Adaptive Fuzzy Controller for a Class of Uncertain Nonlinear Systems

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In general, the state feedback control gain can be obtained by solving certain linear matrix inequalities (LMIs) when using the Takagi-Sugeno (T-S) fuzzy model to develop a control system. In this paper, the reconstruction error between the real system to be controlled and its T-S fuzzy model, which consists of parameter uncertainties and external disturbance, is considered. As a result, we arrive at an adaptive controller that has two parts: one is obtained by solving certain LMIs (fixed part) and another one is acquired by an adaptive law (variable part). The proposed controller can guarantee the control state to converge and uniformly bounded while maintaining all the signals involved stable. Also, the convergence and boundedness in terms of relaxing the LMIs conservatism are discussed. An inverted pendulum is provided to demonstrate the effectiveness of the proposed adaptive fuzzy controller.

Keywords : Linear matrix inequality, adaptive fuzzy controller, conservatism, uniformly boundedness

1. Introduction

One of the essential elements of the control problem is the model of the dynamical system to be controlled. In many cases a mathematical model of the system is unavailable or incomplete, or the equations that we believe are adequate to represent the behavior of the system are too complicated for the design purpose. This is the most likely reason why the application of fuzzy set theory to control problems has been the focus of numerous studies. The reason is that the fuzzy set theory provides an alternative to the traditional modeling and design of control systems, where knowledge at the dynamic model of the system in the traditional sense is uncertain and time-varying. In recent years, there have been significant advances in the study of the stability analysis and controller synthesis for the so-called Takagi-Sugeno (T-S), also known as the Takagi-Sugeno-Kang (TSK) fuzzy systems [1][2], which have been used to represent certain complex nonlinear systems. In the T-S fuzzy model, the local dynamics in different state-space regions are represented by linear models such as $\dot{x}(t) = A_i x(t) + B_i u(t)$, where A_i, B_i are certain known matrices with some appropriate dimensions. The overall model of the system is obtained by the fuzzy blending of these

local models. The control design is carried out based on the fuzzy model by the so-called parallel distributed compensation (PDC) scheme [3]. For each local linear model, a linear feedback control is designed. The resulting overall controller is again a fuzzy blending of the individual linear controllers. Originally, Tanaka and his colleagues have provided certain conditions that are sufficient for the stability of the T-S fuzzy systems in the sense of Lyapunov [3]-[5]. The conditions for the existence of a common Lyapunov function are obtained by solving linear matrix inequalities (LMIs). Currently, the stability issue about the T-S fuzzy systems has been extensively studied.

However, only a few results concerning the modeling error, which often is referred to as reconstruction error, between the real system to be controlled and its T-S fuzzy model, are reported [4][6]-[8]. Basically, the reconstruction error is caused by the uncertainties in the real (nonlinear) system including external disturbance, and the transformation from the real system to the T-S fuzzy model. In [4], they have concluded some LMI conditions to confine the influence of the reconstruction error to a certain region. More popularly, the reconstruction error is treated by involving some norm-bounded matrices with certain structure properties in each local T-S fuzzy model

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such as $\dot{x}(t) = (A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t)$, where ΔA_i , ΔB_i represent the norm-bounded matrices [6] [8], consequently, the LMI technique can be applied to obtain the fuzzy controller. However, the norm-bounded matrices are not easy to be estimated since the mathematical model of the real system is unavailable in most cases, and often being set large to safely cover the reconstruction error. At the same time, the large norm-bounded matrices inserted in each local T-S fuzzy model may lead the final LMIs to being conservative. In [7], Zhang and his colleagues have firstly considered an external disturbance term in addition to the norm-bounded matrices in each local T-S fuzzy model such as $\dot{x}(t) = (A_i + \Delta A_i)x(t) + B_i u(t) + D_i$, where D_i denotes the external disturbance, and proposed a robust fuzzy controller that can deal with both ΔA_i and D_i . However, the results may be limited as we expected since they did not consider uncertainties in the gain matrix B_i .

In this paper, in order to effectively deal with the reconstruction error, each local T-S fuzzy model is formed of $\dot{x}(t) = (A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) + D_i$, and an adaptive controller, which has two parts: one is obtained by solving certain LMIs (fixed part) and the another one is acquired by an adaptive law (variable part), is proposed. The proposed controller can guarantee the control state to converge and uniformly bounded while maintaining all the signals involved stable. Also, the convergence and boundedness in terms of relaxing the LMIs conservatism are discussed. An inverted pendulum is provided to demonstrate the effectiveness of the proposed adaptive fuzzy controller.

2. Problem Statement

2.1 System and its T-S Fuzzy Model

Consider the following uncertain nonlinear system:

$$\begin{aligned} \dot{x}(t) = & f(x, t) + \Delta f(x, t, \xi) \\ & + (g(x, t) + \Delta g(x, t, \xi)) u(t) + \psi(x, t, \xi), \end{aligned} \quad (1)$$

where $x(t) \in \mathcal{R}^n$ is the vector of state variables; $u(t) \in \mathcal{R}^m$, the vector of control inputs; ξ , a vector of uncertain parameters that is restricted to a prescribed bounded set Ξ ; f and g , nonlinear functions of (x, t) , continuously differentiable with respect to (x, t) ;

Δf and Δg , the model uncertainties; and ψ , the external disturbance.

With the uncertainties and disturbance in consideration, such a system can be expressed in the terms of r fuzzy rules in the T-S fuzzy model as follows.

Plant Rule i :

IF $\theta_1(t)$ is F_1^i AND $\cdots$ AND $\theta_p(t)$ is F_p^i , THEN

$$\begin{aligned} \dot{x}(t) = & (A_i + \Delta A_i(x, t, \xi)) x(t) \\ & + (B_i + \Delta B_i(x, t, \xi)) u(t) + D_i(x, t, \xi), \end{aligned}$$

where constant matrices A_i and B_i are of appropriate dimensions, ΔA_i , ΔB_i represent the uncertainties in system matrices, and D_i stands for external disturbance. The overall fuzzy model is of the following form accordingly:

$$\begin{aligned} \dot{x}(t) = & \sum_{i=1}^r \alpha_i(t) [(A_i + \Delta A_i(x, t, \xi)) x(t) \\ & + (B_i + \Delta B_i(x, t, \xi)) u(t) + D_i(x, t, \xi)], \end{aligned} \quad (2)$$

where

$$\begin{aligned} \alpha_i(t) = & \frac{w_i(t)}{\sum_{i=1}^r w_i(t)} \geq 0, \quad \sum_{i=1}^r \alpha_i(t) = 1, \\ w_i(t) = & \prod_{j=1}^p \mu_j^i(\theta_j(t)). \end{aligned}$$

2.2 Matching Conditions

In this paper, the uncertainties ΔA_i and ΔB_i are assumed to be of the following structure [6] [9]:

$$[\Delta A_i(x, t, \xi) \quad \Delta B_i(x, t, \xi)] = M_i \Theta(x, t, \xi) [N_{1i} \quad N_{2i}], \quad (3)$$

where M_i , N_{1i} and N_{2i} are known constant matrices with compatible dimensions, and $\Theta(x, t, \xi)$ is unknown nonlinear time-varying matrix function satisfying

$$\|\Theta(x, t, \xi)\| \leq 1, \quad (4)$$

where the elements of $\Theta(x, t, \xi)$ are assumed to be Lebesgue measurable, and $\|\cdot\|$ denotes the Euclidean norm of a matrix (or vector) in this paper. This type of uncertainty is an effective representation of some nonlinear uncertainties. As mentioned in [9], there are several reasons for assuming that the system un-

certainty has the structure given in (3). First, a linear interconnection of a nominal plant with the uncertainty $\Theta(x, t, \xi)$ leads to a structure of the form (3). In addition, there are many physical systems in which the uncertainty can be modeled in this manner, e.g., systems satisfying “matching conditions”.

As to disturbance, all the external disturbances $D_i(x, t, \xi)$ ($i = 1, 2, \dots, r$) are norm-bounded and satisfy the following matching condition:

$$D_i(x, t, \xi) = B_i d_i(x, t, \xi) \quad (5)$$

$$\|d_i(x, t, \xi)\| \leq d_i^*, \quad (6)$$

where the upper bounds d_i^* are not necessary to be known in this paper. As shown latter on, its estimate $\hat{d}_i$ will be given based on an adaptive law.

To proceed with our development, we state an assumption stipulated on the plant:

$$\bar{b} \leq 2\underline{b}, \quad (7)$$

where $\bar{b} = \max \|B_i\|_{i=1}^r$, $\underline{b} = \min \|B_i\|_{i=1}^r$.

In case the matching condition (7) does not hold, we can modify B_i 's in certain local linear models, since the modified parts are able to be covered by d_i 's in (5), which is estimated by an adaptive law in this paper.

From here, unless confusion arises arguments such as t and ξ will be omitted just for notational convenience.

2.3 Control Objective

The control objective is to design the following PDC controller [3][4] for the fuzzy model (2).

Control Rule i :

IF $\theta_1(t)$ is F_1^i AND $\dots$ AND $\theta_p(t)$ is F_p^i ,

THEN $u(t) = K_i x(t)$, (8)

where $K_i \in \mathcal{R}^n$ ($i = 1, 2, \dots, r$) are the controller gain matrices to be designed. The overall state feedback control law is finally represented as

$$u = \sum_{i=1}^r \alpha_i K_i x \quad (9)$$

With this fuzzy controller, the closed-loop system is written as

$$\begin{aligned} \dot{x} = & \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \{ (A_i + B_i K_j) \\ & + \Delta A_i + \Delta B_i K_j \} x + D_i \} \end{aligned} \quad (10)$$

The aim of this paper is to design r local linear state feedback law (8) for the T-S fuzzy model (2) such that the solutions of (10) are uniformly bounded while maintaining all signals involved stable.

3 . Main Results

3.1 Controller Design

In order to develop our fuzzy control system, the following preliminary lemma is required [10].

Lemma 1 Let $Q \in \mathcal{R}^{n \times n}$ be any matrix, we have, for any $k > 0$ any symmetric positive matrix $T > 0$, that

$$2x^T Q y \leq kx^T Q T^{-1} Q^T x + \frac{1}{k} y^T T y \quad (11)$$

for $\forall x, y \in \mathcal{R}^n$.

Now, for given feedback control gain K_i in (8), we give the following theorem for the system stability conditions of (10).

Theorem 1 For a given fuzzy model (2) with uncertainties and disturbance subject to the matching conditions (3)-(7), if there exist certain symmetric matrices $P > 0$, $T_{1i} > 0$, $T_{2i} > 0$ ($i = 1, 2, \dots, r$), and matrices $\mathcal{K}_j$ ($j = 1, 2, \dots, r$), such that the following LMIs:

$$\begin{bmatrix} \begin{pmatrix} (A_i + B_i \mathcal{K}_j)^T P + P(A_i + B_i \mathcal{K}_j) & (N_{1i} + N_{2i} \mathcal{K}_j)^T \\ + P M_i T_{1i} M_i^T P + P M_i T_{2i} M_i^T P \end{pmatrix} & \\ N_{1i} + N_{2i} \mathcal{K}_j & -T_{1i} \end{bmatrix} < 0 \quad (12)$$

hold, then the feedback control gain K_i in (8):

$$K_i = \mathcal{K}_i + \Phi_i, \quad (13)$$

where

$$\Phi_i = -\delta \hat{d}_i^2 B_i^T P \quad (14)$$

$$\dot{\hat{d}}_i(t) = -\alpha_i \beta_i \gamma_i \hat{d}_i(t) + \alpha_i \gamma_i (\bar{b} \|Px\|)^{1/2} \quad (15)$$

$$\delta \leq \frac{2\bar{b} - \bar{b}}{\hat{d}_i^2 \lambda_{\max} \bar{b}} \quad (16)$$

$\lambda_{\max} = \{\lambda(N_{2i}^T T_{2i}^{-1} N_{2i})\}_{i=1}^r$ which is the maximum eigenvalue (scalar) of square matrices $N_{2i}^T T_{2i}^{-1} N_{2i}$

($i=1, 2, \dots, r$), and $\beta_i > 0, \gamma_i > 0$ are design parameters, will make the solutions of the closed-loop system (10) uniformly bounded while maintaining all signals involved stable. At the same time, the solutions exponentially converge at rate $\frac{\eta}{2}$ to a ball centered at the equilibrium point with radius τ , where η is given in (37), (34), (35), (24), (26), and τ is given in (40) and (32).

Proof: Define a Lyapunov function as follows.

$$V(z(t)) = x^T P x + \sum_{i=1}^r \gamma_i^{-1} \tilde{d}_i^2, \quad (17)$$

where $z = [x, \tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_r]^T$, $\tilde{d}_i = d_i^* - \hat{d}_i$. The time derivative of $V(z)$ on any trajectory of the closed-loop system is

$$\begin{aligned} \dot{V} &= 2\dot{x}^T P x - 2 \sum_{i=1}^r \gamma_i^{-1} \tilde{d}_i \dot{\tilde{d}}_i \\ &= 2 \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \left\{ x^T (A_i + \Delta A_i + B_i K_j \right. \\ &\quad \left. + \Delta B_i K_j)^T + D_i^T \right\} P x - 2 \sum_{i=1}^r \gamma_i^{-1} \tilde{d}_i \dot{\tilde{d}}_i \\ &= \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j x^T \left\{ (A_i + B_i K_j)^T P + P(A_i + B_i K_j) \right\} x \\ &\quad + 2 \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \left\{ x^T (\Delta A_i + \Delta B_i K_j + \Delta B_i \Phi_j + B_i \Phi_j)^T + D_i^T \right\} P x \\ &\quad - 2 \sum_{i=1}^r \gamma_i^{-1} \tilde{d}_i \dot{\tilde{d}}_i. \end{aligned} \quad (18)$$

According to the matching condition (3), (4) and Lemma 1, we have

$$\begin{aligned} &2x^T (\Delta A_i + \Delta B_i K_j)^T P x \\ &= 2x^T (M_i \Theta (N_{1i} + N_{2i} K_j))^T P x \\ &= 2x^T (N_{1i} + N_{2i} K_j)^T \Theta^T M_i^T P x \\ &\leq x^T (N_{1i} + N_{2i} K_j)^T \Theta^T T_{1i}^{-1} \Theta (N_{1i} + N_{2i} K_j) x \\ &\quad + x^T P M_i T_{1i} M_i^T P x \\ &\leq x^T (N_{1i} + N_{2i} K_j)^T T_{1i}^{-1} (N_{1i} + N_{2i} K_j) x \\ &\quad + x^T P M_i T_{1i} M_i^T P x. \end{aligned} \quad (19)$$

By the same token, it follows that

$$2x^T (\Delta B_i \Phi_j)^T P x \leq x^T \Phi_j^T N_{2i}^T T_{2i}^{-1} N_{2i} \Phi_j x + x^T P M_i T_{2i} M_i^T P x. \quad (20)$$

Substituting (19) and (20) in (18), we have

$$\dot{V} \leq V_1 + V_2, \quad (21)$$

where

$$\begin{aligned} V_1 &= x^T \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \left\{ (A_i + B_i K_j)^T P + P(A_i + B_i K_j) \right. \\ &\quad \left. + P M_i T_{1i} M_i^T P + P M_i T_{2i} M_i^T P \right. \\ &\quad \left. + (N_{1i} + N_{2i} K_j)^T T_{1i}^{-1} (N_{1i} + N_{2i} K_j) \right\} x \end{aligned} \quad (22)$$

$$\begin{aligned} V_2 &= \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \left[x^T \left\{ 2(B_i \Phi_j)^T P \right. \right. \\ &\quad \left. \left. + \Phi_j^T N_{2i}^T T_{2i}^{-1} N_{2i} \Phi_j \right\} x + 2D_i^T P x \right] - \sum_{i=1}^r 2\gamma_i^{-1} \tilde{d}_i \dot{\tilde{d}}_i. \end{aligned} \quad (23)$$

We first consider V_1 . Let

$$\begin{aligned} -S_{ij} &= (A_i + B_i K_j)^T P + P(A_i + B_i K_j) \\ &\quad + P M_i T_{1i} M_i^T P + P M_i T_{2i} M_i^T P \\ &\quad + (N_{1i} + N_{2i} K_j)^T T_{1i}^{-1} (N_{1i} + N_{2i} K_j). \end{aligned} \quad (24)$$

Then, by the Schur complement LMIs (12) are equivalent to $S_{ij} > 0$. Therefore, we have

$$V_1 = -x^T S x < 0, \quad (25)$$

where

$$S = \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j S_{ij} > 0. \quad (26)$$

Next, let us consider V_2 . Substituting matching condition (5), variable control gain (14) and adaptive law (15) in (23), it follows that

$$\begin{aligned} V_2 &\leq \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \left\{ -2\delta \tilde{d}_j^2 x^T P B_j B_i^T P x + \delta^2 \tilde{d}_j^4 \lambda_{\max} \bar{b} \|P x\| + 2\tilde{d}_i^* (\bar{b} \|P x\|)^{\frac{1}{2}} \right\} \\ &\quad + \sum_{i=1}^r \alpha_i \left\{ 2\beta_i \tilde{d}_i \dot{\tilde{d}}_i - 2\tilde{d}_i (\bar{b} \|P x\|)^{\frac{1}{2}} \right\} \\ &\leq \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \left\{ -2\delta \tilde{d}_j^2 \bar{b} \|P x\| + \delta^2 \tilde{d}_j^4 \lambda_{\max} \bar{b} \|P x\| + 2\tilde{d}_i^* (\bar{b} \|P x\|)^{\frac{1}{2}} \right\} \\ &\quad + \sum_{i=1}^r \alpha_i \left\{ 2\beta_i \tilde{d}_i \dot{\tilde{d}}_i - 2\tilde{d}_i (\bar{b} \|P x\|)^{\frac{1}{2}} \right\} \\ &= \sum_{i=1}^r \alpha_i \left\{ -2\delta \tilde{d}_i^2 \bar{b} \|P x\| + \delta^2 \tilde{d}_i^4 \lambda_{\max} \bar{b} \|P x\| + 2\tilde{d}_i^* (\bar{b} \|P x\|)^{\frac{1}{2}} \right\} \\ &\quad + \sum_{i=1}^r \alpha_i \left\{ 2\beta_i \tilde{d}_i \dot{\tilde{d}}_i - 2\tilde{d}_i (\bar{b} \|P x\|)^{\frac{1}{2}} \right\} \\ &= \sum_{i=1}^r \alpha_i \left\{ -2\delta \tilde{d}_i^2 \bar{b} \|P x\| + \delta^2 \tilde{d}_i^4 \lambda_{\max} \bar{b} \|P x\| + 2\tilde{d}_i^* (\bar{b} \|P x\|)^{\frac{1}{2}} \right. \\ &\quad \left. + 2\beta_i \tilde{d}_i \dot{\tilde{d}}_i - 2\tilde{d}_i (\bar{b} \|P x\|)^{\frac{1}{2}} \right\} \\ &= \sum_{i=1}^r \alpha_i \left\{ -2\delta \tilde{d}_i^2 \bar{b} \|P x\| + \delta^2 \tilde{d}_i^4 \lambda_{\max} \bar{b} \|P x\| + 2\tilde{d}_i^* (\bar{b} \|P x\|)^{\frac{1}{2}} + 2\beta_i \tilde{d}_i \dot{\tilde{d}}_i \right\}, \end{aligned} \quad (27)$$

where the facts that $\sum_{i=1}^r \alpha_i = \sum_{j=1}^r \alpha_j = 1$, $\tilde{d}_i = d_i^* -$

$\hat{d}_i$ are used. Let

$$\Delta b = \bar{b} - b. \quad (28)$$

Substituting $\underline{b} = \bar{b} - \Delta b$ in (27), it becomes

$$\begin{aligned} V_2 \leq & \sum_{i=1}^r \alpha_i \left\{ -2\delta \hat{d}_i^2 \bar{b} \|Px\| + 2\hat{d}_i (\bar{b} \|Px\|)^{\frac{1}{2}} \right. \\ & \left. + 2\delta \hat{d}_i^2 \Delta b \|Px\| + \delta^2 \hat{d}_i^4 \lambda_{max} \bar{b} \|Px\| + 2\beta_i \tilde{d}_i \hat{d}_i \right\}. \end{aligned} \quad (29)$$

Since the facts that

$$\begin{aligned} & -\delta \hat{d}_i^2 \bar{b} \|Px\| + 2\hat{d}_i (\bar{b} \|Px\|)^{\frac{1}{2}} \\ & = -\left\{ \hat{d}_i (\delta \bar{b} \|Px\|)^{\frac{1}{2}} - \delta^{-\frac{1}{2}} \right\}^2 + \frac{1}{\delta} \\ & 2\beta_i \tilde{d}_i \hat{d}_i = 2\beta_i \tilde{d}_i d_i^* - 2\beta_i \tilde{d}_i^2 \\ & = -\beta_i \tilde{d}_i^2 - \beta_i (\tilde{d}_i^2 - 2\tilde{d}_i d_i^* + d_i^{*2}) + \beta_i d_i^{*2} \\ & \leq -\beta_i \tilde{d}_i^2 + \beta_i d_i^{*2} \end{aligned}$$

hold, it yields that

$$\begin{aligned} V_2 \leq & \sum_{i=1}^r \alpha_i \left\{ \frac{1}{\delta} - \delta \hat{d}_i^2 \bar{b} \|Px\| - \beta_i \tilde{d}_i^2 + \beta_i d_i^{*2} \right. \\ & \left. + 2\delta \hat{d}_i^2 \Delta b \|Px\| + \delta^2 \hat{d}_i^4 \lambda_{max} \bar{b} \|Px\| \right\} \\ & = \sum_{i=1}^r \alpha_i \left\{ \frac{1}{\delta} - \beta_i \tilde{d}_i^2 + \beta_i d_i^{*2} \right. \\ & \left. - \delta \hat{d}_i^2 \|Px\| (\bar{b} - 2\Delta b - \delta \hat{d}_i^2 \lambda_{max} \bar{b}) \right\}. \end{aligned} \quad (30)$$

Now, let us pay attention to the last term in the left side of the above inequality. From (16) and (28), it is clear that $\delta \hat{d}_i^2 \|Px\| (\bar{b} - 2\Delta b - \delta \hat{d}_i^2 \lambda_{max} \bar{b}) \geq 0$. Thus,

$$\begin{aligned} V_2 \leq & \sum_{i=1}^r \alpha_i \left(-\beta_i \tilde{d}_i^2 + \frac{1}{\delta} + \beta_i d_i^{*2} \right) \\ & = -\sum_{i=1}^r \alpha_i \beta_i \tilde{d}_i^2 + \varepsilon, \end{aligned} \quad (31)$$

where

$$\varepsilon = \sum_{i=1}^r \alpha_i \left(\frac{1}{\delta} + \beta_i d_i^{*2} \right). \quad (32)$$

Substituting (25) and (31) in (21), it follows that

$$\begin{aligned} \dot{V} \leq & -x^T \mathcal{S} x - \sum_{i=1}^r \alpha_i \beta_i \tilde{d}_i^2 + \varepsilon \\ \leq & -\lambda_{\min}(\mathcal{S}) \lambda_{\max}^{-1}(P) x^T P x - \sum_{i=1}^r \alpha_i \beta_i \gamma_i \gamma_i^{-1} \tilde{d}_i^2 + \varepsilon. \end{aligned} \quad (33)$$

Let

$$\eta_1 = \lambda_{\min}(\mathcal{S}) \lambda_{\max}^{-1}(P) \quad (34)$$

$$\eta_2 = \min\{\alpha_1 \beta_1 \gamma_1, \alpha_2 \beta_2 \gamma_2, \dots, \alpha_r \beta_r \gamma_r\}. \quad (35)$$

Then,

$$\begin{aligned} \dot{V} \leq & -\eta_1 x^T P x - \sum_{i=1}^r \eta_2 \gamma_i^{-1} \tilde{d}_i^2 + \varepsilon \\ \leq & -\eta \left(x^T P x + \sum_{i=1}^r \gamma_i^{-1} \tilde{d}_i^2 \right) + \varepsilon \\ = & -\eta V + \varepsilon, \end{aligned} \quad (36)$$

where

$$\eta = \min\{\eta_1, \eta_2\}. \quad (37)$$

From (36) we have

$$V \leq \frac{\varepsilon}{\eta} + \left(V(0) - \frac{\varepsilon}{\eta} \right) e^{-\eta t}, \quad (38)$$

where $V(0) = V(z(t))|_{t=0}$. Since that $\eta > 0$, the second block on the right side of (38) tends to 0 when $t \rightarrow \infty$. Therefore, we can get that there exists $\mathcal{T}$ such that for $t \geq \mathcal{T}$ it satisfies

$$\|x\| \leq \tau \quad (39)$$

where

$$\tau = \frac{1}{\lambda_{\min}(P)} \left(\frac{\varepsilon}{\eta} \right)^{\frac{1}{2}}. \quad (40)$$

This completes the proof.

The stability conditions presented in Theorem 1 can be used to obtain an LMI-based design for the feedback gain K_i in (8). If we define $Q = P^{-1}$, the following theorem comes.

Theorem 2 For a given fuzzy model (2) with uncertainties and disturbance subject to the matching conditions (3)-(7), if there exist certain symmetric matrices $Q > 0$, $T_{1i} > 0$, $T_{2i} > 0$ ($i=1, 2, \dots, r$), and matrices G_j ($j=1, 2, \dots, r$) such that the following LMIs:

$$\begin{bmatrix} \left(\begin{array}{c} A_i Q + Q A_i^T + B_i G_j + G_j^T B_i^T \\ + M_i T_{1i} M_i^T + M_i T_{2i} M_i^T \end{array} \right) & (N_{1i} Q + N_{2i} G_j)^T \\ N_{1i} Q + N_{2i} G_j & -T_{1i} \end{bmatrix} < 0 \quad (41)$$

hold, then the feedback control gain K_i in (8):

$$K_i = \mathcal{K}_i + \Phi_i, \quad (42)$$

where

$$\mathcal{K}_j = G_j Q^{-1} \quad (43)$$

$$\Phi_i = -\delta \hat{d}_i^2 B_i^T Q^{-1} \quad (44)$$

$$\dot{\hat{d}}_i(t) = -\alpha_i \beta_i \gamma_i \hat{d}_i(t) + \alpha_i \gamma_i (\bar{b} \|Q^{-1} x\|)^{1/2} \quad (45)$$

$$\delta \leq \frac{2b-\bar{b}}{\hat{d}_{i\max}^2 \lambda_{\max} \bar{b}} \quad (46)$$

$\lambda_{\max} = \{\lambda(N_{2i}^T T_{2i}^{-1} N_{2i})\}_{i=1}^r$ which is the maximum eigenvalue (scalar) of square matrices $N_{2i}^T T_{2i}^{-1} N_{2i}$ ($i=1, 2, \dots, r$), and $\beta_i > 0$, $\gamma_i > 0$ are design parameters, will make the solutions of the closed-loop system (10) uniformly bounded while maintaining all signals involved stable. At the same time, the solutions exponentially converge at rate $\frac{\eta}{2}$ to a ball centered at the equilibrium point with radius τ , where η and τ are provided as follows.

$$\eta = \min\{\lambda_{\min}(\mathcal{S})\lambda_{\max}^{-1}(Q^{-1}), \beta_1\gamma_1, \beta_2\gamma_2, \dots, \beta_r\gamma_r\} \quad (47)$$

$$\begin{aligned} \mathcal{S} = & -\sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \{A_i Q + Q A_i^T + B_i G_j + G_j^T B_i^T \\ & + M_i T_{1i} M_i^T + M_i T_{2i} M_i^T \\ & + (N_{1i} Q + N_{2i} G_j)^T T_{1i}^{-1} (N_{1i} Q + N_{2i} G_j)\} \end{aligned} \quad (48)$$

$$\tau = \frac{1}{\lambda_{\min}(Q^{-1})} \left(\frac{\varepsilon}{\eta} \right)^{\frac{1}{2}} \quad (49)$$

$$\varepsilon = \sum_{i=1}^r \alpha_i \left(\frac{1}{\delta} + \beta_i d_i^{*2} \right). \quad (50)$$

3.2 Relaxing Conservatism

In theorems 1 and 2, it is desirable to have the largest η and the smallest τ so that the system state x can shrink most quickly in a ball centered at the equilibrium point with radius τ . Therefore, the results obtained above are basically conservative since they took no account of the matter.

On the other hand, to our best knowledge, there is not an analytical approach to solve the optimization problem with respect to η and τ . Therefore, the only way left to us is that we do our best to increase η and reduce τ . From (34), (35) and (37), we observe that a bigger β_i , γ_i , $\mathcal{S}$, and a smaller P can help make a larger η . At the same time, from (32) and (40) we

know that a bigger β_i leads to a bigger ε , further to a bigger τ as well as a smaller P . Therefore, in order to make η big, we can do something on γ_i , and $\mathcal{S}$ as follows.

- Choosing a bigger γ_i . In the adaptive law (15), since γ_i is a free design parameter, basically it can be chosen arbitrarily large. However, there is a limit that a large γ_i can lead to a big η_1 , further to a big η if $\eta_1 < \eta_2$ in (37).
- Making $-\mathcal{S}_{ij}$ minimum. It is clear that the smallest $-\mathcal{S}_{ij}$ is equivalent to the largest $\mathcal{S}_{ij}$, further to the largest $\mathcal{S}$.

Next, let us consider the problem of making τ small. As mentioned previously, since making a bigger P is in conflict with increasing η , therefore, how to reduce ε is a key point except for increasing η in (40). From (32), it follows that a large δ and small β_i lead to a small ε . However, choosing a small β_i is also in conflict with making η_2 big, therefore, the only way left is trying to make δ as large as possible. From (46), we arrive at the following point.

- Making T_{2i}^{-1} minimum. It is equivalent to making $\lambda_{\max}$ minimum, which leads to a smaller λ .

Summarizing above, in order to increase η and reduce τ , making both $-\mathcal{S}_{ij}$ and T_{2i}^{-1} minimum are necessary, which can be transformed in

$$-\mathcal{S}_{ij} + I_M T_{2i}^{-1} I_M^T \rightarrow \min, \quad (51)$$

where $I_M = \text{block diag}(I, 0)$ is a block triangular matrix with compatible dimensions. Finally, by the Schur complement, we formulate the requirement of (51) as the following generalized eigenvalue minimization under LMI constraints:

$$\begin{aligned} & \min \lambda \\ \text{s.t.} \quad & \begin{bmatrix} -\mathcal{S}_{ij} & I_M \\ I_M^T & -T_{2i} \end{bmatrix} < \lambda \mathcal{M} \end{aligned} \quad (52)$$

where $\mathcal{M} > 0$ is an arbitrarily given matrix with compatible dimension. Therefore, the following theorem yields in consideration of relaxing conservatism in Theorem 2.

Theorem 3 For a given fuzzy model (2) with uncertainties and disturbance subject to the matching conditions (3)–(7), if there exist certain symmetric matrix $Q > 0$, $T_{1i} > 0$,

$$\begin{bmatrix} (A_1 Q + B_1 G_1) + (A_1 Q + B_1 G_1)^T + M_1(T_{11} + T_{21})M_1^T & (N_{11}Q + N_{21}G_1)^T & I_M \\ N_{11}Q + N_{21}G_1 & -T_{11} & 0 \\ I_M^T & 0 & -T_{21} \end{bmatrix} < \lambda M \quad (53)$$

$T_{2i} > 0$ ($i = 1, 2, \dots, r$), and matrices G_j ($j = 1, 2, \dots, r$) such that the following LMIs:

$$\begin{aligned} \min \lambda \\ \text{s.t.} \quad \begin{cases} \text{LMI (41)} \\ \text{LMI (53)} \end{cases} \end{aligned} \quad (54)$$

hold, then the feedback control gain K_i obtained in (42)-(46) will make the solutions of the closed-loop system (10) uniformly bounded while maintaining all signals involved stable. Also, the solutions exponentially converge at a bigger rate $\frac{\lambda}{2}$ to a ball centered at the equilibrium point with a smaller radius τ .

4. Simulation

Consider the problem of balancing and swing-up of an inverted pendulum on a cart. The equations of motion for the pendulum are

$$\dot{x}_1 = x_2 \quad (55)$$

$$\dot{x}_2 = \frac{g \sin(x_1) - \frac{amlx_2^2 \sin(2x_1)}{2} - \mu a \cos(x_1)u}{\frac{4l}{3} - aml \cos^2(x_1)} + \psi(t), \quad (56)$$

where x_1 denotes the angle of the pendulum from the vertical; x_2 , the angular velocity; $g = 9.8 \text{ m/s}^2$, the gravity constant; m , the mass of the pendulum; M , the mass of the cart (in kilo-Newton); $\psi(t)$, the external disturbance that is assumed to be a square wave with amplitude ± 5.0 and period 2π ; $a = \frac{1}{m+M}$ and $\mu = 1000$. We choose $m = 2.0 \text{ kg}$, $M = 8.0 \text{ kg}$, $l = 0.5 \text{ m}$ in this simulation.

The control objective is to balance the inverted pendulum. Here, we follow [6] to approximate the system by the following two-rule fuzzy model:

Rule 1 : IF x_1 is $\tilde{0}$ THEN

$$\dot{x} = (A_1 + \Delta A_1)x + (B_1 + \Delta B_1)u + D_1$$

Rule 2 : IF x_1 is $\pm \pi/2$ THEN

$$\dot{x} = (A_2 + \Delta A_2)x + (B_2 + \Delta B_2)u + D_2,$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{\frac{4l}{3} - aml\beta^2} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \frac{-\mu a}{\frac{4l}{3} - aml} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ \frac{g}{\pi(\frac{4l}{3} - aml\beta^2)} & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{-\mu a}{\frac{4l}{3} - aml\beta^2} \end{bmatrix},$$

$\beta = \cos(86^\circ)$; ΔA_i , ΔB_i ($i = 1, 2$) are uncertainties that describe modeling errors; D_i , the counterpart of the external disturbance ψ in (56). The membership functions for fuzzy sets $\tilde{0}$, and $\pi/2$ are chosen as follows.

$$\mu_{\tilde{0}}(x_1) = \begin{cases} \cos(x_1), & -\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\pi/2}(x_1) = \begin{cases} 1 - \cos(x_1), & -\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

which are shown in Fig.1. Since the fact that $\|B_1\| = 176.4706$ and $\|B_2\| = 150.1096$, which implies that $\bar{b} = 176.4706$ and $\underline{b} = 150.1096$, it is clear that the fuzzy model satisfies the assumption, $\bar{b} \leq 2\underline{b}$, shown in (7). In what follows, we consider the modeling errors ΔA_i and ΔB_i , which are assumed to satisfy the matching condition as shown in (3) and (4):

Fig.1 Membership functions of $\mu_{\tilde{0}}(x_1)$ and $\mu_{\pi/2}(x_1)$

$$\Delta A_1 = M_1 \Theta N_{11}, \quad \Delta B_1 = M_1 \Theta N_{12},$$

$$\Delta A_2 = M_2 \Theta N_{21}, \quad \Delta B_2 = M_2 \Theta N_{22},$$

where $\Theta = \text{diag}(\theta_1, \theta_2, \theta_3)$ with $|\theta_i| \leq 1$, and

$$M_1^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, N_{11} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{12} \\ 0 & 0 \end{bmatrix}, N_{12} = \begin{bmatrix} 0 \\ 0 \\ b_1 \end{bmatrix}$$

$$M_2^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, N_{21} = \begin{bmatrix} a_{21} & 0 \\ 0 & a_{22} \\ 0 & 0 \end{bmatrix}, N_{22} = \begin{bmatrix} 0 \\ 0 \\ b_2 \end{bmatrix},$$

in which parameter a_{11} , a_{12} , a_{21} , a_{22} , b_1 , and b_2 are to be determined. The resulting fuzzy system is given by

$$\begin{aligned} \dot{x} = & (\alpha_1(A_1 + \Delta A_1) + \alpha_2(A_2 + \Delta A_2))x \\ & + (\alpha_1(B_1 + \Delta B_1) + \alpha_2(B_2 + \Delta B_2))u \\ & + \alpha_1 D_1 + \alpha_2 D_2, \end{aligned} \quad (57)$$

which is equivalent to

$$\begin{aligned} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & \left(\frac{g}{\frac{4l}{3} - aml} \alpha_1 + \frac{2g}{\pi(\frac{4l}{3} - aml\beta^2)} \alpha_2 \right) x_1 \\ & + \left(\frac{-\mu a}{\frac{4l}{3} - aml} \alpha_1 + \frac{-\mu a}{\frac{4l}{3} - aml\beta^2} \alpha_2 \right) u \\ & + (\alpha_1 a_{11} + \alpha_2 a_{21}) \theta_1 x_1 + (\alpha_1 a_{12} + \alpha_2 a_{22}) \theta_2 x_2 \\ & + (\alpha_1 b_1 + \alpha_2 b_2) \theta_3 u + \alpha_1 d_1 + \alpha_2 d_2. \end{aligned} \quad (58)$$

Leaving ΔA_i , ΔB_i , and d_i out of account, the difference between the resulting fuzzy system (58) and the actual pendulum model (56) is

$$\Delta f = f_{x_1} x_1 + f_{x_2} x_2 + f_u u,$$

where

$$\begin{aligned} f_{x_1} = & \frac{g \sin(x_1)}{(\frac{4l}{3} - aml \cos^2(x_1)) x_1} \\ & - \left(\frac{g}{\frac{4l}{3} - aml} \alpha_1 + \frac{2g}{\pi(\frac{4l}{3} - aml\beta^2)} \alpha_2 \right) \end{aligned}$$

$$\begin{aligned} f_{x_2} = & \frac{-\frac{amlx_2 \sin(2x_1)}{2}}{\frac{4l}{3} - aml \cos^2(x_1)} \\ f_u = & \frac{-\mu a \cos(x_1)}{\frac{4l}{3} - aml \cos^2(x_1)} \\ & + \frac{\mu a}{\frac{4l}{3} - aml} \alpha_1 + \frac{\mu a}{\frac{4l}{3} - aml\beta^2} \alpha_2. \end{aligned}$$

Here, we constrain $x_2 \leq 10$. If we set $a_{11} = a_{21} = a_1$, $a_{12} = a_{22} = a_2$ and $b_1 = b_2 = b$, then they can be obtained as follows.

$$a_1 = \max_{x_1 \in [-\pi/2, \pi/2]} |f_{x_1}| = 7.9358$$

$$a_2 = \max_{x_1 \in [-\pi/2, \pi/2], x_2 \in [-10, 10]} |f_{x_2}| = 0.8124$$

$$b = \max_{x_1 \in [-\pi/2, \pi/2]} |f_u| = 176.4706.$$

With the above parameters for M_i , N_{i1} and N_{i2} , we are ready for solving LMIs (54), or (41). Here, we use the software package Robust Control Toolbox (MATLAB) to solve (54), and obtain the following results:

$$\begin{aligned} \mathcal{K}_1 = & [3.6236 \ 0.8430], \quad \mathcal{K}_2 = [3.6236 \ 0.8430], \\ Q^{-1} = & \begin{bmatrix} 0.6094 & 0.1365 \\ 0.1365 & 0.0332 \end{bmatrix} \times 10^{-3}, \quad \lambda_{\min} = 0.0150, \end{aligned}$$

where $\mathcal{M}$ is chosen as an 8×8 identity matrix. On the other hand, in order to obtain the variable gain Φ_i in (44), it is necessary to solve the adaptive law in (45), the related parameters are chosen as $\beta_1 = \beta_2 = 0.1$, $\gamma_1 = \gamma_2 = 2.0$, $\delta = \frac{2b-b}{d_i^2 \lambda_{\max} b} - c$ with $c = 0.5$, where c is automatically replaced by $c/10$ if $\delta < 0$. In this simulation, the initial state is set: $x_{10} = 87^\circ$, $x_{20} = 0$ rad/s.

To make a comparison between the proposed fuzzy controller and the regular one when using T-S fuzzy model, the inverted pendulum first is controlled by $\mathcal{K}_i$, which means the fuzzy controller K_i is only composed of $\mathcal{K}_i$ and without Φ_i . Fig.2 shows the control results, where the solid line, and dashed lines are associated with x_1 , and x_2 , respectively. The oscillations around time $3.14 = \pi$ and $6.28 = 2\pi$ are caused by the 2π -periodic square wave disturbance. Clearly, the pendulum is not be controlled at all without Φ_k since x_1 exceeded $\pi/2$, which is the upper limit in consideration of the actual pendulum. By contrast, the control performance conducted by $K_i = \mathcal{K}_i + \Phi_i$ depicted in Figs. 3-4, where Fig.3 shows the responses for the state;

Fig.4, control input. This simulation shows that the proposed fuzzy controller is able to balance the pendulum with external disturbance in range of $[-87^\circ, 87^\circ]$.

Fig.2 State x_1 and x_2 controlled by $K_i = K_c$

Fig.3 State x_1 and x_2 controlled by $K_i = K_c + \Phi_i$

Fig.4 Control input composed of $K_i = K_c + \Phi_i$

5. Conclusions

In this paper, in order to effectively deal with modeling error between the real system to be controlled and its T-S fuzzy model, an adaptive controller was proposed. Compared with the regular controller when using T-S fuzzy model, the proposed controller puts a variable part updated by an adaptive law besides the fixed one obtained by solving certain LMIs. The controller were applied to an inverted pendulum, and the simulation results confirmed the effectiveness.

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