} \bar{\epsilon}^e, \quad (51)$$

where δ_{ij} is the Kronecker symbol. Using these notations, the generalized Hooke's law can be described by

$$\tau_{ij} = 2\mu \gamma_{ij}^e, \quad \bar{\sigma} = 3k \bar{\epsilon}^e, \quad (52)$$

where μ is the shear modulus of elasticity and k is the bulk modulus. Since it is assumed that the plastic deformation is volume preserving, the following equations hold:

$$D_{ii}^P = 0, \quad \bar{\epsilon}^P = \epsilon_{ii}^P/3 = 0, \quad (53)$$

where $\bar{\epsilon}^P$ is the plastic mean normal strain. Utilizing Eqs. (53), the following equations are readily obtained:

$$\bar{\epsilon} = (\epsilon_{ii}^e + \epsilon_{ii}^P)/3 = \bar{\epsilon}^e, \quad (54)$$

$$\gamma_{ij}^e = \epsilon_{ij} - \epsilon_{ij}^P - \delta_{ij} \bar{\epsilon} = \gamma_{ij} - \epsilon_{ij}^P, \quad (55)$$

where $\bar{\epsilon}$ is the total mean normal strain, and γ_{ij} are the deviatoric components of the total strain tensor ϵ_{ij} . With Eqs. (54) and (55), the generalized Hooke's law is rewritten as

$$\tau_{ij} = 2\mu(\gamma_{ij} - \epsilon_{ij}^P), \quad \bar{\sigma} = 3k \bar{\epsilon}. \quad (56)$$

Equations (52) or Eqs. (56) are the constitutive equations observed with reference to the current coordinate system. The same constitutive equations observed with reference to the reference coordinate system are derived below.

The following equations define the *referential stress tensor* $\hat{\sigma}_{AB}$:

$$\hat{\sigma}_{AB} = R_{iA} R_{jB} \sigma_{ij}. \quad (57)$$

The Cauchy stress tensor discussed so far may be called the *current stress tensor* in contrast to the referential stress tensor.

The referential mean normal stress $\hat{\sigma}$ and the referential deviatoric stress tensor $\hat{\tau}_{AB}$ are defined as

$$\hat{\sigma} = \hat{\sigma}_{AA}/3, \quad \hat{\tau}_{AB} = \hat{\sigma}_{AB} - \delta_{AB} \hat{\sigma}. \quad (58)$$

These definitions readily lead the following relationships:

$$\bar{\sigma} = \hat{\sigma}, \quad \tau_{ij} = R_{iA} R_{jB} \hat{\tau}_{AB}, \quad (59)$$

$$\bar{\epsilon} = \bar{\epsilon}^e = \hat{\epsilon} = \hat{\epsilon}^e, \quad \gamma_{ij}^e = R_{iA} R_{jB} \hat{\gamma}_{AB}^e, \quad (60)$$

where the mean normal component $\hat{\epsilon}$ and the deviatoric components $\hat{\gamma}_{AB}^e$ of the referential strain tensor $\hat{\epsilon}_{AB}$ are introduced.

Substituting Eqs. (59) and (60) into Eqs. (52), the generalized Hooke's law with reference to the reference coordinate system is obtained as

$$\hat{\tau}_{AB} = 2\mu \hat{\gamma}_{AB}^e, \quad \hat{\sigma} = 3k \hat{\epsilon}^e = 3k \hat{\epsilon}. \quad (61)$$

According to the generalized Hooke's law, it is apparent that the elastic strain tensor plays a substantial role in calculating the stress tensor, and the total strain and the plastic strain tensor are used only for obtaining the elastic strain tensor starting from the elastic-plastic decomposition of the rate of deformation tensor described by Eq. (34). The validity of these constitutive equations for stress tensor,

therefore, solely depends on the validity of the elastic strain tensor. The notion of the plastic strain tensor is needed only for obtaining the elastic component of the strain tensor, and rigorous validity of the notion of the plastic strain tensor is irrelevant to the correctness of the constitutive equations for stress tensor.

Now it is already confirmed that the definition of finite strain tensor proposed in this paper holds precise validity when it is used to describe a motion composed of finite rotation and infinitesimal deformation, and the elastic strain dealt in this paper can be regarded as the strain corresponding to such a motion. Hence it is concluded that the constitutive equation for stress tensor obtained so far is valid in the context of finite deformation as far as the elastic deformation is infinitesimal.

Utilizing the notion of finite strain tensor and the constitutive equations of stress tensor, a useful description for plastic strain tensor and stress tensor can be obtained through analytically integrating the constitutive equations for plastic strain rate tensor, which will be shown in the subsequent paper.

In addition it can be readily shown that the current formulation is theoretically equivalent to the rate type formulation with the Green-Naghdi rate of the Cauchy stress tensor.

6. CONCLUSION

As a notion of strain compatible with the theory of plasticity having the plastic potential of von Mises type with the normality principle as flow rule, the strain tensor is defined by integrating the rate of deformation tensor considering the material spin with the rotation tensor.

It has been proved that the strain tensor coincides with the strain tensor defined from the right stretch tensor in case of the infinitesimal deformation with fi-

nite rotation.

Confining our attention to a group of elastic-plastic continua which exhibit infinitesimal elastic deformation while the plastic deformation could be fairly large, elastic-plastic decomposition of the strain tensor is established in accordance with the definition of the strain tensor and the elastic-plastic decomposition of the rate of deformation tensor.

Since the elastic strain tensor is to describe a motion composed of finite rotation and infinitesimal deformation within our scope, the stress tensor can be calculated through the generalized Hooke's law with the elastic strain tensor defined here without any ambiguity.

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