Abstract

In our previous study, we enhanced the predictive power of the principal component portfolio (PCP) model by applying a nonlinear prediction model. However, here we point out that this modification destroys the no-correlation relationship among the principal components, and accordingly the portfolio effect of risk reduction is weakened. To solve this problem, we mixed the advantages of the PCP model and our nonlinear portfolio model. To confirm the validity of this, we performed some investment simulations with real stock data and confirmed that our new portfolio model improves the predictive power and risk-reduction power simultaneously, that is, it improves the efficiency and safety of portfolio management.

1. Introduction

In portfolio theory, is reduced by including uncorrelated stocks in a portfolio. The first portfolio model that was developed was the mean-variance portfolio (MVP) model [1], which estimates the future return rate and risk using the simple moving average of historical stock prices. However, some stocks have positive correlations with each other in real financial markets, and for this reason the principal component portfolio (PCP) model [2] was proposed, which is composed of uncorrelated principal components to enhance the risk reduction effect. However, the above portfolio models are not active enough to predict price movements. Therefore, to improve the predictive power, in our previous study we proposed the nonlinear principal component portfolio (NPCP) model [3], which uses a nonlinear time-series prediction in analogy with the nonlinear portfolio (NP) model [4]. The NPCP and NP models use historical prediction errors to estimate investment risks, but the correlation among these prediction errors has not yet been investigated. If these prediction errors have some correlation, the risk reduction effect cannot work well in the NP and NPCP models. In the present study, we focus on the NPCP model and investigate the correlation among the prediction errors given by the principal components of return rates. If some correlation is confirmed, we modify the estimation of the risk for the NPCP model.

2. Previous Portfolio Models

We denote $x_i(t)$ as the price of the $i$th stock ($i = 1, 2, \cdots, N$) at time $t$, and then the return rate $r_i(t)$ is given by

$$r_i(t) = \frac{x_i(t) - x_i(t - 1)}{x_i(t - 1)} \quad (1)$$

2.1 Mean-variance portfolio (MVP) model

The mean-variance portfolio model [1] estimates the expected return rate $\bar{r}_i(t + 1)$, the expected risk $\bar{\sigma}_i^2(t + 1)$, and the covariance $\bar{\sigma}_{ij}(t + 1)$ using the moving average during the last $T$ periods as follows:

$$\bar{r}_i(t + 1) = \bar{r}_i(t) = \frac{1}{T} \sum_{a=0}^{T-1} r_i(t - a) \quad (2)$$

$$\bar{\sigma}_i^2(t + 1) = \sigma_i^2(t) = \frac{1}{T} \sum_{a=0}^{T-1} [r_i(t - a) - \bar{r}_i(t)]^2 \quad (3)$$

$$\bar{\sigma}_{ij}(t + 1) = \sigma_{ij}(t) = \frac{1}{T} \sum_{a=0}^{T-1} [r_i(t - a) - \bar{r}_i(t)][r_j(t - a) - \bar{r}_j(t)] \quad (4)$$

where $\sigma_{ii} = \sigma_i^2$. Next, the expected return rate $\bar{r}_p(t + 1)$ and the expected risk $\bar{\sigma}_p^2(t + 1)$ of the MVP model are given by

$$\bar{r}_p(t + 1) = \sum_{i=1}^{N} w_i \bar{r}_i(t + 1) \quad (5)$$

$$\bar{\sigma}_p^2(t + 1) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \bar{\sigma}_{ij}(t + 1) \quad (6)$$

where $w_i$ is the allocation rate of the $i$th stock and $\sum_{i=1}^{N} w_i = 1$.

If all of the investment stocks are uncorrelated with each other, $\bar{\sigma}_{ij}(t + 1)$ becomes 0. Therefore, Eq. (6) can be rewritten as

$$\bar{\sigma}_p^2(t + 1) = \sum_{i=1}^{N} w_i^2 \bar{\sigma}_i^2(t + 1) \quad (7)$$
Here, if we denote the upper limit of $\hat{\sigma}_p^2(t+1)$ as $C$, then
\[
\hat{\sigma}_p^2(t+1) \leq (w_1^2 + \cdots + w_N^2)C
\] (8)
In addition, if each allocation rate is $w_i = 1/N$, then
\[
0 \leq \hat{\sigma}_p^2(t+1) \leq \frac{C}{N}
\] (9)
Here, when the number of stocks $N$ is large enough, the risk of the portfolio $\hat{\sigma}_p^2$ approaches 0. This risk reduction is called the portfolio effect, which is derived from uncorrelated stocks. However, in real financial markets, some stocks have positive correlations with each other. For this reason, this risk reduction often does not work well with the MVP model.

Finally, to compose a portfolio, we optimize the allocation rates $\{w_i\}$ at each time $t$ by maximizing the Sharpe ratio:
\[
S_t(t) = \frac{\hat{r}_p(t+1) - r_f}{\hat{\sigma}_p(t+1)}
\] (10)
This maximization corresponds to maximizing $\hat{r}_p(t+1)$ and minimizing $\hat{\sigma}_p(t+1)$ simultaneously. Here, $r_f$ is a risk-free rate.

2.2 Principal component portfolio (PCP) model

The PCP model [2] was proposed to improve the risk reduction effect by using the principal components of all stocks, which are completely uncorrelated with each other.

First, let us denote the covariance matrix of Eq. (4) as $\Sigma(t)$, and we diagonalize it as follows:
\[
E^{-1}\Sigma(t)E = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}
\] (11)
where $E = [e_1, e_2, \ldots, e_N]$ is the eigenvector matrix and $\lambda_i$ indicates the eigenvalue corresponding to $e_i$. In addition, $[\cdot]^{-1}$ denotes the inverse matrix. Then, according to principal component analysis, the $l$th principal component score is calculated from the $l$th eigenvector $e_l$ as follows:
\[
r_l^T(t) = e_l^T r(t)
\] (12)
where $r(t) = [r_1(t), r_2(t), \ldots, r_N(t)]^T$ and $[\cdot]^T$ denotes the transposed matrix. The variance of the $l$th principal component score $r_l$ is equal to the $l$th eigenvalue $\lambda_l$.

Next, we compose a portfolio with these principal component scores $\{r_l\}$ and denote their allocation rates as $\{w_l\}$. Because each principal component score is completely uncorrelated with the other scores, Eq. (6) can be rewritten as follows:
\[
\hat{\sigma}_p^2(t+1) = \sum_{l=1}^{N} w_l^2 \lambda_l
\] (13)
Here, the relationship between $w_l^2$ and $w$ is
\[
w_l = e_l^T w
\] (14)
where $w = [w_1, w_2, \ldots, w_N]^T$.

Then, the contribution of the $l$th component to the total risk is defined by
\[
\phi_l(t) = \frac{w_l^2 \lambda_l}{\hat{\sigma}_p^2(t+1)}
\] (15)
Finally, Ref. [2] optimizes $\{w_l\}$ (i.e., $w$ in Eq. (14)) at each time $t$ by changing $\{\phi_l\}$ to maximize the following entropy:
\[
N_{E_{\text{ent}}}(t) = \exp \left( - \sum_{l=1}^{N} \phi_l(t) \log \phi_l(t) \right)
\] (16)

2.3 Nonlinear principal component portfolio (NPCP) model

The MVP model predicts $\hat{r}_i(t+1)$ using the simple moving average in Eq. (2), and the PCP model does not predict the future return rate. However, if the future movement is related to past movements in a complex manner, this relationship can be approximated by a higher-order nonlinear function $F$. For this reason, the NPCP model applies a nonlinear prediction method to the uncorrelated principal components $\{r_l\}$.

As a preprocess, we reconstruct an attractor $v_l(t)$ from the learning data $r_l^T(t)$ in a multidimensional state space using Takens’ embedding theorem:
\[
v_l(t) = [r_1^T(t), r_1^T(t-\tau_1), \ldots, r_l^T(t-(k_l-1)\tau_l)]
\] (17)
where $\tau_1$ is the delay time and $k_l$ is the embedding dimension. Then, we predict the future return rate of $r_l^T(t)$ using a prediction model as follows:
\[
\hat{r}_l^T(t+1) = [v_l(t), 1] \cdot F
\] (18)
where $F$ corresponds to a vector of the model parameters. If $F$ expresses nonlinear dynamics, the model parameters are changed locally depending on the inputs $v_l(t)$ to the function. Then, because similar inputs generate similar outputs, similar past data of $v_l(t)$ is important to estimate $F$ locally, and thus we apply the weighted least-squares method using the distance $y_l(t, a)$:
\[
y_l(t, a) = \| v_l(t) - v_l(t-a) \|
\] (19)
where $v_l(t)$ is the target input and $v_l(t-a)$ is different learning data during the $L$ most recent periods. Then, the weighting factor $\omega_l(t, a)$ is given by
\[
\omega_l(t, a) = \exp(-y_l(t, a))
\] (20)
Next, we compose the weighted matrix as
\[
\tilde{F} = [X^T W^T W]^{-1} X^T W^T W Y
\] (22)
where
\[
X = [v_1^T(t-1) \ v_1^T(t-2) \ \cdots \ v_l^T(t-L+(k_l-1)\tau_l)]^T
\]
\[
Y = [r_1^T(t-1), r_1^T(t-2), \ldots, r_l^T(t-L+(k_l-1)\tau_l+2)]^T
\]
This estimation of \( \hat{F} \) is approximated locally using a hyperplane, but the slope of each plane changes locally according to the state of each input \( v_i^0(t) \). Namely, this approximation corresponds to nonlinear regression with a global hypersurface.

Then, we can obtain the predicted value \( \hat{r}_p(t+1) \) by Eq. (18) using the estimated \( \hat{F} \), and calculate the expected return rate of a portfolio \( \hat{\eta}(t+1) \) by substituting \( \hat{r}_p(t+1) \) into Eq. (5):

\[
\hat{\eta}(t+1) = \sum_{i=1}^{N} w_i^0 \hat{r}_i^0 (t+1)
\]

(23)

Furthermore, the covariance of Eq. (4) is rewritten as follows:

\[
\hat{\sigma}_{lm}(t+1) = \sum_{a=0}^{T-1} \frac{1}{T} \left[ \sigma_{l(t-a)}^2 - \hat{r}_l(t-a) \right] \left[ \sigma_{m(t-a)}^2 - \hat{r}_m(t-a) \right]
\]

(24)

where \( \sigma_{l(t)} \) is the correct value of the predicted value \( \hat{r}_l \). Namely, in the NPCP model, the source of the investment risk is considered to be the prediction error of each principal component \( r_l^0 - \hat{r}_l \). Then, the total portfolio risk given by Eq. (6) can be rewritten as follows:

\[
\hat{\sigma}_p^2(t+1) = \sum_{i=1}^{N} \sum_{m=1}^{N} w_i^0 w_m^0 \hat{\sigma}_{lm}(t+1)
\]

(25)

Finally, similarly to the MVP model, we optimize the application rates \( \{w_i\} \) at each time \( t \) so as to maximize the Sharpe ratio \( S_r(t) \) in Eq. (10).

### 2.4 Modification of the NPCP model

As mentioned in Sect. 1, the correlation among the prediction errors given by each principal component \( r_l^0 - \hat{r}_l \) has not yet been investigated. If there are some positive correlations, the NPCP model cannot reduce the total risk \( \hat{\sigma}_p^2(t+1) \) even though principal component analysis is applied.

Figure 1(a) shows the correlation between \( \eta_l \) and \( \eta_m \), where \( \eta_l = r_l^0 - \hat{r}_l \). The results were calculated using 50 major stocks \( (N = 50) \) listed on the Tokyo Stock Exchange from 1997 to 2001. It can be seen that, there are some positive correlations. For comparison, we also show the correlation between \( r_l^0 \) and \( \hat{r}_m \) in Fig. 1(b). These principal components are used for the PCP model to reduce the total risk given by Eq. (13) because the correlation between them is zero. This fact can be confirmed in Fig. 1(b). Namely, the NPCP model might not be able to reduce the total risk as effectively as the PCP model.

To solve this problem, we apply the original definition of the risk given by the PCP model to the NPCP model, that is, we replace Eq. (25) by Eq. (13). We call this the modified NPCP (mNPCP) model. Although the PCP model does not estimate the future return rate, our mNPCP model attempts to make a prediction using the nonlinear prediction model.

To confirm the validity of our modification, we performed an investment simulation using 590 major stocks listed on the first section of the Tokyo Stock Exchange. To select more predictable stocks, the dataset from 1997 to 2001 was used for the learning period, and then we selected the \( N \) most predictable stocks for each portfolio model. After that, the dataset from 2002 to 2004 was used for the investment period to examine the investment performance. Here, we composed a long-only portfolio, which means that all of the allocation rates \( w_l \) were optimized by giving them positive values in each portfolio model. In addition, similarly to in the previous study [3], we set the historical data length \( T \) as 245, which is the number of business days in a year, and set \( L \) as 1200, i.e., about five years. For the nonlinear prediction in Sect. 2.3, we set \( \tau_1 = 1 \) so that Eq. (17) can compose an orthogonal state space because the autocorrelation of \( r_l^0(t) \) is almost zero at a lag of 1. Then, we set \( \kappa_l = 4 \) so that the orthogonal state space can be multidimensional. Here, it is better not to use a larger embedding dimension because \( L \) is limited. In particular, a well-known effect of the curse of dimensionality is that local neighbors of the predictee \( v(t) \) are no longer neighbors as the dimension becomes larger. If so, all of the weighted factors in Eq. (20) become flat, and therefore we set the embedding dimension as shown above.

Next, we evaluated the investment performance using the following measures: the asset amplification \( \zeta_A \) [times] based on simple interest is calculated as

\[
\zeta_A = \frac{M(\text{end})}{M(0)} \quad \text{where} \quad M(t+1) = M(t) + M(0) r_p(t+1)
\]

(26)

where \( M(\text{end}) \) is the final asset value and \( M(0) \) is the initial
Table 1: Investment performance of each portfolio model composed with the 50 most predictable stocks: The bold figure gives the best score in each category.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\zeta_A$</th>
<th>$\zeta_D$</th>
<th>$\zeta_W$</th>
<th>$\zeta_R$</th>
<th>$\zeta_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MVP</td>
<td>1.20</td>
<td>25.37%</td>
<td>52.50%</td>
<td>0.98</td>
<td>1.06</td>
</tr>
<tr>
<td>PCP</td>
<td>1.21</td>
<td>24.13%</td>
<td>47.17%</td>
<td>1.20</td>
<td>1.07</td>
</tr>
<tr>
<td>NPCP</td>
<td>2.26</td>
<td>11.48%</td>
<td>53.67%</td>
<td>1.13</td>
<td>1.33</td>
</tr>
<tr>
<td>mNPCP</td>
<td>4.99</td>
<td>3.77%</td>
<td>66.83%</td>
<td>1.65</td>
<td>3.01</td>
</tr>
</tbody>
</table>

The maximum drawdown $\zeta_D(\%)$ is calculated as

$$\zeta_D = \max_{1 \leq t' \leq t} D(t')$$

(27)

where $D(t') = 100 \cdot \left(1 - \frac{M(t')}{\max_{1 \leq t' \leq t} M(t')}\right)$

This measure means the cumulative losses since the previous largest asset value. Namely, when $\zeta_D$ is smaller, the portfolio management is safer. Then, the winning percentage is denoted as $\zeta_W(\%)$. The risk reward ratio $\zeta_R(\text{times})$ is calculated as the ratio between the average profit and average loss, and therefore it evaluates the efficiency of the investment.

$$\zeta_R = \langle \{r_p | r_p \geq 0\} \rangle / \langle \{r_p | r_p < 0\} \rangle$$

(28)

where $\langle \cdot \rangle$ means the average value. Finally, the profit factor $\zeta_P(\text{times})$ is calculated as the ratio between the total profit and total loss, namely, it evaluates the profitability of the investment.

$$\zeta_P = \Sigma \{r_p | r_p \geq 0\} / \Sigma \{r_p | r_p < 0\}$$

(29)

The given investment performance is shown in Table 1 and Fig. 2. We can confirm that our mNPCP model shows the best performance in terms of all the measures. In particular, the maximum drawdown $\zeta_D$ is the smallest and the profit factor $\zeta_P$ is the largest. Therefore, the mNPCP model simultaneously improves the safety and profitability of the other portfolio models. On the other hand, the NPCP model shows an unstable and large drawdown $\zeta_D$ of around 40(%) as shown in Fig. 2 because the positive correlations shown in Fig. 1(a) weaken the risk reduction effect. This difference shows the validity of our modification in Sect. 2.4. Moreover, regarding the suitable value of $N$ for our mNPCP model, roughly 30 stocks are sufficient as shown in Fig. 2. This is because a larger number of stocks can enhance the portfolio effect, but the composed portfolio includes less predictable stocks when $N$ is larger. Therefore, it is better not to use a too large number of stocks for our mNPCP model.

4. Conclusions

By analyzing real stock data, we found that the prediction errors given by the NPCP model have some positive correlations. Therefore, even though we apply principal component analysis, the portfolio effect for risk reduction cannot work well. To solve this problem, we mixed the advantages of the NPCP and PCP models. Namely, we adopt principal component analysis to improve the risk reduction power and apply nonlinear prediction to improve the predictive power of future return rates. Here, if this prediction is impossible, our portfolio model is exactly the same as the PCP model. However, we were able to confirm the improvement of these powers by performing investment simulations with real stock data from the viewpoint of in terms of the safety and efficiency of portfolio management.

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References