Formal Linearization by Chebyshev Interpolation for Both State and Measurement Equations of Nonlinear Scalar-Measurement Systems and Its Application to Nonlinear Filter

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Abstract  This paper is concerned with a formal linearization based on Chebyshev interpolation for nonlinear dynamic and scalar-measurement systems with Gaussian white noise and its application to a filter design. A linearization function that consists of the Chebyshev polynomials up to the higher order is defined, and a given nonlinear dynamic system is transformed into an augmented linear one with respect to this linearization function by applying Chebyshev interpolation. Furthermore, an augmented measurement vector that consists of polynomials of measurement data is also defined and a measurement equation is transformed into an augmented linear one with respect to the linearization function in the same way. To these augmented linearized systems, a linear estimation theory is applied to design a new nonlinear filter. With this method, the formal linearization is easily incorporated into many practical systems by simple calculation using a computer, and a nonlinear filter with higher accuracy than those using conventional methods can be designed.

Keywords: nonlinear system, formal linearization, nonlinear filter, Chebyshev interpolation, linearizing function

1. Introduction

One of the solutions for treating estimation problems for nonlinear systems that are defined by nonlinear differential equations with measurement noise is based on the linearized filter, and there are several ways of accomplishing the linearization [2]-[7]. The extended Kalman filter [1] is one of the standard methods, but it is usually not useful for systems with higher nonlinearity.

The purpose of this work is to design a nonlinear filter via a formal linearization for systems with high nonlinearity. Therefore, the paper is focused on designing a nonlinear filter using a formal linearization based on Chebyshev interpolation, introducing a new augmented state and measurement vectors in order to provide good performance. In the previous work [7], a nonlinear filter prepared by a formal linearization based on Chebyshev interpolation was presented. This filter design used the same order as the measurement vector when the given measurement equation was linearized. In this paper, a new technique [8] involving an augmented measurement vector is introduced in order to improve the performance of the resulting filter. The advantages of this method are that coefficients of the linearized system are simply obtained by summation because of the orthogonality for a finite sum, and inversion is carried out by simple calculation.

Experimental results show that the performance of this method is superior to that of the extended Kalman filter and the previous method [7].

2. Statement of Problem

In the estimation problem, it is assumed that the nonlinear dynamic and noisy measurement equations are described by

\[ \Sigma_1 : \dot{x}(t) = f(x(t)) , \quad x(0) = x_0 \in \mathcal{D} \quad (1) \]

\[ \eta(t) = h(x(t)) + v(t) \quad (2) \]

where \( t \) denotes time, \( \dot{} = d/dt \), \( x \) is an \( n \times 1 \) state vector, \( f \in \mathbb{R}^n \) is a nonlinear vector-valued function
and is continuously differentiable, $\eta$ is a scalar measurement, $h \in \mathbb{R}$ is a nonlinear function and continuously differentiable, $v(t)$ is white Gaussian noise of $\mathcal{N}(0, V)$, $V$ is a variance, and $T$ denotes a transpose. The problem is to determine the state of a nonlinear dynamic system from the given measurement data $\eta$.

3. Nonlinear Filter Obtained by Formal Linearization

3.1 Formal linearization for dynamic system

With reference to the previous work [7], the given nonlinear system (Eq. (1)) is linearized by Chebyshev interpolation. The state vector $x$ is changed into $y$ so that $y$ has the basic domain of the Chebyshev polynomials $\mathcal{D}_0 = \bigcup_{i=1}^{n} [-1, 1]$ and $y$ is rewritten as

$$y = P^{-1}(x - L) \in \mathcal{D}_0$$

where

$$L = \begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix}, P = \begin{pmatrix} p_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The given dynamic system (Eq. (1)) becomes

$$\dot{y}(t) = P^{-1}(f(Py(t) + L))$$

The Chebyshev polynomials $\{T_r(\cdot)\}$ are defined as

$$T_r(y_i) = \cos(r \cdot \cos^{-1} y_i), \quad (r = 0, 1, 2, \cdots)$$

or

$$T_0(y_i) = 1, \quad T_1(y_i) = y_i, \quad T_2(y_i) = 2y_i^2 - 1,$$

$$T_3(y_i) = 4y_i^3 - 3y_i, \quad T_4(y_i) = 8y_i^4 - 8y_i^2 + 1, \cdots$$

The recurrence formula is

$$T_{q+1}(y_i) = 2y_iT_q(y_i) - T_{q-1}(y_i), \quad (q \geq 1)$$

$$T_0(y_i) = 1, \quad T_1(y_i) = y_i$$

Therefore, the derivative of the Chebyshev polynomials

$$S_q(y_i) = \frac{dT_q(y_i)}{dy_i}$$

is given by

$$S_{q+1}(y_i) = 2T_q(y_i) + 2y_iS_q(y_i) - S_{q-1}(y_i), \quad (q \geq 1)$$

$$S_0(y_i) = 0, \quad S_1(y_i) = 1$$

Using these Chebyshev polynomials, an $N$th order linearization function $\phi(\cdot) = \phi(y(\cdot))$, which consists of the Chebyshev polynomials, is defined as

$$\phi = [\phi_1, \phi_2, \cdots, \phi_1, \cdots, \phi_{(N+1)^n-1}]^T$$

$$= [T_{(10\cdots0)}(y), T_{(01\cdots0)}(y), \cdots, T_{(0\cdots0)}(y), T_{(11\cdots0)}(y), T_{(10\cdots0)}(y), \cdots, T_{(10\cdots1)}(y), T_{(20\cdots0)}(y), T_{(02\cdots0)}(y), \cdots, T_{(r_1\cdots r_n)}(y), \cdots, T_{(N\cdots N)}(y)]^T$$

where

$$T_{(r_1\cdots r_n)}(y) = \prod_{i=1}^{n} T_{r_i}(y_i)$$

The derivative of each element of $\phi$, along with the solution of the given nonlinear system (Eq. (1)), becomes

$$\dot{\phi}(y) = \dot{T}_{(r_1\cdots r_n)}(y) = \frac{\partial T_{(r_1\cdots r_n)}(y)}{\partial y^T} \dot{y}$$

$$= [S_{r_1}(y_1)T_{r_2}(y_2) \cdots T_{r_{n-1}}(y_{n-1})T_{r_n}(y_n), \cdots, T_{r_1}(y_1)T_{r_2}(y_2) \cdots T_{r_{n-1}}(y_{n-1})S_{r_n}(y_n)] \dot{T}(Py + L)$$

$$\equiv G_{(r_1\cdots r_n)}(y), \quad \alpha = \alpha(r_1, \cdots, r_n)$$

Applying Chebyshev interpolation up to the $N$th order, this $G_{(r_1\cdots r_n)}(y)$ is approximated by

$$\hat{G}_{(r_1\cdots r_n)}(y) = \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C_{(q_1\cdots q_n)}^{(r_1\cdots r_n)} T_{(q_1\cdots q_n)}(y)$$

where

$$C_{(q_1\cdots q_n)}^{(r_1\cdots r_n)} = \frac{2^{n-\gamma}}{n} \sum_{j_1=0}^{N} \cdots \sum_{j_n=0}^{N} \prod_{i=1}^{n} (N + 1)$$

$$\sum_{j_1=0}^{N} \cdots \sum_{j_n=0}^{N} G_{(r_1\cdots r_n)}(y_{j_1}, y_{j_2}, \cdots, y_{j_n})$$

$$T_{q_1}(y_{j_1})T_{q_2}(y_{j_2}) \cdots T_{q_n}(y_{j_n})$$

$$\gamma = \{\text{the number of } q_i = 0 : 1 \leq i \leq n\}$$

The interpolating points $\{y_{j_i}\}$ are set to be

$$y_{j_i} = \cos\left(\frac{2j_i + 1}{2N + 2}\pi\right), \quad (i = 1, \cdots, n, \quad j_i = 0, \cdots, N)$$

Substituting this $\hat{G}_{(r_1\cdots r_n)}(y)$ into Eq. (11) yields

$$\dot{\phi}(y) \approx A \phi(y) + b$$

where

$$[A_\alpha \beta] = [C_{(1\cdots 1)}^{(r_1\cdots r_n)}] \in R^{((N+1)^n-1) \times ((N+1)^n-1)}$$

$$[b_\alpha] = [C_{(0\cdots 0)}^{(r_1\cdots r_n)}] \in R^{((N+1)^n-1)}$$
Thus, a formal linear state differential equation is derived as
\[ \Sigma_2 : \dot{z}(t) = Az(t) + b \] \[ z(0) = \phi(y(0)) = \phi(P^{-1}(x(0) - L)) \] (16)
From Eqs. (3) and (9), the inversion is carried out as
\[ \dot{x}(t) = P[I 0 \cdots 0]z(t) + L \] (17)
where \( I \) is an \( n \times n \) unit matrix.

3.2 Formal linearization for measurement equation

In this paper, we consider the polynomial \( \eta \) of Eq. (2) as follows. For simplicity, noises are assumed to be produced from noise generators,
\[ N_{g_j} = \{ v^0_j : v^0_j \in N(0, \sigma^2) \} \quad (j = 1, 2, \ldots, M) \] (18)
where \( v^0_j \) is white Gaussian noise of variance
\[ \sigma^2 = \frac{1}{\| h \|^2} \] (19)
and is independent of the state \( x \) and other noises, \( v^0_j \perp v^0_k \perp x_i \quad (j \neq k) \), and \( M \leq N \). Here, \( \| \cdot \| \) is a norm such that
\[ \| z \| = \sup_{x \in D} \sqrt{z^T(x)z(x)} \] (20)
for any vector \( z \), and
\[ \| z \| = \sup_{x \in D} | z(x) | \] (21)
for a scalar \( z \).

We define the functions
\[ g_j(x) = \frac{h^j(x)}{\| h^j \|} \quad (j = 1, 2, \ldots, M) \] (22)
which satisfy
\[ | g_j(x) | \leq 1 \quad \text{for all} \quad x \in D \] (23)
and \( g_j \in C^\infty \) if \( h \in C^\infty \). Similarly, we also define the \( j \)th polynomial function of \( \eta \) as
\[ p_j(\eta) = \frac{\eta^j}{\| h^j \|} \quad (j = 1, 2, \ldots, M) \] (24)
by considering \( \eta \approx h(x) \) in the case of \( v \approx 0 \), because \( \eta = h(x) + v \). Let us assume that the polynomial function \( p_j(\eta) \) can be described by the sum of the measurement function \( g_j(x) \) and noise \( v^0_j \). Therefore, we have the following polynomial equations of \( \eta \) by setting
\[ v_j = \| h^j \| v^0_j \quad (j = 1, 2, \ldots, M) \] (25)
When \( j = 1 \), we have \( p_1(\eta) = g_1(x) + v^0_1 \) or \( \frac{\eta}{\| h \|} = \frac{h(x)}{\| h \|} + v^0_1 \), thus
\[ \eta = h(x) + \| h \| v^0_1 = h(x) + v_1 = h(x) + v \] (26)
In this case, \( v_1 = v \).

When \( j = 2 \), we have \( p_2(\eta) = g_2(x) + v^0_2 \) or \( \frac{\eta^2}{\| h^2 \|} = \frac{h^2(x)}{\| h^2 \|} + v^0_2 \), thus
\[ \frac{\eta^2}{\| h^2 \|} = h^2(x) + \| h^2 \| v^0_2 = h^2(x) + v_2 \] (27)
When \( j = j \), we have \( p_j(\eta) = g_j(x) + v^0_j \) or \( \frac{\eta^j}{\| h^j \|} = \frac{h^j(x)}{\| h^j \|} + v^0_j \), thus
\[ \frac{\eta^j}{\| h^j \|} = h^j(x) + \| h^j \| v^0_j = h^j(x) + v_j \] (28)
When \( j = M \), we have \( p_M(\eta) = g_M(x) + v^0_M \) or \( \frac{\eta^M}{\| h^M \|} = \frac{h^M(x)}{\| h^M \|} + v^0_M \), thus
\[ \frac{\eta^M}{\| h^M \|} = h^M(x) + \| h^M \| v^0_M = h^M(x) + v_M \] (29)
Using these polynomials \( \{ \eta^j \} \), an augmented measurement vector \( Y \) and noise vector \( v \) are introduced as
\[ Y = \begin{pmatrix} \eta^1 \\ \eta^2 \\ \vdots \\ \eta^M \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{pmatrix} \] (30)
Thus,
\[ Y = \begin{pmatrix} \eta^1 \\ \eta^2 \\ \vdots \\ \eta^M \end{pmatrix} = \begin{pmatrix} h_1(x) + v_1 \\ h_2(x) + v_2 \\ \vdots \\ h_M(x) + v_M \end{pmatrix} = \begin{pmatrix} \eta^1 \\ \eta^2 \\ \vdots \\ \eta^M \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{pmatrix} \] (31)

We apply Chebyshev interpolation up to the \( N \)th order to each \( \{ h^j \} \). Thus, \( h^j(x) = h^j(Py + L) \) is approximated by
\[ h^j(Py + L) = \sum_{q_1=0}^{N} \cdots \sum_{q_N=0}^{N} C^{(j)}_{(q_1 \cdots q_N)} T_{(q_1 \cdots q_N)}(y) \] (32)
where
\[ C^{(j)}_{(q_1 \cdots q_N)} = \frac{2^{N-q'-1}}{n} \sum_{i=1}^{N} \prod_{j=1}^{N} (N + 1)_{j_1=0}^{j_2=0} \cdots \prod_{j_N=0}^{N} h^j(y_{1j_1} y_{2j_2} \cdots y_{nj_N}) \times T_{q_1}(y_{1j_1}) T_{q_2}(y_{2j_2}) \cdots T_{q_N}(y_{nj_N}) \] (33)
\[ \gamma' = \{ \text{the number of} \quad q_i = 0 : 1 \leq i \leq n \} \]
Substituting this \( h^j(y) \) into Eq. (31), the augmented measurement equation becomes
\[ Y \approx D\phi(y) + e + v \] (34)
where
\[ [D_{j \beta}] = [C^{(j)}_{(0,0,q_{j})}] \in \mathbb{R}^{M \times ((N+1)^n-1)} \]
\[ [e_j] = [C^{(j)}_{(0,0,0)}] \in \mathbb{R}^M, \beta' = \beta'(q_1, \ldots, q_n) \]

Thus a formal linear measurement equation is derived as
\[ Y(t) = Dz(t) + e + v(t) \quad (35) \]
This noise \( v \) has the following property. The expectation of \( v \) becomes
\[
E(v) = E \left( \begin{array}{c}
v_1 \\
v_j \\
\vdots \\
v_M \\
\end{array} \right) = E \left( \begin{array}{c}
h \| v_0^j \| \\
h \| v_0^j \| \\
\vdots \\
h \| v_0^j \| \\
\end{array} \right) = \left( \begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array} \right) \quad (36)
\]
The covariance is
\[
V = E\{[v - E(v)][v - E(v)]^T\} = E(vv^T) = E \left( \begin{array}{cccc}
v_1^2 & \cdots & v_1v_M \\
v_1v_M & \cdots & v_Mv_M \\
\vdots & \cdots & \vdots \\
v_Mv_M & \cdots & v_Mv_M \\
\end{array} \right)
\]
\[
= E \left( \begin{array}{cccc}
h \| v_0^j \|^2 & \cdots & h \| v_0^j \| v_0^k \\
h \| v_0^j \| v_0^k & \cdots & h \| v_0^j \|^2 \\
\vdots & \cdots & \vdots \\
ah \| v_0^j \|^2 & \cdots & h \| v_0^j \|^2 \\
\end{array} \right) = \left( \begin{array}{cccc}
h \| v_0^j \|^2 & \cdots & h \| v_0^j \| v_0^k \\
h \| v_0^j \| v_0^k & \cdots & h \| v_0^j \|^2 \\
\vdots & \cdots & \vdots \\
h \| v_0^j \|^2 & \cdots & h \| v_0^j \|^2 \\
\end{array} \right)
\]

3.3 Design of nonlinear filter

The Kalman filter method is applied to the above linearized system (Eqs. (16) and (35)), so that the filter becomes
\[ \dot{z}(t) = A\hat{z}(t) + b + K(t)(Y(t) - (D\hat{z}(t) + c)) \quad (38) \]
\[ \hat{z}(0) = \hat{z}_0 \]
where \( K(t) \) is the filter gain given by
\[ K(t) = \Sigma(t)^{-1}D^TR^{-1} \]
\( \Sigma(t) \) satisfies the matrix Riccati equation
\[ \dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A^T - \Sigma(t)D^TR^{-1}DS\Sigma(t) \]
\( \Sigma(0) \) is a positive definite matrix, and \( R \) satisfies \( R = \Sigma \) of Eq. (37). From Eq. (17), the estimate of the nonlinear filter is obtained by the inversion
\[ \hat{z}(t) = P[I \ 0 \ \cdots \ 0]z(t) + L \quad (39) \]

4. Numerical Experiments

Numerical experiments on nonlinear filters for cases of scalar and multidimensional systems are illustrated to explore the effectiveness of the proposed design. The extended Kalman filter [1] is also depicted as a conventional filter for comparison.

4.1 Nonlinear filter for scalar system

Consider the scalar system
\[ \dot{x} = -\sin x, \quad x(0) = 1, \quad D = [0,1] \subset \mathbb{R} \]
\[ \eta = \frac{1}{x + 1} + v \quad (40) \]

Fig. 1 Estimates \( \hat{z}(t) \) of the scalar system with various orders and measurement data.
Fig. 2 Integral square errors of estimation of the scalar system for various orders

Fig. 1 shows the true value \( x \) of Eq. (40), the approximated values \( \hat{x} \) of Eq. (39) when \( N \) is 3 and \( M \) is varied from 1 to 3, and \( \eta \). \( \hat{x}(E.K.) \) refers to a result obtained using the extended Kalman filter. The other values of the proposed filter (Eq. (38)) are set as \( \hat{x}(0) = 0 \), \( L = 0.5 \), \( P = 0.51 \), \( R(t) = \text{diag}(0.005, 0.005, 0.005) \) and \( \Sigma(0) = I \) for \( \hat{x}(N = 3, M = 3) \); \( R(t) = \text{diag}(0.005, 0.005) \) and \( \Sigma(0) = I \) for \( \hat{x}(N = 3, M = 2) \); \( R(t) = 0.005 \) and \( \Sigma(0) = I \) for \( \hat{x}(N = 3, M = 1) \); and \( R(t) = 0.005 \) and \( \Sigma(0) = 1 \) for \( \hat{x}(E.K.) \). \( \hat{x}(N = 3, M = 1) \) is a result taken from the previous work [7].

Fig. 2 shows the integral square errors of estimation
\[
J(t) = \int_0^t (x(\tau) - \hat{x}(\tau))^2 d\tau
\]  
(42)
for various orders and the conventional method \( (E.K.) \).

4.2 Nonlinear filter for multidimensional system

As a multidimensional system, we consider a pendulum in which the bob is connected to a rod of zero mass. Let \( \theta \) denote the angle subtended by the rod and the vertical axis through the pivot point. This system is written as
\[
\frac{d^2}{dt^2} \theta + a_1 \frac{d}{dt} \theta + a_2 \sin(\theta) = 0
\]
(43)
We assume that the position of the bob is measured from above and the measurement equation is
\[
\eta = a_3 \sin(\theta) + \nu
\]
(44)
The system parameters are set as \( a_1 = 0.5, a_2 = \frac{980.7}{400} \), and \( a_3 = 1 \), the state variables are \( x_1 = \theta \) and \( x_2 = \dot{\theta} \), and the other values are \( L = \begin{pmatrix} 0.4 \\ -0.4 \end{pmatrix}, P = \begin{pmatrix} 1.3 & 0 \\ 0 & 1.5 \end{pmatrix}, x(0) = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix}, \) and \( \hat{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

Fig. 3 Estimates \( \hat{x}_1 \) of the multidimensional system for various orders

The values of the proposed filter (Eq. (38)) are set as \( R(t) = \text{diag}(0.05, 0.05, 0.05) \) and \( \Sigma(0) = I \) for \( \hat{x}(N = 3, M = 3) \); \( R(t) = \text{diag}(0.05, 0.05) \) and \( \Sigma(0) = I \) for \( \hat{x}(N = 3, M = 2) \); \( R(t) = 0.05 \) and \( \Sigma(0) = I \) for

Fig. 4 Estimates \( \hat{x}_2 \) of the multidimensional system with various orders and measurement data

Fig. 5 Integral square errors of estimation of the multidimensional system for various orders
\hat{x}(N = 3, M = 1); and \( R(t) = 0.05 \) and \( \Sigma(0) = 10 \) for \( \hat{x}(E.K.) \).

Figs. 3 and 4 show the true value \( x_i(t) \) of Eq. (43) along with the approximated values \( \hat{x}_i(t) \) for \( i = 1 \) and 2, respectively, when \( N \) is 3 and \( M \) is varied from 1 to 3, and \( \eta \). \( \hat{x}(E.K.) \) refers to the result obtained using the extended Kalman filter.

Fig. 5 shows the integral square errors of estimation
\[
J(t) = \int_0^t (x(\tau) - \hat{x}(\tau))^T (x(\tau) - \hat{x}(\tau)) d\tau
\]
(45)
for the various orders. \( J(t)(E.K.) \) is the error for the extended Kalman filter, and \( J(t)(N = 3, M = 1) \) is that in the previous work [7].

5. Conclusions

We developed a filter design for a nonlinear system by a formal linearization method exploiting Chebyshev interpolation for both state and measurement equations. By this method, a nonlinear filter was synthesized easily using a computer.

Numerical experiments showed that our method is better than those in previous works, and the accuracy is improved as the order of the augmented measurement vector increases.

References


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