Extended Continuous-Time Image Reconstruction System for Binary and Continuous Tomography

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Abstract

For solving inverse problems in computed tomography (CT), we have presented a continuous-time image reconstruction (CIR) system described by a switched nonlinear dynamical system with a piecewise smooth vector field. Recently, we have also proposed an extended CIR (ECIR) system with a modification of the nonlinear vector field, for the purpose of solving inverse problems in binary tomography, which is the process of reconstructing a binary image from a finite number of projections. In this study, we show, by means of numerical experiments, that the ECIR system is effective not only for binary tomography, but also for continuous tomography, such that the maximum pixel value to be reconstructed is previously given. For projection data with insufficient views, we found that the ECIR system exhibited a better convergence performance than the maximum-likelihood expectation-maximization method.

1. Introduction

The most common algorithms for image reconstruction from projections in computed tomography (CT) are a procedure using filtered back-projection (FBP) [1], which is a transform method, and an iterative image reconstruction method [2], based on difference equations. The FBP procedure is widely used for reconstructing images because its computation speed is higher than that of software-based iterative methods. However, compared with the FBP method, iterative methods can produce high-quality reconstructed images with fewer artifacts even for a small number of projection data that have low signal-to-noise ratios.

On the other hand, for solving inverse problems in CT, we have presented a continuous-time image reconstruction (CIR) system [3, 4] described by a switched nonlinear dynamical system with a piecewise smooth vector field. We have proved theoretically, for a consistent case, the stability of an equilibrium corresponding to the desired image to be reconstructed, in the CIR system using a common Lyapunov function based on the Kullback–Leibler divergence [5]. Moreover, we have shown that the CIR system for an ill-posed inverse problem could reconstruct better-quality images compared with the FBP procedure through numerical experiments [6]. The system described by ordinary differential equations could be implemented in hardware as an analog electronic circuit [4, 7], i.e., it provided fast image reconstruction and completely parallel computing of reconstructing images. Recently, we have also proposed an extended CIR (ECIR) system with a modification of the nonlinear vector field, for the purpose of solving inverse problems in binary tomography [8], which is the process of reconstructing a binary image from a finite number of projections.

In this study, we show, by means of numerical experiments, that the ECIR system is effective not only for binary tomography, but also for continuous tomography, such that the maximum pixel value to be reconstructed is previously given. In particular, for projection data with insufficient views, we found that the ECIR system exhibited a better convergence performance than the maximum-likelihood expectation-maximization (ML-EM) method, which is widely known as having the best quality among conventional iterative reconstruction methods with respect to the convergence property.

2. System

Let \(x \in \tilde{\Omega} \subset R^J\) be an unknown variable for pixel values satisfying

\[
y = Ax
\]

(1)

where \(y \in R^I\) and \(A \in R^{I \times J}\) denote the projection value and a normalized projection operator corresponding to the Radon transform, respectively, and \(R_i\) indicates the set of non-negative real numbers. Since projection data are, in practice, noisy and incomplete, the inverse problem in CT becomes generally ill-posed, which means that its solution is not unique or does not exist. We consider continuous tomography such that the maximum pixel value to be reconstructed is previously given. By assuming that the maximum value is one, without loss of generality, the reconstruction problem is...
to find $x$ from an optimization minimizing an appropriate cost function $V(x)$ regarding the linear system in Eq. (1), with $x$ being the closure of the open hypercube $\Omega = (0, 1)^J$. We also treat binary tomography, which is a reconstruction problem of finding a binarization of the value of $x \in \Omega$.

Before defining the cost function for the inverse problem, we introduce the generalized Kullback–Leibler (KL) divergence of two non-negative vectors $\alpha$ and $\beta$:

$$\text{KL}(\alpha, \beta) = \sum_k \beta_k \log \frac{\beta_k}{\alpha_k} + \alpha_k - \beta_k$$

(2)

where $\alpha_k$ and $\beta_k$ denote the $k$th elements of $\alpha$ and $\beta$, respectively. The divergence $\text{KL}(\alpha, \beta)$ for the vectors $\alpha$ and $\beta$ of non-negative real numbers is non-negative with $\text{KL}(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.

To solve the inverse problem of binary and continuous tomography, we shall consider the minimization problem:

$$\min_{x(\cdot) \in \mathbb{R}^J_+} V(x(t)), \quad t \in \mathbb{R}$$

(3)

$$V(x) := \text{KL}(x, e) + \text{KL}(u - x, u - e)$$

where $e \in \mathbb{R}^J_+$ is a locally unique solution corresponding to Eq. (1) and $u$ denotes the all-ones vector $(1, 1, \ldots, 1)^T$ of length $J$. For the purpose of obtaining a time evolution $x(t)$ that converges to a local minimum of the function $V(x(t))$, we previously proposed [8] a switched nonlinear system consisting of the family of subsystems

$$\frac{dx}{dt} = -X(U - X)\Lambda^\top \Lambda_{\mathrm{m}} (A_{\mathrm{m}} x - y_{\mathrm{m}}),$$

(4)

$$t - k \tau \in [t_{m-1}, t_m), \quad t \in \mathbb{R}_+, \quad x(0) = x^0$$

for a series of times $0 = t_0 < t_1 < t_2 < \ldots < t_M = \tau$ and non-negative integer $k$, where $X := \text{diag}(x)$ indicates the diagonal matrix in which the diagonal entries starting in the upper left corner are the elements of $x$, and $U$ denotes the identity matrix, while $A_{\mathrm{m}} \in \mathbb{R}^{M \times J}_{\mathrm{m}}$ and $y_{\mathrm{m}} \in \mathbb{R}^{J}_{\mathrm{m}}$ are respectively a submatrix consisting of $I_m$ partial rows of $A$ and a subvector of $y$ with the same corresponding rows of $A_{\mathrm{m}}$, for $m = 1, 2, \ldots, M$, with $M$ denoting the total number of divisions. The system of Eq. (4) describes an extended system with a modified vector field in the CIR, and minimizes the value of Eq. (3) [8]. Moreover, if we set an initial value to $x^0 \in \Omega$, then the solution $x(t)$ is in $\Omega$ for all $t$ [8].

3. Experiments

In order to show the superior efficiency of the ECIR system compared with the ML-EM method, some numerical experiments using binary and continuous phantom images were performed. The simulated phantom $e$ is made of $87 \times 87$ pixels ($J = 7569$) of a binary image (continuous image), each of which has a value of either 0 or 1 (between 0 and 1), as shown in Fig. 1(a) (1(b)). Three kinds of reconstructions were performed using 128 detectors per projection and 180-degree scanning with sampling every 30, 20, and 10 degrees, which correspond to the numbers of projections $J = 768, 1152$, and $2304$, respectively.

To simplify the comparison, we dealt with an unblocked ECIR system derived from Eq. (4) with $M = 1$. On the other hand, the iterative step of the ML-EM method is defined by

$$z_j(n + 1) = z_j(n)s_j^{-1} \sum_{i=1}^M A_{ij}y_{i} (Ax(n))_i,$$

(5)

$$n = 0, 1, \ldots, \quad z_j(0) = x_0^0$$

for $j = 1, 2, \ldots, J$, where $s_j = \sum_{i=1}^M A_{ij}$, and $A_{ij}$ denotes an $(i,j)$ element of $A$. By choosing a positive initial value $x^0$, we have $z_j(n+1) > 0$ for $n = 0, 1, \ldots$, and $j = 1, 2, \ldots, J$.

To seek an iterative solution in $\Omega$, we modify the ML-EM method by replacing the $z_j(n + 1)$ given by Eq. (5) with a smaller value between one and $z_j(n + 1)$.

![Figure 1: Phantom images](image)

We are interested in the convergence property rather than the computation time of the continuous- and discrete-time minimization methods, because the execution time depends on the implementation technique of hardware and/or software. In order to compare the convergence properties of the continuous-time and discrete-time systems, we select the points of times with the same measure, which can be observed via the projection data. Using the solutions $x(t)$ to Eq. (4) and $z(n)$ to Eq. (5) emanating from the same initial value $x^0 \in \Omega$, the set of a pair of times is defined as

$$\Gamma := \{(n, t) \in \mathbb{Z}_+ \times \mathbb{R}_+ : K(z(n)) = K(x(t)), \quad K(z(k - 1)) > K(z(k)) \text{ for } k = 1, 2, \ldots, n, \quad K(x(\tau)) \text{ is monotonically decreasing for } \tau \in [0, t]\}$$

(6)

with $\mathbb{Z}_+$ denoting the set of positive integer numbers and $K(w) := \text{KL}(Aw, y)$. Now, let us consider the distances

$$D_2(z)(n), \ D_x(z)(n) := \left( \frac{||z(n) - e||_1}{||e||_1}, \frac{||x(t) - e||_1}{||e||_1} \right),$$

(7)

$$(n, t) \in \Gamma$$

164 Journal of Signal Processing, Vol. 17, No. 4, July 2013
In the case of binary tomography, we also use the Hamming distance between the binarized reconstructed image and the phantom image, defined as

$$H_v(n) = \sum_{j=1}^{J} |B(v_j(n)) - e_j|$$

where \(v\) is either \(x\) or \(z\), and \(B(a), a \in R_+\), is a binarization operator:

$$B(a) = \begin{cases} 1, & \text{if } a \geq 0.5 \\ 0, & \text{otherwise} \end{cases}$$

Figure 2 shows plots of the sequences of \(D_x(n)\) and \(D_z(n)\) along the solutions to Eqs. (4) and (5) with initial values \(x^0\), where the solid and dashed curves correspond to the ECIR system and the ML-EM method, respectively. The reconstructed images are shown in Fig. 3. Each image shows a snapshot at the right end of the abscissa in each graph of Fig. 2. We can visually confirm that the ECIR system reconstructs better quality images with fewer artifacts than the ML-EM method. Table 1 summarizes the results of distances for both binary and continuous tomography. Indeed, a comparison of the values shows that the quality of images made with the ECIR method is better than that of the images made with the ML-EM method at \((n, t) \in \Gamma\). In particular, \(H_x\) was zero for the image made with the ECIR method and all projection views we examined; i.e., the reconstructed binary image and the phantom image were exactly the same, despite an extremely small number of projection views, such as six.

Table 1: Comparison of (a) \(H_x\) and \(H_z\) for binarized reconstructed images and (b) \(D_x\) and \(D_z\) for the continuous images

(a) Binary tomography

<table>
<thead>
<tr>
<th>Projection views</th>
<th>6</th>
<th>9</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>80</td>
<td>86</td>
<td>109</td>
</tr>
<tr>
<td>ECIR: (H_x(n))</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>ML-EM: (H_z(n))</td>
<td>63</td>
<td>38</td>
<td>11</td>
</tr>
</tbody>
</table>

(b) Continuous tomography

<table>
<thead>
<tr>
<th>Projection views</th>
<th>6</th>
<th>9</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>64</td>
<td>77</td>
<td>101</td>
</tr>
<tr>
<td>ECIR: (D_x(n))</td>
<td>0.118</td>
<td>0.065</td>
<td>0.046</td>
</tr>
<tr>
<td>ML-EM: (D_z(n))</td>
<td>0.252</td>
<td>0.151</td>
<td>0.105</td>
</tr>
</tbody>
</table>

From Fig. 2, we see that the values of \(D_x(n)\) and \(D_z(n)\) decrease as \(n\) increases for each case in both binary and continuous tomography, as well as for each projection data set. Moreover, since \(D_x(n)\) is always less than \(D_z(n)\), it is suggested that the ECIR system can produce images more similar to the phantom image than those produced by the ML-EM method. We also found that the contribution of the ML-EM method to the minimization of the absolute relative error for the phantom image was smaller than that of the ECIR system in spite of it minimizing \(KL(\mathbf{A} \mathbf{w}, y)\).

4. Conclusion

We showed that the ECIR system is effective not only for binary tomography but also for continuous tomography such that the maximum pixel value to be reconstructed is previously given. In particular, for projection data with insufficient views, we found that the ECIR system exhibits better convergence performance than the ML-EM method. Because the ECIR system can be created as an analog electronic circuit or digital signal processor, its implementation in actual hardware yields faster image reconstructions than software-based methods. Consequently, the numerical results presented in this paper indicate that the use of hardware customized for the ECIR system can reduce X-ray doses to a human body in clinical CT scanning in the case of binary and continuous tomography with a small number of projection views.

References


Figure 2: Distances $D_x(n)$ and $D_z(n)$ for (a) binary and (b) continuous tomography

Figure 3: Reconstructed images for (a) binary and (b) continuous tomography